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Journal of Algebra 281 (2004) 247–263
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Hyperfocal subalgebras of source algebras of blocks over small-ground fields

Yun Fan, Yuanyang Zhou *

Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PR China

Received 15 October 2003

Communicated by Michel Broué

1. Introduction

1.1. The local theory of blocks of finite groups was proposed originally by J. Alperin and M. Broué in [1], and developed by L. Puig [12], where the source algebra of a block is introduced as the smallest algebra which carries the local information of the block. One of the classical applications of the theory is the research on *nilpotent blocks* (see [2,9]). Recently, understanding the *fusions* of local pointed groups, L. Puig in [7] and [8] introduces the *hyperfocal subalgebra* in the source algebra of a block, and proves its existence and uniqueness up to conjugation. The local information of nilpotent blocks are the simplest case, and the structure theorem of their source algebras in [9] is the simplest case of the Puig's work on hyperfocal subalgebras.

Noting that Puig obtains his results in large enough coefficient fields, in this paper we make a research on the hyperfocal subalgebras of *source algebras* of blocks over small ground-fields.

1.2. Let G always be a finite group. Let p be a prime number, and $\mathcal{O}$ be a complete discrete valuation ring with a fraction field $\mathcal{K}$ of characteristic zero and a perfect residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p. All $\mathcal{O}$-algebras considered in this paper are associative and unitary, and $\mathcal{O}$-free of finite rank; but subalgebras of an algebra are not necessarily unitary, i.e., the identity element of a subalgebra may be different from the identity element

* Corresponding author.
E-mail addresses: yunfan@whu.edu.cn (Y. Fan), zhouyuanyang@msn.com (Y. Zhou).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2004.07.031

of the algebra. For an algebra A, we denote by $J(A)$, $Z(A)$ and A^* the Jacobson radical of A, the center of A and the set of all invertible elements of A respectively. A G-algebra means an algebra A with a group homomorphism $G \to \mathrm{Aut}(A)$, where the latter denotes the automorphism group of the algebra A. An interior G-algebra means an algebra A with a group homomorphism $G \to A^*$.

For a G-algebra A and a subgroup P of G, by A^P we denote the unitary subalgebra of A consisting of the P-fixed elements of A; and denote

$$A(P) = k \otimes_{\mathcal{O}} \left(A^P \Big/ \sum_{Q \lneqq P} A_Q^P \right),$$

where Q runs on the set of the proper subgroups of P and A_Q^P denotes the image of the *relative trace map* $\mathrm{Tr}_Q^P : A^Q \to A^P$; and we call the canonical surjective homomorphism $\mathrm{Br}_P^A : A^P \to A(P)$ the *Brauer homomorphism* associated with P. By the way, we remark that for any $\mathcal{O}G$-module M, the $\mathcal{O}$-submodule M^G, the trace map $\mathrm{Tr}_Q^P : M^Q \to M^P$, and M_Q^P, $M(P)$ and the Brauer map $\mathrm{Br}_P^M : M^P \to M(P)$, are defined similarly.

1.3. Recall that a *pointed group* H_α on a G-algebra A means a pair (H, α), where H is a subgroup of G and α is a conjugate class of primitive idempotents of the algebra A^H; a pointed group K_β is said to be *contained* in H_α, denoted by $K_\beta \leqslant H_\alpha$, if $K \leqslant H$ and there exist $i \in \alpha$ and $j \in \beta$ such that $ij = j = ji$. A pointed group P_γ is said to be *local* if $\mathrm{Br}_P^A(\gamma) \neq \{0\}$. Then all the maximal local pointed groups P_γ which are contained in a pointed group H_α form exactly one H-conjugate class; and they are called *defect pointed groups* of H_α. Thus the stabilizer $N_H(P_\gamma)$ in H of the defect pointed group P_γ of H_α is unique up to conjugation. We set $E_H(P_\gamma) = N_H(P_\gamma)/PC_H(P)$. And, for $i \in \gamma$, we set $A_\gamma = iAi$, and call it a *source algebra* of H_α, see [12].

1.4. In the following, let $A = \mathcal{O}G$ be the group algebra over $\mathcal{O}$ of the finite group G. Obviously, the conjugate action of G induces a G-algebra structure on A. Let $G_{\{b\}}$ be a pointed group on A; then b is called an *$\mathcal{O}$-block* of G. Let P_γ be a defect pointed group of $G_{\{b\}}$ and $i \in \gamma$, and set $A_\gamma = iAi$, which admits an obvious $\mathcal{O}P$-interior algebra structure. Since $\mathrm{Br}_P^A(\gamma)$ is a point of $A(P) \cong kC_G(P)$, it determines a unique block $\bar{b}_\gamma$ of $kC_G(P)$ such that $\bar{b}_\gamma \mathrm{Br}_P^A(\gamma) \neq 0$. Further, the surjective homomorphism $\mathcal{O}C_G(P) \to kC_G(P)$ induces a surjective homomorphism $Z(\mathcal{O}C_G(P)) \to Z(kC_G(P))$, hence $\bar{b}_\gamma$ can be lifted to a unique central primitive idempotent b_γ of $\mathcal{O}C_G(P)$. Set $\bar{C}_G(P) = C_G(P)/Z(P)$, and let $\bar{\bar{b}}_\gamma$ be the image of b_γ in $\mathcal{O}\bar{C}_G(P)$. By [6, 4.3], we have that

$$\hat{\mathcal{O}} = Z\big(\mathcal{O}\bar{C}_G(P)\bar{\bar{b}}_\gamma\big) \tag{1.4.1}$$

is an *unramified Galois extension* of $\mathcal{O}$, that is, the fraction field $\hat{\mathcal{K}}$ of $\hat{\mathcal{O}}$ is a Galois extension of $\mathcal{K}$ and the residue filed $\hat{k}$ of $\hat{\mathcal{O}}$ is a separable Galois extension of k, and they have the same Galois group $\Gamma = \mathrm{Gal}(\hat{\mathcal{K}}/\mathcal{K}) = \mathrm{Gal}(\hat{\mathcal{O}}/\mathcal{O}) = \mathrm{Gal}(\hat{k}/k)$, which is in fact cyclic (see [4, 2.2.2]). Moreover, by [6, 4.3] again, $\mathcal{O}\bar{C}_G(P)\bar{\bar{b}}_\gamma$ is a full matrix algebra over $\hat{\mathcal{O}}$. Since

A_γ is embedded into $\mathcal{O}G$ as interior P-algebras and the embedding is compatible with Brauer homomorphisms, we have that $A_\gamma^P/J(A_\gamma^P)$ is embedded into $k \otimes_{\mathcal{O}} \mathcal{O}\bar{C}_G(P)\bar{\bar{b}}_\gamma$, thus $A_\gamma^P/J(A_\gamma^P) \cong \hat{k}$.

