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Fixed-parameter tractability for the subset feedback set problem and the S-cycle packing problem

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ABSTRACT

We investigate generalizations of the following well-known problems in the framework of parameterized complexity: the feedback set problem and the cycle packing problem. Our problem setting is that we are given a graph and a vertex set S called "terminals". Our purpose here is to consider the following problems:

1. The feedback set problem with respect to the terminals S. We call it *the subset feedback set problem*.
2. The cycle packing problem with respect to the terminals S, i.e., each cycle has to contain a vertex in S (such a cycle is called an *S-cycle*). We call it *the S-cycle packing problem*.

We give the first fixed parameter algorithms for the two problems. Namely;

1. For fixed k, we can either find a vertex set X of size k such that $G - X$ has no S-cycle, or conclude that such a vertex set does not exist in $O(n^2m)$ time, where n is the number of vertices of the input graph and m is the number of edges of the input graph.
2. For fixed k, we can either find k vertex-disjoint S-cycles or conclude that such k disjoint cycles do not exist in $O(n^3)$ time.

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1. Introduction

Packing and covering vertex-disjoint cycles is one of the central areas in both graph theory and theoretical computer science. The starting point of this research area goes back to the following well-known theorem due to Erdős and Pósa [8] in early 1960s.

Theorem 1.1. *(See Erdős and Pósa [8].) For any k and any graph G, either G contains k vertex-disjoint cycles or a vertex set X of order at most $f(k)$ (for some function f of k) such that $G - X$ is acyclic.*

Theorem 1.1 concerns about both "packing", i.e., k vertex-disjoint cycles and "covering", i.e., at most $f(k)$ vertices that hit all the cycles in G. Starting with this result, there are a host of the results in this direction. Packing appears almost everywhere in extremal graph theory. Also, the cycle packing problem, which asks to find maximum number of vertex-disjoint (or edge-disjoint) cycles in an input graph G, is a well-known problem too. For example, Krivelevich et al. [19] give an $O(\sqrt{\log n})$-approximation algorithm for the edge-disjoint cycle packing problem and show some hardness results.

Also, "covering" leads to the well-known concept "feedback set" in theoretical computer science. The problem of finding a minimum feedback vertex set in a graph, i.e., the smallest set of vertices whose deletion makes the graph acyclic, has many applications and its history can be traced back to the early 60's (see the survey of Festa et al. [10]). It is also one of the classical NP-complete problems from Karp's list [16]. Thus not surprisingly, for several decades, many different algorithmic approaches were tried on this problem including approximation algorithms [2,3], linear programming [5], polyhedral combinatorics [4,12], exact algorithm [11] and parameterized complexity [14].

Natural generalizations of the feedback set problem and the cycle packing problem have been studied extensively in theoretical computer science.

The problem called "subset feedback set" is that we are given a graph G and a subset S of its vertices, and the goal is to find a vertex set X of minimum order such that $G - X$ has no S-cycle (for $S \subseteq V$, an *S-cycle* is a cycle which has a vertex in S). For this problem, Even et al. [9] give an 8-approximation algorithm for the subset feedback set problem.

The problem called "S-cycle packing" is that we are given a graph G and a subset S of its vertices, and the goal is to find among the cycles that intersect S a maximum number of vertex-disjoint (or edge-disjoint) ones. See [19] for the history of the cycle packing problem. As pointed out there, this problem is rather close to the well-known problem "the disjoint paths problem" [20], and approximation algorithms to find an S-cycle packing have been studied extensively.

In this paper, we are interested in the framework of parameterized complexity developed by Downey and Fellows [7] for both the packing problem and the feedback set problem. The standard goal of parameterized analysis is to take the parameter out of the exponent in the running time. A problem is called *fixed-parameter tractable* (FPT) if it can be solved in time $O(f(k)n^c)$, where n is the number of vertices of the input graph, c is a constant not depending on k, and f is an arbitrary function. An algorithm with such a running time is also called FPT.

We can trivially determine whether or not G has a vertex set X of order at most k such that $G - X$ has no S-cycle in $O(n^{k+2})$ time by enumerating all k vertices of G. Although this is polynomial time for each fixed k, it is practically too slow for large inputs, even if k is relatively small. Our first main result is the first FPT algorithm for the subset feedback set problem.

Theorem 1.2. *For a graph $G = (V, E)$, a terminal set $S \subseteq V$, and a fixed integer k, we can either find a vertex set X of size k such that $G - X$ has no S-cycle, or conclude that such a vertex set does not exist in $O(n^2 m)$ time, where n is the number of vertices and m is the number of edges.*

Note that, in 2010, a FPT algorithm for the subset feedback set problem is also given in [6] independently.

Second, we are interested in the packing problem. We give the first FPT algorithm for the S-cycle packing problem.

Theorem 1.3. *For a graph $G = (V, E)$, a terminal set $S \subseteq V$, and a fixed integer k, we can either find k disjoint S-cycles or conclude that such k disjoint cycles do not exist in $O(n^3)$ time.*

Let us observe that if $S = V(G)$, then we can find k disjoint cycles in linear time for fixed k, if they exist. Indeed, if a given graph has large tree-width, then we can do this from the existence of a large grid minor, and otherwise we can use the dynamic programming to find disjoint cycles.

On the other hand, this would not work for the S-cycle packing problem. In fact, the problem setting is closer to the well-known "the disjoint paths problem for fixed number of terminals" [20] as pointed out in [19]. Using the result in [20], we can determine whether or not G has k disjoint S-cycles in $O(n^{2k+3})$ time as follows: we enumerate all k pairs of vertices such that each pair contains at least one vertex in S, and for such pairs we apply Robertson–Seymour's $O(n^3)$ time algorithm for finding k disjoint paths [20]. We emphasize here that even obtaining a $n^{O(k)}$ time algorithm is non-trivial without using the graph minor theory, which implies that the S-cycle packing problem is harder than the subset feedback set problem. Thus we shall use some tools from the graph minor theory.

2. Preliminaries

2.1. Basic notations

Let $G = (V, E)$ be a graph with a vertex set V and an edge set E. In this paper, n and m always mean the number of vertices of a given graph and the number of edges of a given graph, respectively. For a subgraph H of G, the vertex set and the edge set of H are denoted by $V(H)$ and $E(H)$, respectively. For $X \subseteq V$, the *subgraph induced by* X, denoted by $G[X]$, is the subgraph $G' = (X, F)$, where F consists of all edges in E with both ends in X. Let $N(X)$ be the neighbor of X, i.e., the set of all vertices adjacent to X. For an edge $e \in E$, *contracting* e means the operation that deletes e, identifies the end vertices of e, and removes parallel edges (if exist). For an edge set $F \subseteq E$, let G/F denote the graph obtained from G by contracting all edges in F. Similarly, contracting a subgraph H means contracting $E(H)$ and contracting a vertex set X means contracting $G[X]$. For two graphs G and H, we say that H *is a minor of* G (or G *has a* H*-minor*), if there exists an edge set $F \subseteq E(G)$ such that G/F contains H as a subgraph.

For an integer p, K_p is the complete graph with p vertices. A graph G contains a K_p*-model* if there exists a function σ with domain $V(K_p) \cup E(K_p)$ such that

1. for each vertex $v \in V(K_p)$, $\sigma(v)$ is a tree of G, and the trees $\sigma(v)$ ($v \in V(K_p)$) are pairwise vertex-disjoint, and
2. for each edge $e = uv \in E(K_p)$, $\sigma(e)$ is an edge $f \in E(G)$, such that f is incident in G with a vertex in $\sigma(u)$ and with a vertex in $\sigma(v)$.

Thus G contains a K_p-minor if and only if G contains a K_p-model. We call the tree $\sigma(v)$ ($v \in V(K_p)$) the *node* of the K_p-model. The image of σ, which is a subgraph of G, is called the K_p*-model*.

Let $H = (V_H, E_H)$ be a subgraph of $G = (V, E)$. For $X \subseteq V$ and $F \subseteq E$, let $G - X$ be the subgraph of G induced by $V \setminus X$, and let $G - F = (V, E \setminus F)$. For subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ of G, define $H_1 \cup H_2 = (V_1 \cup V_2, E_1 \cup E_2)$, $H_1 \cap H_2 = (V_1 \cap V_2, E_1 \cap E_2)$, and $H_1 - H_2 = H_1 - V(H_2)$. A *separation* of a graph G is a pair of subgraphs (A, B) of G such that $G = A \cup B$ and $E(A \cap B) = \emptyset$. The *order* of the separation (A, B) is $|V(A) \cap V(B)|$.

For a vertex set $S \subseteq V$, a cycle is called an S*-cycle* if it contains a vertex in S. We say that a vertex set $X \subseteq V$ is an S*-cycle feedback set* if $G - X$ contains no S-cycles. For $S, T \subseteq V$ with $S \cap T = \emptyset$, an S*-path with respect to* T is a path with end vertices in T that contains a vertex in S.

2.2. Tree-width and wall

The *tree-width* of a graph is defined as follows.

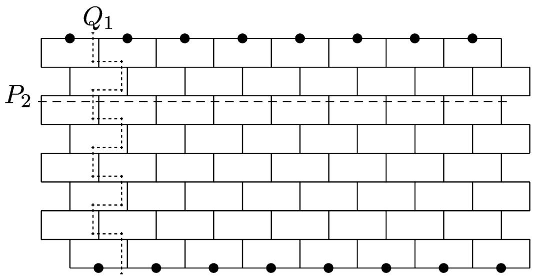

Fig. 1. An elementary wall of height 8.

Definition 2.1. Let G be a graph, T a tree and let $\mathcal{V} = \{V_t \subseteq V(G) \mid t \in V(T)\}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following three conditions:

- $V(G) = \bigcup_{t \in T} V_t$,
- for every edge $e \in E(G)$ there exists a $t \in T$ such that both ends of e lie in V_t,
- if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'', then $V_t \cap V_{t''} \subseteq V_{t'}$.

The *width* of $(T, \mathcal{V})$ is the number $\max\{|V_t| - 1 \mid t \in T\}$ and the *tree-width* $\mathrm{tw}(G)$ of G is the minimum width of a tree-decomposition of G.

An elementary wall of height eight is depicted in Fig. 1. An *elementary wall* of height h for $h \geqslant 3$ is similar. It consists of h levels each containing h bricks, where a brick is a cycle of length six. A *wall* of height h is obtained from an elementary wall of height h by subdividing some of the edges, i.e., replacing the edges with internally vertex disjoint paths with the same endpoints (see Fig. 2). The *nails* of a wall are the vertices of degree three within it.

Any wall has a unique planar embedding. The *perimeter* of a wall W, denoted *per*(W) is the unique face in this embedding which contains more than six nails. For any wall W in a given graph G, there is a unique component U of $G - \mathrm{per}(W)$ containing $W - \mathrm{per}(W)$. The *compass* of W, denoted $\mathrm{comp}(W)$, is the subgraph of G induced by the vertex set $V(U) \cup V(\mathrm{per}(W))$.

A *subwall* of a wall W is a wall which is a subgraph of W. A wall is *flat* if its compass does not contain two vertex-disjoint paths connecting the diagonally opposite corners. Note that if the compass of W has a planar embedding whose infinite face is bounded by the perimeter of W then W is clearly flat. It is shown in [22,23] that a wall W is flat if and only if there are pairwise disjoint sets $A_1, \dots, A_l \subseteq V(\mathrm{comp}(W))$ $(l \geqslant 0)$ containing no corners of W such that

(1) for $1 \leqslant i, j \leqslant l$ with $i \neq j$, $N(A_i) \cap A_j = \emptyset$,
(2) for $1 \leqslant i \leqslant l$, $|N(A_i)| \leqslant 3$, and
(3) if W' is the graph obtained from $\mathrm{comp}(W)$ by deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$ for each i, then W' may be drawn in a plane so that all corners of W are on the outer face boundary.

If such $A_1, \dots, A_l$ exist, we say that $\mathrm{comp}(W)$ *can be embedded into a plane up to 3-separations*, and an embedding as in (3) is called a *flat embedding*.

2.3. Folio

As we mentioned in the end of Section 1, we use some tools from the graph minor theory. In this subsection, we state some results of Robertson and Seymour.

In [20], Robertson and Seymour gave a polynomial-time algorithm for the vertex-disjoint paths problem for fixed number of terminals. Actually, they solved a generalized problem called *folio*. For

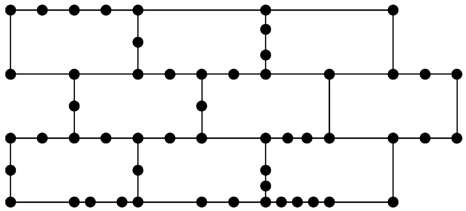

Fig. 2. A wall of height 3.

a vertex set X, a partition $\mathcal{X} = \{X_1, \dots, X_q\}$ of X is *realizable* if there are disjoint trees $T_1, \dots, T_q$ in G such that $X_i \subseteq V(T_i)$ for $i = 1, \dots, q$. We say that a vertex $v \in V \setminus X$ is *irrelevant with respect to* X when a partition of X is realizable in $G - v$ if and only if it is also realizable in G. The list of realizable partitions of X is called the *folio relative to* X, and the problem of computing it is also called the *folio*.