1.5. Let $\hat{A} = \hat{\mathcal{O}}G = \hat{\mathcal{O}} \otimes_{\mathcal{O}} A$ and $P_{\hat{\gamma}}$ be a pointed group of $\hat{A}$ such that there exists $\hat{i} \in \hat{\gamma}$ such that $i\hat{i} = \hat{i} = \hat{i}i$. Then $P_{\hat{\gamma}}$ determines a unique $\hat{\mathcal{O}}$-block $\hat{b}$ of $\hat{A}$ such that $b\hat{b} = \hat{b}$ and we set $\hat{A}_{\hat{\gamma}} = \hat{i}\hat{A}\hat{i}$; since the Brauer homomorphisms Br_P^A and $\mathrm{Br}_P^{\hat{A}}$ induce an isomorphism $\hat{k} \otimes_k A(P) \cong \hat{A}(P)$, it is easily checked that $P_{\hat{\gamma}}$ is a defect pointed group of $G_{\{\hat{b}\}}$. Because $\hat{A}_{\hat{\gamma}}$ is embedded into $\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ as $\hat{\mathcal{O}}P$-interior algebras and

$$\left(\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma^P\right)/J\left(\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma^P\right) \cong \hat{\mathcal{O}} \otimes_{\mathcal{O}} \left(A_\gamma^P/J\left(A_\gamma^P\right)\right) \cong \hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{k} \cong \hat{k}^{|\Gamma|}$$

is a direct sum of $|\Gamma|$ copies of $\hat{k}$; hence $\hat{A}_{\hat{\gamma}}^P/J(\hat{A}_{\hat{\gamma}}^P)$ is isomorphic to $\hat{k}$. Similarly, $Z(\hat{A}\hat{b})/J(Z(\hat{A}\hat{b}))$ is isomorphic to $\hat{k}$. That is, $\hat{i}$ and $\hat{b}$ are absolutely primitive in $\hat{A}^P$ and in $Z(\hat{A})$ respectively.

Let $\tilde{k}$ be an algebraic closure of $\hat{k}$ and $\tilde{\mathcal{O}}$ be an unramified extension of $\hat{\mathcal{O}}$ with the residue field $\tilde{k}$. Then from [6, 2.13], we can conclude that $\tilde{b} = \hat{b}$ is an $\tilde{\mathcal{O}}$-block of G and $\hat{\gamma}$ is contained in a point $\tilde{\gamma}$ of P on $\tilde{\mathcal{O}}G$; moreover $P_{\tilde{\gamma}}$ is a defect pointed group of the block $\tilde{b}$.

1.6. Recall that a *self-centralizing* pointed group $Q_{\tilde{\delta}}$ on $\tilde{\mathcal{O}}G$ is a local pointed group on $\tilde{\mathcal{O}}G$ such that $Z(Q)$ is a defect group of the block $b_{\tilde{\delta}}$ of $\tilde{\mathcal{O}}C_G(Q)$ determined by $\tilde{\delta}$ (i.e., $b_{\tilde{\delta}}\,\mathrm{Br}_Q(\tilde{\delta}) \neq \{0\}$); and, an *essential* pointed group $R_{\tilde{\varepsilon}}$ on $\tilde{\mathcal{O}}G$ is a self-centralizing pointed group on $\tilde{\mathcal{O}}G$ such that the quotient $E_G(R_{\tilde{\varepsilon}})$ contains a proper subgroup M satisfying that p divides $|M|$ but does not divide $|M \cap M^x|$ for any $x \in E_G(R_{\tilde{\varepsilon}}) - M$. And recall that the *hyperfocal subgroup* $\tilde{Q}$ of $P_{\tilde{\gamma}}$ (see [8, 1.3] or [7, 13.2]) is generated by the commutators $[K, R]$, where $R_{\tilde{\varepsilon}} \leqslant P_{\tilde{\gamma}}$ is either essential or equal to $P_{\tilde{\gamma}}$ and K runs over the set of p'-subgroups of $N_G(R_{\tilde{\varepsilon}})$.

1.7. Let Q be the normal subgroup of P generated by $\tilde{Q}$ and the commutators $[K, \tilde{Q}]$ where K runs over the p'-subgroups of $N_G(P_\gamma)$.

Our main result is as follows, where D is called a *hyperfocal subalgebra* of the $\mathcal{O}$-block b.

Theorem 1.8. *With notation as above, and assume that $E_G(P_\gamma)$ is a p'-group. Then there exists a P-stable unitary $\mathcal{O}$-subalgebra D of A_γ such that*

$$D \cap Pi = Qi \quad \textit{and} \quad A_\gamma = \bigoplus_u Du, \tag{1.8.1}$$

where u runs on a set of representatives for P/Q in P; and all such subalgebras of A_γ are conjugate to each other by $1 + J(A_\gamma^P)$.

Remark 1.9. The idempotent i is the identity element of A_γ; and $P \cong Pi \subset (A_\gamma)^*$ because $\mathcal{O}G$ is a projective $\mathcal{O}P$-module. The subalgebra D described in (1.8.1) inherits an $\mathcal{O}Q$-*interior P-algebra* structure from the interior P-algebra A_γ, so the second equality means that A_γ is a *crossed product* of P/Q by D. More precisely, $A_\gamma \cong D \otimes_{\mathcal{O}Q} \mathcal{O}P$ as $\mathcal{O}P$-interior algebras, where $D \otimes_{\mathcal{O}Q} \mathcal{O}P$ is endowed with multiplication

$$(d \otimes x)\big(d' \otimes x'\big) = d\big(d'^{x^{-1}}\big) \otimes xx', \quad \forall d, d' \in D,\ x, x' \in P;$$

we denote $D \otimes_Q P = D \otimes_{\mathcal{O}Q} \mathcal{O}P$, and call it *twisted Q-group algebra of P over D*. Thus (1.8.1) can be restated as

$$D \cap Pi = Qi \quad \text{and} \quad A_\gamma \cong D \otimes_Q P. \tag{1.9.1}$$

For details, please see [6, 1.6].