It is known that the folio can be solved in polynomial time if the tree-width is bounded.

Theorem 2.2. *(See [1,20].) For integers w and k, there exists a $(k + w)^{O(k+w)} O(n^2)$ time algorithm for computing the folio relative to a set of k vertices in graphs of tree-width w. Furthermore, if w and k are fixed, there exists an $O(n)$ time algorithm.*

When tree-width is large, in Robertson–Seymour's algorithm for the disjoint paths problem or the folio, they first find a large clique minor or a large "almost flat" wall. The precise description of their theorem is as follows.

Theorem 2.3. *(See Robertson–Seymour [20, Theorem (9.8)].) For any p and any h there are computable constants $g_1(p, h)$ and $g_2(p, h)$ such that, if a given graph G has tree-width at least $g_1(p, h)$, then there is an $O(nm)$ time algorithm to find either a K_p-minor or a pair (X, W) satisfying the following conditions*:

(C1) *X is a vertex set with $|X| \leqslant \binom{p}{2}$,*
(C2) *W is a flat wall of height h in $G - X$,*
(C3) *all the components $A_1, \dots, A_l$ (as in the definition "up to 3-separations") have tree-width at most $g_2(p, h)$.*

Note that the statement of [20, Theorem (9.8)] is stated in terms of "branch-width" instead of tree-width, it does not cause any problems because branch-width differs only by a constant factor from tree-width. We also note that there is now an $O(n)$ time algorithm to either a K_p-minor or a pair (X, W) satisfying (C1)–(C3) in [18].

Robertson and Seymour find an irrelevant vertex if the graph contains a large clique minor or a large flat wall. The following theorem plays a crucial role in their algorithm when the graph has a large clique minor. Actually, this theorem is used to find an irrelevant vertex in a clique model in $O(m)$ time.

Theorem 2.4. *(From [20, Theorem (5.3)] by Robertson–Seymour.) Let Z be a vertex set with $|Z| = 2k$ in a given graph G. Suppose that there is a clique model K of order at least $3k$ in G, and there is no separation (A, B) of order at most $2k - 1$ in G such that A contains Z and $B - A$ contains at least one node of the clique model. Then, we can find mutually disjoint connected subgraphs $H_1, \dots, H_{2k}$ of G such that $|V(H_i) \cap Z| = 1$ for every i and there is an edge between H_i and H_j for every $i \neq j$, in $O(m)$ time. Moreover, they satisfy the following.*

- *The edges between H_i and H_j are contained in $E(K)$.*
- *If the order of K is at least $3k + 1$, then we can take $H_1, \dots, H_{2k}$ not intersecting with some node of K, and the vertices in the node of K are irrelevant to the folio relative to Z.*

We note that symbols k, ξ, μ, and G_i in [20, Theorem (5.3)] correspond to $3k, 2k, 0$, and a node of K in the above statement, respectively. If we have a large flat wall, we can find an irrelevant vertex by the following theorem.

Theorem 2.5. *(See Robertson and Seymour [21], see also [20, Theorem (10.2)].) For fixed integers k, p, there is a computable constant $h_2(k, p)$ satisfying the following: if there is a subset $X \subseteq V(G)$ of order at most p such that there is a flat wall W of height $h_2(k, p)$ in $G - X$, then there is a vertex v in W such that v is irrelevant to the folio relative to a set of k vertices. Furthermore, if all the components $G[A_1], \ldots, G[A_l]$ have tree-width bounded by a fixed constant, where $A_1, \ldots, A_l$ $(l \geqslant 0)$ are as in the flat embedding of* $\mathrm{comp}(W)$*, we can find in $O(m)$ time the irrelevant vertex v.*

Note that Robertson and Seymour actually showed that the "middle" vertices of a large flat wall are irrelevant. With these theorems, Robertson and Seymour gave a polynomial-time algorithm for the folio.

Theorem 2.6. *(See Robertson and Seymour [20].) For a fixed integer k, the folio with k terminals in graphs can be solved in $O(n^3)$ time.*

Note that the running time of their algorithm is improved to $O(n^2)$ time in [18]. We now introduce a new concept S-*folio*, which is similar to the folio. Let $G = (V, E)$ be a graph, $S \subseteq V$ be a terminal set, and $X \subseteq V$ be a vertex set. Let $\mathcal{X} = \{(X_1, s_1, t_1, \delta_1), \ldots, (X_q, s_q, t_q, \delta_q)\}$ be a set of quadruples, where $X_1, X_2, \ldots, X_q$ are mutually disjoint subsets of X, s_i and t_i are distinct vertices in X_i, and $\delta_i \in \{0, 1\}$ for $i = 1, 2, \ldots, q$. We say that $\mathcal{X}$ is S-*realizable* if there are paths $P_1, \ldots, P_q$ in G such that

- $X \cap V(P_i) = X_i$,
- end vertices of P_i are s_i and t_i, and
- P_i contains a vertex in S if and only if $\delta_i = 1$

for $i = 1, 2, \ldots, p$. The S-*folio* relative to X in G is the set of all S-realizable sets of quadruples.

In the same way as Theorem 2.2, the S-folio can be computed in polynomial time in graphs of bounded tree-width.

Theorem 2.7. *(See [1,20].) For integers w and k, there exists a $(k + w)^{O(k+w)} O(n^2)$ time algorithm for computing the S-folio relative to a set of k vertices in graphs of tree-width w. Furthermore, if w and k are fixed, there exists an $O(n)$ time algorithm.*

3. Algorithm for finding a feedback vertex set

Suppose we are given a graph $G = (V, E)$ and a terminal set $S \subseteq V$. Recall that a vertex set $X \subseteq V$ is said to be an S-*cycle feedback vertex set* if $G - X$ contains no S-cycle. The objective of this section is to prove Theorem 1.2, that is, to give an $O(n^2 m)$ algorithm for the following problem.

Subset Feedback Vertex Set.
Input. A graph $G = (V, E)$, a terminal set $S \subseteq V$, and a fixed integer k (parameter).
Problem. Find an S-cycle feedback vertex set $X \subseteq V$ of size k, or conclude that such a vertex set does not exist.