In Section 2 we prove the theorem for the case that $\hat{\mathcal{O}} = \mathcal{O}$; note that $E_G(P_\gamma)$ is always a p'-group if $\hat{\mathcal{O}} = \mathcal{O}$ (see [6, 4.4.2]). In Section 3 we show some general properties of hyperfocal subalgebras of a block; then we prove the theorem for the case that $\mathcal{O} < \hat{\mathcal{O}}$ in Section 4.

2. Hyperfocal subalgebras in the case that $\hat{\mathcal{O}} = \mathcal{O}$

2.1. First we mention two general facts; then from 2.3 on we turn to our objects.

Let X be a group and Y be a normal subgroup of X such that $X/Y \cong G$, i.e., X is an *extension* of G by Y. The conjugation of elements of X induces a group homomorphism $G \to \widetilde{\mathrm{Aut}}(Y)$ where $\widetilde{\mathrm{Aut}}(Y)$ denotes the outer automorphism group of Y. Such a group Y which is endowed with a group homomorphism $G \to \widetilde{\mathrm{Aut}}(Y)$ is called a *G-acted group*. Recall that a G-acted group Y is said to be *uniquely split* if any extension of G by Y splits and all the splittings are pairwise conjugate. Let $\{Y_n\}_{n\in\mathbb{N}}$ be a *normal filtration of Y*, i.e., a family of normal subgroups of Y indexed by the set $\mathbb{N}$ of the natural numbers such that $Y_0 = Y$ and $Y_{n+1} \subset Y_n$ for any $n \in \mathbb{N}$; then we have a canonical group homomorphism c from Y to the projective limit $\varprojlim \{Y/Y_n\}_{n\in\mathbb{N}}$. We say that $\{Y_n\}_{n\in\mathbb{N}}$ is *a completing filtration* of Y if c is an isomorphism. A normal filtration $\{Y_n\}_{n\in\mathbb{N}}$ of Y is called *interior* if for any $n \in \mathbb{N}$ the image of Y in $\mathrm{Aut}(Y_n/Y_{n+1})$ coincides with the inner automorphism group $\mathrm{Int}(Y_n/Y_{n+1})$ of Y_n/Y_{n+1}. Please see [6, §3] for details.

Lemma 2.2. *Let $\tilde{k}$ be an algebraic closure of k (recall that k is perfect), and $\tilde{\mathcal{O}}$ be an unramified extension of $\mathcal{O}$ such that $\tilde{\mathcal{O}}/J(\tilde{\mathcal{O}}) = \tilde{k}$. If $\tilde{A}$ is a G-algebra over $\tilde{\mathcal{O}}$ and $\tilde{B}$ is a G-stable subalgebra of $\tilde{A}$, then there are an $\bar{\mathcal{O}} \subset \tilde{\mathcal{O}}$ which is a finite Galois extension over $\mathcal{O}$ and a G-algebra $\bar{A}$ over $\bar{\mathcal{O}}$ and a G-stable subalgebra $\bar{B}$ of $\bar{A}$ such that $\tilde{A} = \tilde{\mathcal{O}} \otimes_{\bar{\mathcal{O}}} \bar{A}$ and $\tilde{B} = \tilde{\mathcal{O}} \otimes_{\bar{\mathcal{O}}} \bar{B}$.*

Proof. Let $\tilde{\mathcal{K}}$ be a fraction field of $\tilde{\mathcal{O}}$. Let $\{a_1, a_2, \ldots, a_n\}$ be an $\tilde{\mathcal{O}}$-basis of $\tilde{A}$, and $\{d_1, d_2, \ldots, d_m\}$ be an $\tilde{\mathcal{O}}$-basis of $\tilde{B}$. Assume that

$$a_i a_j = \sum_{k=1}^{n} \lambda_{ijk} a_k, \qquad d_i = \sum_{k=1}^{n} \mu_{ik} a_k,$$

$$d_i d_j = \sum_{k=1}^{m} \zeta_{ijk} d_k, \qquad d_i^x = \sum_{k=1}^{m} \eta_{x,ik} d_k, \quad x \in G,$$

where all $\lambda_{ijk}, \mu_{ik}, \zeta_{ijk}, \eta_{x,ik} \in \tilde{\mathcal{O}}$ are algebraic over $\mathcal{O}$. Let $\bar{\mathcal{K}}$ be the normal closure of the extension of $\mathcal{K}$ generated by all the $\lambda_{ijk}, \mu_{ik}, \zeta_{ijk}, \eta_{x,ik}$; and let $\bar{\mathcal{O}}$ be the integral closure of $\mathcal{O}$ in $\bar{\mathcal{K}}$. Then $\bar{\mathcal{K}}$ and $\bar{\mathcal{O}}$ are finite Galois extensions of $\mathcal{K}$ and $\mathcal{O}$ respectively, and $\bar{A} = \sum_{i=1}^{n} \bar{\mathcal{O}} a_i$ and $\bar{B} = \sum_{i=1}^{m} \bar{\mathcal{O}} d_i$ are desired algebras. □

Remark. It is clear that the conclusion still holds for finitely many subalgebras of $\tilde{A}$.

2.3. From now on to the end of this section we keep the notation in 1.2, 1.4, 1.5, and 1.7, and always assume that $\hat{\mathcal{O}} = \mathcal{O}$; note that in this case $E_G(P_\gamma)$ is always a p'-group (see [6, 4.4.2]). Then for any extension $\mathcal{O} \subseteq \bar{\mathcal{O}} \subseteq \tilde{\mathcal{O}}$, we have that $b = \hat{b}$ is a block idempotent of $\bar{\mathcal{O}}G$, and $i = \hat{i}$ is a primitive idempotent in $(\bar{\mathcal{O}}G)^P$, and $\bar{A}_{\bar{\gamma}} = i\bar{\mathcal{O}}Gi$ is a source algebra of the $\bar{\mathcal{O}}$-block b, where $P_{\bar{\gamma}}$ is a pointed group on $\bar{\mathcal{O}}G$ such that $i \in \bar{\gamma}$.

Lemma 2.4. *With notation as above, assume that $\bar{\mathcal{O}}$ is a finite extension of $\mathcal{O}$ and that $\bar{D}$ is a P-stable $\bar{\mathcal{O}}$-subalgebra of $\bar{A}_\gamma = \bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ fulfilling that $\bar{D} \cap Pi = Qi$ and $\bar{D} \otimes_Q P = \bar{A}_\gamma$. Then $N_{1+J(\bar{A}_\gamma^P)}(\bar{D}) = (1 + J(Z(\bar{A}_\gamma)))(1 + J(\bar{D}^P))$.*

Proof. The proof is inspired by [7]. Let $\tilde{k}$ be an algebraic closure of k and $\tilde{\mathcal{O}}$ be a corresponding unramified extension of $\mathcal{O}$ such that $\tilde{\mathcal{O}}/J(\tilde{\mathcal{O}}) = \tilde{k}$. Then, by in [6, 2.13.5], P_γ determines a defect pointed group $P_{\tilde{\gamma}}$ of the $\tilde{\mathcal{O}}$-block b; then by [14, 38.10], $(\tilde{A}_{\tilde{\gamma}})(P) \cong \tilde{k}Z(P)$, and further we have that $(\bar{A}_\gamma)(P) \cong \bar{k}Z(P)$, where $\bar{k} = \bar{\mathcal{O}}/J(\bar{\mathcal{O}})$. Moreover, $\bar{D}(P)$ is a direct summand of $(\bar{A}_\gamma)(P)$ as $\bar{k}C_Q(P)$-modules, and for any $u \in Z(P)$, we have $(\bar{D}u)(P) \cong \bar{D}(P)$; consequently $\bar{D}(P) \cong \bar{k}C_Q(P)$. Let U be a set of representatives of P/Q in P. For any $a \in N_{1+J(\bar{A}_\gamma^P)}(\bar{D})$, we can write $a = \sum_{u\in U} a_u$, where $a_u \in \bar{D}u$; then $\sum_{u\in U\cap QZ(P)} \mathrm{Br}_P(a_u) \in \mathrm{Br}_P(i) + J((\bar{A}_\gamma)(P))$, and thus there exists a suitable $z \in U \cap QZ(P)$ such that $\mathrm{Br}_P(a_z)$ is not contained in $J((\bar{A}_\gamma)(P))$. In particular, there exists $\lambda \in \bar{\mathcal{O}}^*$ such that $\lambda a_z z^{-1} \in i + J(\bar{D}^P)$.

Set $c = \lambda^{-1} z(a_z)^{-1} a$; then $c \in N_{1+J(\bar{A}_\gamma^P)}(\bar{D})$. Write $c = i + \sum_{u\in U-Q} c_u$, where $c_u \in \bar{D}u$; for any $\bar{d} \in \bar{D}$, there exists $\bar{d}' \in \bar{D}$ such that

$$(\bar{d} \otimes 1)\Bigl(i + \sum_{u\in U-Q} c_u\Bigr) = \Bigl(i + \sum_{u\in U-Q} c_u\Bigr)(\bar{d}' \otimes 1),$$

thus $\bar{d} = \bar{d}'$ and further we have $(\bar{d} \otimes 1)c_u = c_u(\bar{d} \otimes 1)$ for any $u \in U - Q$. In conclusion, $c \in i + J(Z(\bar{A}_\gamma))$. □

Lemma 2.5. *With notation as above, assume that $\bar{\mathcal{O}}$ is a unramified Galois extension of $\mathcal{O}$ with the Galois group $\bar{\Gamma}$ and that $\bar{D}$ is a P-stable $\bar{\mathcal{O}}$-subalgebra of $\bar{A}_\gamma = \bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ fulfilling that $\bar{D} \cap Pi = Qi$ and $\bar{D} \otimes_Q P = \bar{A}_\gamma$. Then $N_{1+J(\bar{A}_\gamma)}(\bar{D})$ is a uniquely split $\bar{\Gamma}$-acted group.*

Proof. In fact, the family $\{1 + J(\bar{D}^P)^{n+1}\}_{n\in\mathbb{N}}$ is clearly an interior completing filtration of $1 + J(\bar{D}^P)$, and for any $n \geqslant 1$ the map $r \mapsto 1 + r$ induces a group isomorphism

$$J\big(\bar{D}^P\big)^n / J\big(\bar{D}^P\big)^{n+1} \cong \big(1 + J\big(\bar{D}^P\big)^n\big) / \big(1 + J\big(\bar{D}^P\big)^{n+1}\big),$$

by [6, 3.8], $1 + J(\bar{D}^P)$ is a uniquely split $\bar{\Gamma}$-acted group. Applying [6, Theorem 3.11] to the case that $Y = N_{1+J(\bar{A}_\gamma)}(\bar{D})$ and $X = N_{1+J(\bar{A}_\gamma)}(\bar{D}) \rtimes \bar{\Gamma}$ and $G = \bar{\Gamma}$ and $\bar{\mathcal{O}}$-algebra $\bar{D}^P$ and $M = J(Z(\bar{A}_\gamma))$ and $N = J(Z(D)^P)$, we have that $(1+M)/(1+N)$ is a uniquely split $\bar{\Gamma}$-acted group. By Lemma 2.4 and [6, Corollary 3.6], we conclude that $N_{1+J(\bar{A}_\gamma)}(\bar{D})$ is a uniquely split $\bar{\Gamma}$-acted group. □

2.6. A proof of the existence of Theorem 1.8

Set $\tilde{A} = \tilde{\mathcal{O}}G = \tilde{\mathcal{O}} \otimes_{\mathcal{O}} A$, and consider the source algebra $\tilde{A}_{\tilde{\gamma}} = i\tilde{\mathcal{O}}Gi$ of the block $\tilde{\mathcal{O}}Gb$ with a defect pointed group $P_{\tilde{\gamma}}$ where $i \in \tilde{\gamma}$. By [7, Theorem 15.10] or [8, Theorem 1.8], there exists a P-stable $\tilde{\mathcal{O}}$-subalgebra $\tilde{D}$ of $\tilde{A}_{\tilde{\gamma}}$ such that $\tilde{D} \cap Pi = Qi$ and $\tilde{A}_{\tilde{\gamma}} = \bigoplus_u \tilde{D}u$ with u running on a set of representatives for P/Q in P. By Lemma 2.2, there are an $\bar{\mathcal{O}} \subseteq \tilde{\mathcal{O}}$ which is an unramified finite Galois extension of $\mathcal{O}$ and a P-stable subalgebra $\bar{D}$ of $\bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ such that $\tilde{D} = \tilde{\mathcal{O}} \otimes_{\bar{\mathcal{O}}} \bar{D}$. In particular, we also have that

$$\bar{D} \cap Pi = Qi \quad \text{and} \quad \bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma = \bigoplus_u \bar{D}u$$

with u running on a set of representatives for P/Q in P. Let $\bar{\Gamma}$ be the Galois group of $\bar{\mathcal{O}}$ over $\mathcal{O}$; then $\bar{\Gamma}$ acts on $\bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ in a natural way, and $D = \bar{D}^{\bar{\Gamma}}$ is the desired P-stable $\mathcal{O}$-subalgebra of A_γ.