3.1. Overview

To show Theorem 1.2, we use a standard technique for fixed parameter tractable problems. We use the following proposition, whose proof is given later.

Proposition 3.1. *Suppose that we are given a fixed integer l, a graph $G = (V, E)$, a terminal set $S \subseteq V$, and an S-cycle feedback vertex set T of size l. Then, we can find an S-cycle feedback vertex set T' of size $l - 1$ with $T' \cap T = \emptyset$ or conclude that such a vertex set does not exist in $O(nm)$ time.*

By applying this proposition repeatedly, we obtain Theorem 1.2 as follows.

Proof of Theorem 1.2. We begin with a subgraph G_0 of G with $k+1$ vertices and its S-cycle feedback vertex set[3] T_0 of size k. In each step, by adding one new vertex to the subgraph, we obtain a new subgraph G_1 of G with an S-cycle feedback vertex set T_1' of size $k+1$. We apply Proposition 3.1 for every set $T \subseteq T_1'$ in a graph $G_1 - (T_1' \setminus T)$. Then, we can obtain an S-cycle feedback vertex set T_1 of size k in G_1 or conclude that such a vertex set does not exist. Note that the number of subsets of T_1' is at most 2^{k+1}, which is a constant depending only on k. By repeating this procedure at most n times, in $O(n^2m)$ time, we obtain an S-cycle feedback vertex set of size k if exists. $\square$

We now give a high-level description of the proof of Proposition 3.1. Our framework follows Robertson–Seymour's algorithm for the disjoint paths problem described in Section 2.3.

Suppose that we are given a graph $G = (V, E)$, a terminal set $S \subseteq V$, and an S-cycle feedback vertex set T of size l. The first step of our algorithm is to examine whether or not the tree-width of $G - T$ is large. If it is bounded by a fixed constant, then we can find an S-cycle feedback vertex set T' of size $l-1$ with $T' \cap T = \emptyset$ or conclude that such a vertex set does not exist in $O(n)$ time by applying a standard dynamic programming technique to a tree-decomposition of bounded width similarly to Theorem 2.7. Note that tree-width of G is bounded by tree-width of $G - T$ plus $|T|$. Otherwise, we apply Theorem 2.3 to G (p and h will be given later) and obtain either a large clique minor or a large "almost flat" wall. For both cases, we shall find an "irrelevant" vertex v in $G - T$. Here, we say that a vertex $v \in V \setminus (S \cup T)$ is called *l-irrelevant* when G has an S-cycle feedback vertex set T' of size $l - 1$ with $T' \cap T = \emptyset$ if and only if $G - v$ has an S-cycle feedback vertex set T'' of size $l - 1$ with $T'' \cap T = \emptyset$. We remove an l-irrelevant vertex v and go back to determine whether or not the tree-width is bounded. By repeating this at most n times, we obtain a desired S-cycle feedback vertex set T'.

Therefore, the remaining task in this algorithm is to find an l-irrelevant vertex in the large clique minor or the large "almost flat" wall efficiently. In what follows, we consider these two cases, separately.

3.2. Large clique minor

Suppose that $G - T$ contains a K_p-minor. The objective of this subsection is to show that we can find an l-irrelevant vertex in the clique minor.

We begin with the following lemma. We say a K_p-model K is *minimal* if for every vertex v and for every edge e in K, both $K - v$ and $K - e$ do not have a K_p-model.

Lemma 3.2. *Let p, l be integers, and let K be a minimal K_p-model. Then, for any vertex set U of K with $|U| \leqslant l$, $K - U$ consists of at most $\binom{l}{2} + 1$ connected components.*

Proof. Let $V_1, \ldots, V_p$ be the nodes of the K_p-model. Then, $G[V_i]$ is a tree by the definition of a K_p-model. Since at most l of these sets intersect with U, we may assume that each of $V_1, \ldots, V_{l'}$ intersects with a vertex in U and each of $V_{l'+1}, \ldots, V_p$ does not intersect with any vertex in U for some $l' \leqslant l$. Clearly $V_{l'+1}, \ldots, V_p$ are contained in the same connected component K^* of $K - U$.

[3] Formally, it should be written as "$(S \cap V(G_0))$-cycle feedback vertex set". However, we denote "S-cycle feedback vertex set" for simplicity.

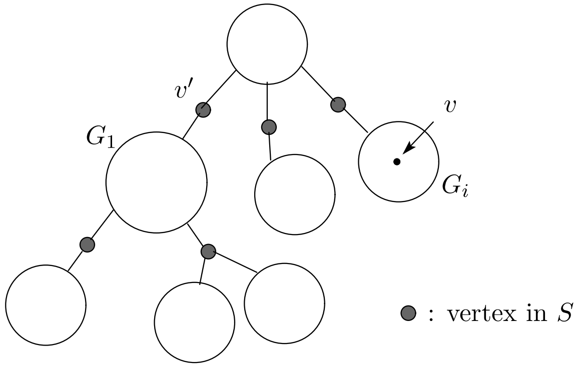

Fig. 3. Structure of $G-T$.

For any $1 \leqslant i \leqslant l'$, if $|V_i \cap U| = t_i$, then there are at most $t_i - 1$ connected components of $G[V_i] - U$ containing no leaf of $G[V_i]$. By the minimality of K, there are at most $\binom{l'}{2}$ leaves of $G[V_i]$ that are not contained in K^*. Hence, the number of connected components of $K - U$ is at most

$$\sum_{i=1}^{l'}(t_i - 1) + \binom{l'}{2} + 1 = |U| - l' + \binom{l'}{2} + 1,$$

which is maximum when $l' = |U| = l$. Thus, $K - U$ consists of at most $\binom{l}{2} + 1$ connected components. $\square$

With this lemma, we can find an l-irrelevant vertex as follows.

Lemma 3.3. *Let $l \geqslant 2$ be an integer. Suppose we are given a graph $G = (V, E)$, a terminal set S, an S-cycle feedback vertex set T of size l, and a clique minor of size $p = \frac{5}{2}l^3 + 1$ in $G - T$. Then, in $O(m)$ time, either we can find an l-irrelevant vertex, or we can conclude that there is no S-cycle feedback vertex set T' of size $l - 1$ with $T' \cap T = \emptyset$.*

Proof. We may assume that G is 2-connected. Let $G_1, \dots, G_r$ be the connected components of $G - T - S$. Let K be a minimal K_p-model in $G - T$. Since $G - T$ contains no S-cycle, and hence the clique model K does not contain any vertex in S, therefore we may assume that K is contained in G_1.

Let G' be the graph obtained from G by removing all edges connecting T and $V(G_1)$. From each vertex t of T, we try to find l^2 paths in G' to K that are mutually vertex disjoint except for t. If such paths exist, then there exist at least $l^2 - |T'|$ paths from t to $K - T'$ for any set $T' \subseteq V(G) \setminus T$ with $|T'| \leqslant l - 1$. Since removing $T' \subseteq V(K)$ splits K into at most $\binom{|T'|}{2} + 1 < l^2 - |T'|$ connected components by Lemma 3.2, some component of $K - T'$ is connected by two internally disjoint paths from t, which means that $G - T'$ contains a cycle containing t. We note that this cycle contains a vertex in S, because every path from t to K contains a vertex in S by the construction of G'.

Thus, we only consider the case when there exists a vertex set C_t for each $t \in T$ such that $|C_t| \leqslant l^2 - 1$ and C_t separates t and K in G'. Since $G - T$ contains no S-cycles, if C_t contains a vertex v not contained in $V(G_1) \cup N(V(G_1))$, then there exists a vertex $v' \in N(V(G_1))$ such that v and G_1 are not connected in $G - T - \{v'\}$ (see Fig. 3), which means that $(C_t \setminus \{v\}) \cup \{v'\}$ also separates t and K in G'. Thus, we may assume that $C_t \subseteq V(G_1) \cup N(V(G_1))$ for each $t \in T$.

Then $C = T \cup \bigcup_{t \in T} C_t$ separates S and K in G, because we assume the 2-connectivity of G. Hence, there is a separation (A', B') of G such that A' contains all vertices in S, B' contains a node of K, and $|V(A') \cap V(B')| = |C| =\leqslant |T| + (l^2 - 1)|T| = l^3$. Note that since $p > l^3$, there must exist a node of K in $B' - A'$. Let (A, B) be a separation of minimum order (at most l^3) such that A contains all vertices in S and $B - A$ contains a node of K. Then $B - A$ contains at least $p - l^3 = \frac{3}{2}l^3 + 1$ nodes of K, and hence it has a K_{p-l^3}-minor.

Since $B - A$ contains no vertices in S, the existence of S-cycles depends on the folio relative to $V(A) \cap V(B)$ in B, and we do not necessarily need all information of B. By applying Theorem 2.4 with $Z = V(A) \cap V(B)$ and this K_{p-l^3}-minor, we can find mutually disjoint connected subgraphs $H_1, \ldots, H_{|Z|}$ in B such that $|V(H_i) \cap Z| = 1$ for every i and there is an edge between H_i and H_j for every $i \neq j$ in $O(m)$ time. Also there is a vertex v in $V(K)$ not contained in any H_i, which is an l-irrelevant vertex. To see this, let v_i be the vertex in $V(H_i) \cap Z$ and let $X \subseteq V - T$ be an S-cycle feedback vertex set in $G - v$ with $|X| \leqslant l - 1$. Then, by the definition of H_i, the vertex set X' defined by

$$X' = \bigl(X \setminus V(B)\bigr) \cup \{v_i \mid X \cap H_i \neq \emptyset\}$$

is an S-cycle feedback vertex set in G with $|X'| \leqslant |X| \leqslant l - 1$. This means that v is l-irrelevant.

Since all these procedures can be done in linear time, the total running time is linear. $\square$

3.3. *Large wall*

Suppose that we have a pair (X, W) in $G - T$ satisfying (C1)–(C3). Then, we can find an l-irrelevant vertex by the following lemma.

Lemma 3.4. *For any integers p and l, there exists an integer $h_1(p, l)$ satisfying the following. Suppose we are given a graph $G = (V, E)$, a terminal set S, an S-cycle feedback vertex set T of size l, and a pair (X, W) in $G - T$ such that*

- *X is a vertex set with $|X| \leqslant \binom{p}{2}$,*
- *W is a flat wall of height $h_1(p, l)$ in $G - T - X$.*

Then, in $O(m)$ time, either we can find an l-irrelevant vertex or conclude that there is no S-cycle feedback vertex set T' of size $l - 1$ such that $T' \cap T = \emptyset$.

Proof. Let $h = l^2 + 1$, and we show that $h_1(p, l) = (\binom{p}{2} + l)(2l + 2)(h + 1)$ satisfies the condition. We may assume that G is 2-connected. Suppose the flat wall W is of height $h_1(p, l) \geqslant |X \cup T|(2l + 2) \times (h + 1)$. Then W contains $|X \cup T|(2l + 2)$ disjoint flat subwalls $W_1, \ldots, W_{|X \cup T|(2l+2)}$ of height h. By the 2-connectivity of G, for each i, if $\mathrm{comp}_{G-T-X}(W_i)$ contains a vertex in S, then there exist a vertex $x \in X \cup T$ and a path P such that P connects x and W_i, $V(P) \subseteq \mathrm{comp}_{G-T-X}(W_i) \cup \{x\}$, and P contains a vertex in S. If such a path P exists, we say that x is *S-attached* to W_i. Now we observe the following.

- Since $G - T$ contains no S-cycle, for each $x \in X$, x is S-attached to at most one of $W_1, \ldots, W_{|X \cup T|(2l+2)}$.
- For each $x \in T$, if x is S-attached to at least $l + 1$ of $W_1, \ldots, W_{|X \cup T|(2l+2)}$, then there exist $l + 1$ internally vertex-disjoint paths from x to W each containing a vertex in S, which shows that there is no S-cycle feedback vertex set T' of size $l - 1$ such that $T' \cap T = \emptyset$.

By these observations, in what follows, we may assume that x is S-attached to at most l of $W_1, \ldots, W_{|X \cup T|(2l+2)}$ for each $x \in X \cup T$. Then, there exist at least $|X \cup T|(2l + 2) - |X \cup T|l = |X \cup T|(l + 2)$ subwalls whose compasses contain no vertex in S. By changing the indices if necessary, let $W_1, \ldots, W_{|X \cup T|(l+2)}$ be such subwalls.

We say that a vertex x in $X \cup T$ is *universal* if x has neighbors in at least $l + 2$ of $\mathrm{comp}_{G-T-X}(W_1), \ldots, \mathrm{comp}_{G-T-X}(W_{|X \cup T|(l+2)})$. By pigeon hole principle, there is a vertex set $X' \subseteq X \cup T$ and a wall W' of height h such that X' is the set of vertices in $X \cup T$ which has neighbors in $\mathrm{comp}_{G-T-X}(W')$, and each vertex in X' is universal. In particular, this condition implies that W' is flat in $G - X'$.