2.7. A proof of the uniqueness of Theorem 1.8

With notation as above, assume that both D and D' are two P-stable $\mathcal{O}$-subalgebras of A_γ fulfilling (1.8.1). Then (1.8.1) also holds in $\tilde{A}_{\tilde{\gamma}}$ for both $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D$ and $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D'$, i.e.,

$$\big(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D\big) \cap Pi = Qi \quad \text{and} \quad \tilde{A}_{\tilde{\gamma}} = \big(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D\big) \otimes_Q P;$$
$$\big(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D'\big) \cap Pi = Qi \quad \text{and} \quad \tilde{A}_{\tilde{\gamma}} = \big(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D'\big) \otimes_Q P.$$

By [7, 14.7] or [8, 1.8], there is an $\tilde{a} \in 1 + J(\tilde{A}_{\tilde{\gamma}}^P)$ such that $(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D)^{\tilde{a}} = (\tilde{\mathcal{O}} \otimes_{\mathcal{O}} D')$. By Lemma 2.2, there are an $\bar{\mathcal{O}} \subseteq \tilde{\mathcal{O}}$ which is an unramified finite Galois extension of $\mathcal{O}$ and an $\bar{a} \in i + J((\bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma)^P)$ such that $(\bar{\mathcal{O}} \otimes_{\mathcal{O}} D)^{\bar{a}} = \bar{\mathcal{O}} \otimes_{\mathcal{O}} D'$. Considering the action on

$\bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ of the Galois group $\bar{\Gamma}$ of $\bar{\mathcal{O}}$ over $\mathcal{O}$, for any $t \in \bar{\Gamma}$ we have $(\bar{\mathcal{O}} \otimes_{\mathcal{O}} D)^{t(\bar{a})} = \bar{\mathcal{O}} \otimes_{\mathcal{O}} D'$.

Set $\bar{A}_\gamma = \bar{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$, $\bar{D} = \bar{\mathcal{O}} \otimes_{\mathcal{O}} D$, and $\bar{D}' = \bar{\mathcal{O}} \otimes_{\mathcal{O}} D'$, and let $\mathcal{S}$ be the set of all the elements $\bar{a}$ of $1 + J(\bar{A}_\gamma^P)$ such that $\bar{D}^{\bar{a}} = \bar{D}'$. Then it is easily checked that the group $N_{1+J(\bar{A}_\gamma)}(\bar{D}) \rtimes \bar{\Gamma}$ acts on the set $\mathcal{S}$ and $N_{1+J(\bar{A}_\gamma)}(\bar{D})$ acts regularly on $\mathcal{S}$. Since $N_{1+J(\bar{A}_\gamma)}(\bar{D})$ acts regularly on $\mathcal{S}$, the stabilizer of any element of $\mathcal{S}$ in $N_{1+J(\bar{A}_\gamma)}(\bar{D}) \rtimes \bar{\Gamma}$ is isomorphic to $\bar{\Gamma}$; then, by Lemma 2.5 and [6, 3.3], $\mathcal{S}^{\bar{\Gamma}}$ is non-empty; moreover, there exists $a \in (1 + J(\bar{A}_\gamma^P))^{\bar{\Gamma}} = 1 + J(A_\gamma^P)$ such that $\bar{D}^a = \bar{D}'$; consequently, $D^a = (\bar{D}^a)^{\bar{\Gamma}} = (\bar{D}')^{\bar{\Gamma}} = D'$.

3. The local structure of hyperfocal subalgebras

3.1. Keep the notation in 1.2 throughout this section. First we recall a general notation, then turn to show some general properties of hyperfocal subalgebras.

Let H be a normal subgroup of G. Assume that A is an H-interior G-algebra, i.e., A is a G-algebra with an interior H-algebra structure compatible with the G-action, cf. [6, 1.6]. Let K_γ and L_δ be pointed groups on A, and assume that $K \subset HL$. Recall that a group exomorphism from K to L is an orbit on the set of the injective group homomorphisms from K to L under the natural action of the product $\mathrm{Int}(K) \times \mathrm{Int}(L)$ of the inner automorphism groups $\mathrm{Int}(K)$ and $\mathrm{Int}(L)$ of K and of L respectively. We say that a group exomorphism determined by an injective group homomorphism $\phi : K \to L$, fulfilling $\phi(y) \in yH$ for all $y \in K$, is an A-*fusion* from K_γ to L_δ if, for some $i \in \gamma$ and some $j \in \delta$, there exists $a \in A^*$ such that $iAi \subset (jAj)^a$ and

$$(ai)^y = \big(y^{-1}\phi(y)\big)ai \quad \text{and} \quad \big(ia^{-1}\big)^y = ia^{-1}\big(y^{-1}\phi(y)\big)^{-1}, \quad \forall y \in K. \tag{3.1.1}$$

By $F_A(K_\gamma, L_\delta)$ we denote the set of the A-fusions from K_γ to L_δ, and write $F_A(K_\gamma)$ instead of $F_A(K_\gamma, K_\gamma)$. Further, suppose $\phi(K) = L$ and let $\Delta_\phi(K) = \{(\phi(x), x)\}_{x \in K}$ be a subgroup of $L \times K$; then jAi admits an $\mathcal{O}\Delta_\phi(K)$-module structure defined by $(\phi(x), x)a = \phi(x)ax^{-1}$ for any $x \in K$ and $a \in jAi$. Note that, if $\phi(K) = L$, ϕ^{-1} also determines an A-fusion from L_δ to K_γ.