We claim that the middle vertex v in W' is l-irrelevant. To show this, it suffices to show that any minimal S-cycle feedback vertex set T' of order l with $T' \cap T = \emptyset$ does not contain v. Suppose for a contradiction that v is in T'. Thus $G-(T'-\{v\})$ has an S-cycle C through $s \in S$. Note that $W'-T'$ still has a (non-proper) subwall W'' of height $h-l$.

Since $G-T'$ has no S-cycle, there exists a separation (A', B') of order one in $G-T'$ such that $B'-A'$ contains $V(W'')$ and $A'-B'$ contains s. We take such a separation such that A' is as small as possible. We can observe that if a nail u of W' is contained in A', then the row and the column of W' containing u have to intersect with T'. With this observation, one can see that A' contains at most l^2 nails of W', because at most l rows and at most l columns of W' intersect with T'. Furthermore, $A'-B'$ cannot contain a universal vertex $x' \in X'$, because there exist $l+2$ internally disjoint paths from a universal vertex x' to the wall W'' in the original graph G.

On the other hand, the existence of C implies that v is adjacent to a vertex in $A'-B'$, which means $A'-B'$ contains a neighbor of v. This neighbor u has to be in $\mathrm{comp}_{G-X'}(W')$, since $u \notin X'$, whereas s is not contained in $\mathrm{comp}_{G-X'}(W')$. By our choice of the separation (A', B'), there is a path P from u to s in $A'-B'$. Since P has to go through at least $\frac{h}{2}$ nested cycles of the wall W', A' has to contain at least $\frac{h}{2} > l^2$ nails of W', which is a contradiction.

Thus v cannot be contained in a minimal S-cycle feedback vertex set T' of order l with $T' \cap T = \emptyset$, which shows that v is l-irrelevant. It is easy to see that we can find the middle vertex v in W' in linear time. $\square$

3.4. *Proof of Proposition 3.1*

Now we are ready to give a proof of Proposition 3.1. Set $p = \frac{5}{2}l^3 + 1$ and $h = h_1(p, l)$. Here is the description of our algorithm.

Algorithm for finding a smaller S-cycle feedback vertex set.
Input. A graph $G = (V, E)$, a terminal set $S \subseteq V$, and an S-cycle feedback vertex set T of size l.
Output. Find an S-cycle feedback vertex set T' of size $l-1$ with $T' \cap T = \emptyset$ or conclude that such a vertex set does not exist.

Step 1. Determine whether tree-width of G is at most $g_1(p,h) + |T|$ or not. If it is at most $g_1(p,h) + |T|$, then solve the problem by a dynamic programming technique in a similar way as Theorem 2.7. Otherwise, go to Step 2.

Step 2. Apply Theorem 2.3 to $G-T$ and obtain either a K_p-minor or a pair (X, W) satisfying the conditions in Lemma 3.4. If we have a K_p-minor, then apply Lemma 3.3 to find an l-irrelevant vertex. Otherwise, apply Lemma 3.4 to find an l-irrelevant vertex. Then, remove the l-irrelevant vertex and go to Step 1.

Since Step 2 can be done in $O(m)$ time, the total running time is $O(nm)$. This completes the proof of Proposition 3.1. $\square$

4. Algorithm for packing S-cycles

In this section, we prove Theorem 1.3, that is, we give an $O(n^3)$ time algorithm for the following problem for fixed k.

S-cycle Packing.
Input. A graph $G = (V, E)$, a terminal set $S \subseteq V$, and a fixed integer k (parameter).
Problem. Find k vertex-disjoint S-cycles in G, or conclude that such cycles do not exist.

In the same way as the procedure for the Subset Feedback Vertex Set, our algorithm for the S-cycle Packing consists of the following two steps. First, we examine whether or not the tree-width of G is large. If it is bounded by a fixed constant, then we can solve the problem by Theorem 2.7. Otherwise, we apply Theorem 2.3 to G (p and h will be given later) and obtain either a K_p-minor or a pair (X, W) satisfying (C1)–(C3). For both cases, we find an *irrelevant* vertex v, i.e., a vertex v such that G has a solution if and only if so does $G - v$, remove v, and go back to determine whether or not the tree-width is bounded.

In what follows, we give an algorithm for finding an irrelevant vertex in the large clique minor or the large "almost flat" wall.

4.1. Large clique minor

We say that a K_p-model is *even* if the K_p-model is a bipartite graph, which is called a *bipartite expansion* in [13]. We also say that a K_p-model is *odd* if for each cycle C in the union of the nodes of the K_p-model, the number of edges in C that belong to nodes of the K_p-model is even. Let $v \in V(K_p)$. A *center* for $\sigma(v)$ is a vertex $t \in V(\sigma(v))$ such that for each component H of $\sigma(v) - t$, the number of edges $e \in E(K_p)$ such that $\sigma(e)$ is incident in G with a vertex of H is at most half the number of edges in K_p incident with v. It is not hard to see that every node $\sigma(v)$ has a center (perhaps more than one). Thus we assume that for each node, one of its centers has been selected, and we often speak of the center of a node without further explanation. Recall that, for $S, T \subseteq V$ with $S \cap T = \emptyset$, an *S-path with respect to T* is a path with end vertices in T containing at least one vertex of S.

Set $p = p' + 3k$ when we apply Theorem 2.3, and suppose that G has a K_p-model K'. We may assume that K' is minimal. If there exists a node of the K_p-model that contains a vertex in S, then we can find an S-cycle and a K_{p-3}-model that are mutually disjoint. By finding a node that contains a vertex in S repeatedly, we can obtain either k vertex-disjoint S-cycles or a $K_{p'}$-model K containing no vertices of S. Therefore, in what follows in this subsection, we assume that we have such a $K_{p'}$-model K.

Now we shall prove the following.

Theorem 4.1. *Suppose there is a K_{36k}-model containing no vertices of S. Let $T = \{t_1, \ldots, t_{36k}\}$ be the centers of the nodes of the K_{36k}-model. If there are $12k$ vertex-disjoint S-paths $P_1, \ldots, P_{12k}$ with respect to T, then G has k vertex-disjoint S-cycles in the union of the nodes of the K_{36k}-model, together with vertex-disjoint S-paths $P_1, \ldots, P_{12k}$.*

Proof. We construct a graph G' from G as follows. We first subdivide every edge with a new vertex, and, for every vertex in S, add an edge between it and all its original neighbors.

It is easy to see that we have an even K_{36k}-model K in G', and all vertices in T are still center. Moreover, if a path connecting two vertices of T in G' is odd, then the corresponding path in G contains a vertex of S, i.e., an S-path. Furthermore, an S-path with respect to T in G gives rise to an odd path connecting two vertices of T in G'. Therefore, G' has k disjoint odd paths with end vertices in T if and only if G has k disjoint S-paths with respect to T.

A path P with end vertices in an even clique model is called *parity breaking* if P together with the even clique model gives rise to an odd cycle. Note that in the graph G' defined as above, a path with end vertices in T is parity breaking if and only if it has odd length. So we can take parity breaking paths $P'_1, \ldots, P'_{12k}$ with respect to K that correspond to $P_1, \ldots, P_{12k}$. We now use the following result in [13] (see also [17]).

Lemma 4.2. *(See Geelen et al. [13].) Let K' be an even K_{12k}-model in G. If there are $4k$ vertex-disjoint parity-breaking paths with respect to K' such that the $8k$ endpoints of these paths are the centers of distinct node of K', then G contains an odd K_k-model in K'.*

By Lemma 4.2, the even K_{36k}-model, together with the paths $P'_1, \ldots, P'_{12k}$ gives rise to an odd K_{3k}-model K''. Hence K'' contains k vertex-disjoint odd cycles such that each of the odd cycles contains exactly three nodes of the odd K_{3k}-model K'. On the other hand, each such an odd cycle has to contain at least one vertex in S by our construction of G'. Let us observe that each vertex in S is contained in a triangle in G', with one vertex of degree two in G'. But this triangle cannot be any of the above k vertex-disjoint odd cycles because each of the odd cycles contains exactly three nodes of the odd K_{3k}-model K'. Thus these k vertex-disjoint odd cycles in G' correspond to k vertex-disjoint S-cycles in G. This completes the proof of Theorem 4.1. $\Box$

We also use the following theorem, which is obtained from the algorithm for finding odd paths in [13] and the reduction from S-paths to odd paths in [15]. Note that a non-bipartite matching algorithm is used in [13]. Since we find an augmenting path at most k times in the algorithm, the running time is bounded by $O(kn^2)$.

Theorem 4.3. *(See Geelen et al. [13] and Kakimura et al. [15].) Let $G = (V, E)$ be a graph with $S, T \subseteq V$ and k be an integer. Then, we can find either*

- *k vertex-disjoint S-paths with respect to T, or*
- *$Z \subseteq V$ with $|Z| \leqslant 2k - 2$ that intersects every such path.*

in $O(kn^2)$ time.

By Theorems 4.1 and 4.3, we have the following.

Theorem 4.4. *Let $G = (V, E)$ be a graph with $S \subseteq V$ and k, p' be integers with $p' \geqslant 36k$. Suppose we have a $K_{p'}$-model containing no vertices in S and let $T = \{t_1, \dots, t_{p'}\}$ be the centers of the nodes of the $K_{p'}$-model. Then, we can find either*

- *k vertex-disjoint S-cycles, or*
- *$Z \subseteq V$ with $|Z| \leqslant 24k - 2$ such that $G - Z$ contains no S-paths with respect to T*

in $O(kn^2)$ time.

Let p' be a sufficiently large integer (the definition will be given later), and we apply this theorem to a $K_{p'}$-model K containing no vertices in S. If we find k vertex-disjoint S-cycles, then we stop the algorithm. Thus, in what follows, we consider the case when we have a subset $Z \subseteq V$ with $|Z| \leqslant 24k - 2$ such that $G - Z$ contains no S-paths with respect to the set T of the centers. Let $S' \subseteq S$ be the set of terminals contained in a connected component of $G - Z$ intersecting with T. By Menger's theorem, for any vertex $s \in S'$, there exists a vertex $\tau(s) \in V$ such that the connected component of $G - Z - \{\tau(s)\}$ containing s, say G_s, does not intersect with any vertex in T. We take such a vertex $\tau(s) \in V - Z$ so that G_s is maximal. We denote $\bigcup_{s \in S'}\{\tau(s)\}$ by $U = \{u_1, \dots, u_q\}$, and let V_i be the vertex set defined by

$$V_i = \bigcup \bigl\{ V(G_s) \bigm| s \in S,\ \tau(s) = u_i \bigr\}$$

for $i = 1, \dots, q$. Then the collection of V_i's is mutually disjoint by the definition of $\tau(s)$. Let $V_0 = V - Z - U - \bigcup_i V_i$. Then $G[V_0]$ intersects with K because $T \subseteq V_0$.

Let $U_0 \subseteq U$ be the vertex set defined by

$$U_0 = \bigl\{ u_i \in U \bigm| G[V_i] \text{ contains an } S\text{-cycle} \bigr\},$$

and define $U_1 = U \setminus U_0$. Note that we can easily compute U_0. If $|U_0| \geqslant k$, then we can immediately find k vertex-disjoint S-cycles, since the collection of V_i's is mutually disjoint.

Suppose that $|U_0| < k$. Since a path internally disjoint from V_0 with end vertices in V_0 must contain at least one vertex in Z, we observe the following:

(1) If G has k vertex-disjoint S-cycles, then they intersect with at most $2|Z|$ sets of $\{V_1, \dots, V_q\}$.

If $|U_1|$ is bounded by a fixed constant, then we can find a vertex that is irrelevant to the existence of k vertex-disjoint S-cycles in an enough large clique model K as follows. We find a separation (A, B) of G of minimum order such that A contains all vertices in S and $B - A$ contains at least one node of K. Then, we find a vertex that is irrelevant to the folio relative to $V(A) \cap V(B)$ in B using Theorem 2.4, which is a desired vertex.

However, the number of elements in U_1 is not necessarily bounded. Our main idea is that we only need $2|Z|$ elements in each "equivalence class" of U_1 by observation (1). We now describe how to divide U_1 into the equivalence classes.

We introduce a new concept *weak folio*, which is a weaker concept than the folio. Let $G = (V, E)$ be a graph and $X \subseteq V$ be a set. We say that (s, t, γ), where s and t are distinct vertices in X and $\gamma \in \{0, 1\}$, is *admissible* if G has a path P from s to t such that P has a vertex of S if $\gamma = 1$. Let $\mathcal{X} = \{(s, t, \gamma) \mid (s, t, \gamma) \text{ is admissible}, s, t \in X\}$, called the *weak folio relative to* X. Note that $|\mathcal{X}|$ is bounded by a function of $|X|$.