Lemma 3.2. *With notation as above, a group isomorphism $\phi : K \cong L$ such that $\phi(x) \in xH$ for all $x \in K$ determines an A-fusion from K_γ to L_δ if and only if*

$$(iAj)^{\Delta_{\phi^{-1}}(L)}(jAi)^{\Delta_\phi(K)} = iA^K i. \tag{3.2.1}$$

Proof. The essential materials of the proof are from [7]. In any case, it is easily checked that the left side of the equality is contained in the right one and it is a two-sided ideal of the right one. If $\tilde{\phi} \in F_A(K_\gamma, L_\delta)$ and $a \in A^*$ fulfills equality (3.1.1), then ai and ia^{-1} belong to $(jAi)^{\Delta_\phi(K)}$ and $(iAj)^{\Delta_{\phi^{-1}}(L)}$ respectively, thus the equality (3.2.1) holds. Conversely, since $iA^K i$ is a local algebra, the equality (3.2.1) implies that we can choose

$c \in (iAj)^{\Delta_{\phi^{-1}}(L)}$ and $d \in (jAi)^{\Delta_\phi(K)}$ such that cd is invertible in iA^Ki; modifying our choice, we may assume that $cd = i$; then dc is a non-zero idempotent of jA^Lj, hence $dc = j$. In particular, i and j are conjugate in A, i.e., $i = j^b$ for a $b \in A^*$. We claim that $a = d + (1 - j)b(1 - i)$ is invertible in A and fulfills equality (3.1.1); indeed, it is easily checked that $c + (1 - i)b^{-1}(1 - j)$ is the inverse of a, and, since $ai = d$ and $ia^{-1} = c$, the equality follows from the fact that $\Delta_\phi(K)$ fixes ai and $\Delta_{\phi^{-1}}(L)$ fixes ia^{-1}. □

3.3. From now on we turn to the notation 1.4, 1.5, and 1.7, and always assume that D is a P-stable unitary subalgebra of A_γ fulfilling (1.8.1). Then A_γ is an interior P-algebra, while D is an $\mathcal{O}Q$-interior P-algebra. Note that $\gamma \cap A_\gamma = \gamma \cap D = \{i\}$, so $P_{\{i\}}$ is a local pointed group on both A_γ and D; we denote the both by P_γ again for convenience. Further, we identify $Pi \subseteq A_\gamma$ with P, and identify $ui \in Pi$ with $u \in P$ for convenience.

Let $\phi : P \to P$ determine a D-fusion of P_γ, i.e., $\tilde{\phi} \in F_D(P_\gamma)$, and assume that $a \in D$ makes (3.1.1) holds; then in A_γ (not in D) (3.1.1) is rewritten as $a^{-1}ya = \phi(y)$, $\forall\, y \in P$. In other words,

$$F_D(P_\gamma) = N_{D^*}(P)\big/\big(N_{D^*}(P) \cap \big(A_\gamma^P\big)^* P\big), \tag{3.3.1}$$

where $N_{D^*}(P) = \{a \in D^* \mid P^a = P\}$. On the other hand, it is known from [11, 2.13 and 3.1] that

$$F_{A_\gamma}(P_\gamma) = N_{A_\gamma{}^*}(P)\big/\big(\big(A_\gamma^P\big)^* P\big) = E_G(P_\gamma). \tag{3.3.2}$$

Since it is shown in the end of 1.4 that $(A_\gamma^P)^*/(i + J(A_\gamma^P)) \cong \hat{k}$, by [13, Chapter II, Proposition 8] we get

$$\big(A_\gamma^P\big)^* \cong \big(i + J\big(A_\gamma^P\big)\big) \rtimes \hat{k}^*; \tag{3.3.3}$$

with a suitable identification we regard $\hat{k}^* \subseteq (A_\gamma^P)^*$ and $(A_\gamma^P)^* = (i + J(A_\gamma^P)) \rtimes \hat{k}^*$. And

$$\hat{E}_G(P_\gamma)^\circ = N_{A_\gamma{}^*}(P)\big/\big(\big(i + J\big(A_\gamma^P\big)\big)P\big) \tag{3.3.4}$$

is an extension of $E_G(P_\gamma)$ by $\hat{k}^*$, we call it a *$\hat{k}^*$-group* with *$\hat{k}^*$-quotient $E_G(P_\gamma)$*.

Lemma 3.4. *Notation as above. Then $F_D(P_\gamma) = F_{A_\gamma}(P_\gamma)$.*

Proof. The essential materials of the proof come from [7]. It is clear that $F_D(P_\gamma) \subseteq F_{A_\gamma}(P_\gamma)$. Let $\tilde{\phi} \in F_{A_\gamma}(P_\gamma)$ and ϕ be a suitable representative of the A_γ-fusion. It follows from Lemma 3.2 that

$$A_\gamma^{\Delta_\phi(P)} A_\gamma^{\Delta_{\phi^{-1}}(P)} = A_\gamma^P,$$

since P_γ is local, this equality implies that the k-linear map

$$(A_\gamma)\big(\Delta_\phi(P)\big) \otimes_k (A_\gamma)\big(\Delta_{\phi^{-1}}(P)\big) \to (A_\gamma)(P)$$

induced by the multiplication in A_γ is surjective. Let T be a set of representatives for P/Q in P, and U be the set of $t \in T$ such that $\phi(y)t^{-1}y^{-1} \in Qt^{-1}$ for any $y \in P$, we have

$$(A_\gamma)\big(\Delta_\phi(P)\big) = \bigoplus_{u \in U}\big(D \otimes u^{-1}\big)\big(\Delta_\phi(P)\big) \quad \text{and}$$
$$(A_\gamma)(\Delta_{\phi^{-1}(P)}) = \bigoplus_{u \in U}(D \otimes u)\big(\Delta_{\phi^{-1}}(P)\big).$$

Consequently, there exist $u, v \in U$ and $c, d \in D$ such that $\Delta_\phi(P)$ fixes $c \otimes u^{-1}$, $\Delta_{\phi^{-1}}(P)$ fixes $d \otimes v$ and the product $(c \otimes u^{-1})(d \otimes v)$ is invertible in A_γ^P; thus, modifying the choice of the second factor, we can assume that $v = u$ and $cd^u = i$. In particular, we get $\phi(Q) = Q$. Since $\Delta_\phi(P)$ fixes $c \otimes u^{-1}$ and $\Delta_{\phi^{-1}}(P)$ fixes $d \otimes v$, it is easily checked that $c \in D^{\Delta_\psi(P)}$ and $d^u \in D^{\Delta_{\psi^{-1}}(P)}$; then, since $cd^u = i$, we get

$$D^{\Delta_\phi(P)} D^{\Delta_{\phi^{-1}}(P)} = D^P.$$

Thus, by Lemma 3.2 again, we have that $\tilde{\phi} \in F_D(P_\gamma)$. □

Lemma 3.5. $D^P/J(D^P) \cong A_\gamma^P/J(A_\gamma^P)$.

Proof. Assume that $T/Q = Z(P/Q)$ and U is a set of the representatives of T in P. Then

$$A_\gamma(P) = \bigoplus_{u \in U}(D \otimes_Q u)(P).$$

It is easy to check that $(D \otimes_Q u)(P)(D \otimes_Q v)(P) \subset (D \otimes_Q uv)(P)$ for any $u, v \in P$; i.e., $(A_\gamma)(P)$ is a T-graded k-algebra. Set $I = (A_\gamma)(P)J(D(P))(A_\gamma)(P)$. By the computation similar to the first and second paragraphs of the proof of [7, Lemma 7.3], we have that $J(D(P)) \subset J(A(P))$ and that I is a T-graded proper ideal of $(A_\gamma)(P)$ with the t-component

$$I_t = \sum_{y \in T}\big(A_\gamma)(P)_y J(D(P)\big)(A_\gamma)(P)_{y^{-1}t},$$

thus $(A_\gamma)(P)/I$ is a T-graded k-algebra with 1-component isomorphic to $D^P/J(D^P)$. Set

$$T' = \big\{t \in T \;\big|\; \big((A_\gamma)(P)/I\big)_t\big((A_\gamma)(P)/I\big)_{t^{-1}} = \big((A_\gamma)(P)\big)_1\big\},$$

then it is easily checked that T' is a subgroup of T (see [5, Lemma 8]), and by [5, Lemma 9], $\bigoplus_{t \in T'}((A_\gamma)(P)/I)_t$ is a crossed product and $\bigoplus_{t \in T-T'}((A_\gamma)(P)/I)_t$ is a nilpotent ideal of $(A_\gamma)(P)/I$. Since $D^P/J(D^P)$ is a perfect field, $\bigoplus_{t \in T'}((A_\gamma)(P)/I)_t$ is isomorphic to the group algebra of T' over $D^P/J(D^P)$; thus $D^P/J(D^P) \cong A_\gamma^P/J(A_\gamma^P)$. □

Remark. By the lemma and 1.4 and [13, Chapter II, Proposition 8], we can lift it to an algebra injection $\hat{\mathcal{O}} \to D^P$; on the other hand, a choice of the subgroup $\hat{k}^*$ of $(A_\gamma^P)^*$ in 3.3 also determines an algebra injection $\hat{\mathcal{O}} \to A_\gamma^P$. But by [3, Lemma 2.3], these two algebra injections are conjugate by $i + J(A_\gamma^P)$; so with a suitable choice, we can assume that they coincide with each other. So, in the following we assume that $\hat{\mathcal{O}} \subseteq D^P$; and, since $\hat{\mathcal{O}}^* = (1 + J(\hat{\mathcal{O}})) \times \hat{k}^*$, we have

$$\left(D^P\right)^* = \left(i + J\left(D^P\right)\right) \rtimes \hat{k}^*. \tag{3.5.1}$$

3.6. From now on, we further always assume that $E_G(P_\gamma)$ is a p'-group.

Since $N_G(P_\gamma)$ stabilizes both $C_G(P)$ and the block b_γ of $\mathcal{O}C_G(P)$, we see that $E_G(P_\gamma)$ acts on $\hat{\mathcal{O}}$ by the equality (1.4.1). Then the actions of $N_G(P_\gamma)$ on P and $\hat{\mathcal{O}}$ determine a group homomorphism

$$E_G(P_\gamma) \to \widetilde{\mathrm{Aut}}\left(\hat{\mathcal{O}}, \hat{\mathcal{O}}P\right), \tag{3.6.1}$$

where $\mathrm{Aut}(\hat{\mathcal{O}}, \hat{\mathcal{O}}P)$ denotes the group of the $\hat{\mathcal{O}}$-semi-linear automorphisms of $\hat{\mathcal{O}}P$, and $\widetilde{\mathrm{Aut}}(\hat{\mathcal{O}}, \hat{\mathcal{O}}P)$ denotes the quotient group of $\mathrm{Aut}(\hat{\mathcal{O}}, \hat{\mathcal{O}}P)$ by the inner automorphism group $\mathrm{Int}(\hat{\mathcal{O}}P)$ of $\hat{\mathcal{O}}P$ induced by all the invertible elements of $\hat{\mathcal{O}}P$.

Because the kernel of the surjective homomorphism $N_G(P)/C_G(P) \to E_G(P_\gamma)$ is a p-group, we can lift it to an injective group homomorphism $E_G(P_\gamma) \to \mathrm{Aut}(P)$. Thus, the actions of $E_G(P_\gamma)$ on both P and $\hat{\mathcal{O}}$ determine a group homomorphism

$$\theta : E_G(P_\gamma) \to \mathrm{Aut}\left(\hat{\mathcal{O}}, \hat{\mathcal{O}}P\right) \tag{3.6.2}$$

such that $\theta(E_G(P_\gamma))$ stabilizes both $\hat{\mathcal{O}}$ and P, and for any $\tilde{x} \in E_G(P_\gamma)$ there is a p'-element $s \in N_G(P_\gamma)$ fulfilling

$$\theta(\tilde{x})(u) = u^s, \quad \forall u \in P. \tag{3.6.3}$$

In the following we fix such a group homomorphism θ in (3.6.2); and note that by the definition 1.7 of Q and (3.6.3) we have the following conclusion:

$$Q \textit{ is stabilized by the } E_G(P_\gamma)\textit{-action on } P \textit{ through } \theta. \tag{3.6.4}$$

We remark that the (3.6.1) can always be lifted to a unique $\mathrm{Int}(\hat{\mathcal{O}}P)$-conjugate class of homomorphisms $E_G(P_\gamma) \to \mathrm{Aut}(\hat{\mathcal{O}}, \hat{\mathcal{O}}P)$, but the lifting which stabilizes P may not exist if $E_G(P_\gamma)$ is not a p'-group, cf [6, 1.14 and 1.15].

Proposition 3.7.

(1) *There is a subgroup $\hat{E}$ of $N_{A_\gamma^*}(P)$ such that $\hat{E} \supseteq \hat{k}^*$ (recall $\hat{k}^* \subseteq (D^P)^* \subset (A_\gamma^P)^*$, see* (3.5.1)) *and* (3.3.4) *induces an isomorphism $\hat{E} \cong \hat{E}_G(P_\gamma)^\circ$; and all such subgroups of $N_{A_\gamma^*}(P)$ are conjugate by $N_{A_\gamma^*}(P \times \hat{k}^*) \cap (i + J(A_\gamma^P))P$.*

(2) *There is a subgroup* $\hat{E}$ *of* $N_{D^*}(P)$ *such that* $\hat{E} \supseteq \hat{k}^*$ *and* (3.3.4) *induces an isomorphism* $\hat{E} \cong \hat{E}_G(P_\gamma)^\circ$; *and all such subgroups of* $N_{D^*}(P)$ *are conjugate by* $N_{D^*}(P \times \hat{k}^*) \cap (i + J(A_\gamma^P))P$.