For each $u_i \in U_1$, we compute the weak folio relative to $Z \cup \{u_i\}$ in $G[V_i \cup Z \cup \{u_i\}]$. This can be computed in $O(m)$ time by checking whether or not each (s, t, γ) is admissible. Let $U_1^1, \dots, U_1^r$ be the partition of U_1 depending on the weak folios, that is, u_i and $u_{i'}$ are in the same set if and only if the weak folio relative to $Z \cup \{u_i\}$ in $G[V_i \cup Z \cup \{u_i\}]$ and that relative to $Z \cup \{u_{i'}\}$ in $G[V_{i'} \cup Z \cup \{u_{i'}\}]$ are the same by exchanging u_i and $u_{i'}$. Note that r is bounded by an exponential function of $|Z| + 1$, say $f_1(|Z|)$.

If $|U_1^j| > 2|Z|$ for some j, then we replace the vertex sets V_i's corresponding to all vertices in U_1^j by $2|Z|$ new vertices, and add all edges between these new vertices and the vertices in U_1^j. Let $U_1'^j$ be the set of these $2|Z|$ vertices, and we replace V_0 and U_1^j by $V_0 \cup U_1^j$ and $U_1'^j$, respectively when we find an irrelevant vertex. After executing this reduction for each j, we can find a vertex that is irrelevant to the existence of k vertex-disjoint S-cycles by Theorem 2.4 in $O(m)$ time, since $|U| \leqslant 2|Z| f_1(|Z|)$. More precisely, we find a separation (A, B) of $G[V_0 \cup Z \cup U]$ of minimum order such that A contains all vertices in $Z \cup U$ and $B - A$ contains at least one node of K. Then, we find a vertex that is irrelevant to the folio relative to $V(A) \cap V(B)$ in B using Theorem 2.4. Note that by the conditions of H_i in Theorem 2.4, we do not need to consider paths (or trees) not intersecting with K when we find an irrelevant vertex in K. Thus, by observation (1), we can see that this vertex is also irrelevant to the existence of k vertex-disjoint S-cycles.

Note that, for the above arguments, we define p' as an integer at least $3 \cdot (2|Z| f_1(|Z|) + |Z|) + 1 \geqslant 3(|U| + |Z|) + 1$, and the total time to find an irrelevant vertex is $O(m)$.

4.2. Large wall

We may assume that each connected component of G is 2-connected. Suppose we are given a graph $G = (V, E)$, a terminal set S, and a pair (X, W') in $G - T$ such that

- X is a vertex set with $|X| \leqslant \binom{p}{2}$,
- W' is a flat wall of height h in $G - X$ (h will be given later).

Then W' contains k disjoint flat walls of height h/k. If the compasses of all these walls contain vertices in S, then we can easily find k vertex-disjoint S-cycles. Otherwise, we can find a flat wall W of height h/k whose compass does not contain vertices in S. We now use the following lemma.

Lemma 4.5. *(See Kakimura et al. [15].) Let k be a positive integer. Assume that G has a cycle C with no vertices of S. If G has $4k \log_2(k + 10)$ vertex-disjoint S-paths with respect to $V(C)$, then there are k vertex-disjoint S-cycles. Moreover, such k disjoint S-cycles can be found in linear time.*

Then, by Theorem 4.3 and Lemma 4.5, we have the following theorem.

Theorem 4.6. *Let $G = (V, E)$ be a graph with $S \subseteq V$ and k be an integer. Suppose we have a flat wall W containing no vertices in S and let $T = V(\mathrm{per}(W))$. Then, we can find either*

- *k vertex-disjoint S-cycles, or*
- *$Z \subseteq V$ with $|Z| \leqslant 8k \log_2(k + 10) - 2$ such that $G - Z$ contains no S-paths with respect to T,*

in $O(kn^2)$ time.

We execute Theorem 4.6 for the wall W of height h/k that contains no terminals. If we have k vertex-disjoint S-cycles, we are done. Hence, we may assume that we obtain a subset $Z \subseteq V$ with $|Z| \leqslant 8k \log_2(k+10) - 2$ such that $G - Z$ contains no S-paths with respect to $T = V(\mathrm{per}(W))$.

By replacing Z with $Z \cup X$, we use the same argument as the previous subsection. Then, there exists a vertex set U such that removal of $Z \cup U \cup X$ separates S and W. Then, by observation (1), the intersection of k vertex-disjoint S-cycles and $G[V_0 \cup Z \cup X \cup U]$ consists of at most $2|Z \cup X|$ paths whose end vertices are in $Z \cup X \cup U$, where $V_0 \subseteq V$ is defined in the same way as Section 4.1. Thus, it suffices to find a vertex in W that is irrelevant to the existence of such $2|Z \cup X|$ paths. By executing the same reductions as the previous subsection, we may assume that $|Z \cup U \cup X|$ is bounded by a function of k, say $f_2(k)$. That is, it suffices to consider the case when there is a separation (A, B) of bounded order such that A contains all vertices in S and $B - A$ contains W. Then if follows from Theorem 2.5 that if we have a wall of size $h_2(f_2(k), |X|)$ in $B - A$, then removing some vertex of the wall does not affect the folio of B relative to $V(A) \cap V(B)$, where h_2 is as in Theorem 2.5. Since $B - A$ contains a wall of height $h - |Z \cup U|$, if we set $h \geqslant k \cdot h_2(f_2(k), |X|) + f_2(k)$, then we can find an irrelevant vertex in $O(m)$ time. Thus, the total time to find an irrelevant vertex is $O(n^2)$.

4.3. Algorithm

Finally in this subsection, we describe our algorithm for the S-cycle Packing. Assume that p and h are given as in Sections 4.1 and 4.2.

Algorithm for the *S*-cycle packing.
Input. A graph $G = (V, E)$, a terminal set $S \subseteq V$, and a fixed integer k.
Output. Find k vertex-disjoint S-cycles in G, or conclude that such cycles do not exist.

Step 1. Determine whether the tree-width of G is at most $g_1(p, h)$ or not. If it is at most $g_1(p, h)$, then solve the problem by Theorem 2.7. Otherwise, go to Step 2.

Step 2. Apply Theorem 2.3 to G and obtain either a K_p-minor or a pair (X, W) satisfying (C1)–(C3). If we have a K_p-minor, then find an irrelevant vertex as in Section 4.1. Otherwise, we find an irrelevant vertex as in Section 4.2. Then, remove the irrelevant vertex and go to Step 1.

Since we can find an irrelevant vertex in $O(n^2)$ time when k is fixed, this algorithm runs in $O(n^3)$ time. This completes the proof of Theorem 1.3. □

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