Proof. (1) Denote by V the centralizer of $\hat{k}^*$ in $J(A_\gamma^P)$; it is clear that V is an $\mathcal{O}$-submodule of $J(A_\gamma^P)$ satisfying that $V.V \subset V$, thus $i + V$ is a subgroup of $i + J(A_\gamma^P)$. Then, by (3.3.2) and (3.3.3), we have $N_{A_\gamma^*}(P \times \hat{k}^*)/((i + V) \times \hat{k}^*)P \cong E_G(P_\gamma)$, thus we have a short exact sequence

$$1 \to (i + V)P\hat{k}^*/\hat{k}^* \xrightarrow{\text{incl}} N_{A_\gamma^*}\big(P \times \hat{k}^*\big)/\hat{k}^* \xrightarrow{\rho} E_G(P_\gamma) \to 1, \tag{3.7.1}$$

where "incl" is the inclusion map and ρ is induced by (3.3.4). However, $P(i + V)\hat{k}^*/(i + V)\hat{k}^*$ is a finite p-group and $E_G(P_\gamma)$ is a finite p'-group, $P(i + V)\hat{k}^*/(i + V)\hat{k}^*$ is a uniquely split $E_G(P_\gamma)$-acted group. On the other hand, since $i + V$ is equal to the subgroup of $y \in i + J(A_\gamma^P)$ such that $\hat{\mathcal{O}}^y = \hat{\mathcal{O}}$, by [10, Lemma 4.10] and [6, Proposition 3.5], $i + V$ is a uniquely split $E_G(P_\gamma)$-acted group. Further, P and $i + V$ centralize each other, by [6, 3.6] we have that $(i + V)P\hat{k}^*/\hat{k}^*$ is a uniquely split $E_G(P_\gamma)$-acted group. Therefore the sequence (3.7.1) is uniquely split, that is, there is a subgroup $\hat{E}/\hat{k}^* \subseteq N_{A_\gamma^*}(P \times \hat{k}^*)/\hat{k}^*$ such that the restriction map $\rho|_{\hat{E}/\hat{k}^*} : \hat{E}/\hat{k}^* \to E_G(P_\gamma)$ is an isomorphism; and all such subgroups of $N_{A_\gamma^*}(P)/\hat{k}^*$ are conjugate to each other by $(i + V)P\hat{k}^*/\hat{k}^*$.

(2) By Lemma 3.4 we have $F_D(P_\gamma) = E_G(P_\gamma)$, thus by (3.3.1) we have an exact sequence

$$1 \to \big(N_{D^*}\big(P \times \hat{k}^*\big) \cap P\big(A_\gamma^P\big)^*\big)/\hat{k}^* \xrightarrow{\text{incl}} N_{D^*}\big(P \times \hat{k}^*\big)/\hat{k}^* \xrightarrow{\rho} E_G(P_\gamma) \to 1. \tag{3.7.2}$$

Set $W = V \cap J(D^P)$; then it is clear that W is the centralizer of $\hat{k}^*$ in $J(D^P)$ and that W is an $\mathcal{O}$-submodule of $J(D^P)$ such that $W.W \subset W$, thus $i + W$ is a subgroup of $i + J(D^P)$. Then similar to the proof below (3.7.1), we also can obtain that

$$N_{D^*}\big(P \times \hat{k}^*\big) \cap \big(i + J\big(A_\gamma^P\big)P\big) \text{ is a uniquely split } E_G(P_\gamma)\text{-acted group}, \tag{3.7.3}$$

thus we get the conclusions of (2). □

Remark. Recall that D is P-stable, from the proposition we have the following conclusion:

$$\begin{aligned} &\text{If } \hat{k}^* \subseteq \hat{E} \subseteq A_\gamma^* \text{ such that (3.3.4) induces an isomorphism } \hat{E} \cong E_G(P_\gamma), \\ &\text{then there is an } a \in i + J\big(A_\gamma^P\big) \text{ such that } \hat{E} \subseteq D^a. \end{aligned} \tag{3.7.4}$$

3.8. Now we follow the idea of [6, §4] to choose $\hat{i}$ and $\hat{b}$ in 1.5 suitably. Let $\hat{j}$ be the primitive idempotent of $\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ which is mapped non-zero by the homomorphism

$\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to \hat{\mathcal{O}}$, $\lambda \otimes \mu \mapsto \lambda\mu$. By [6, Proposition 4.10], there exists an injective unitary homomorphism from $\hat{\mathcal{O}}$ to A_γ, hence we have an injective homomorphism

$$\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to \hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma. \tag{3.8.1}$$

By [6, 4.13.2], $\hat{j}$ determines a primitive idempotent $\hat{i}$ of $\hat{A}_\gamma^P$ through the above homomorphism (3.8.1), and there exists a unique local point $\hat{\gamma}$ of P on $\hat{\mathcal{O}}G$ such that $\hat{i} \in \hat{\gamma}$. Let $\hat{b}$ be the $\hat{\mathcal{O}}$-block of G such that $b\hat{\gamma} = \hat{\gamma}$; by [6, 2.13.5], $P_{\hat{\gamma}}$ is a defect pointed group of $G_{\{\hat{b}\}}$. Set $\hat{A}_{\hat{\gamma}} = \hat{i}\hat{A}\hat{i}$; then $\hat{A}_{\hat{\gamma}}$ is a source algebra of $\hat{\mathcal{O}}G\hat{b}$.

Then, by [6, 1.19.1], the usual trace map Tr_1^Γ on $\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma$ induces a $\hat{k}^*$-group homomorphism $\hat{E}_G(P_{\hat{\gamma}})^\circ \to \hat{E}_G(P_\gamma)^\circ$ which is a lifting of the inclusion map $E_G(P_{\hat{\gamma}}) \subset E_G(P_\gamma)$. Thus by [6, 1.20], $\hat{A}_{\hat{\gamma}}$ admits an $\hat{\mathcal{O}}\hat{E}_G(P_{\hat{\gamma}})^\circ$-interior $\hat{E}_G(P_\gamma)^\circ$-algebra structure, unique up to $(\hat{A}_{\hat{\gamma}}^P)^*$-conjugation, such that the action of $\hat{E}_G(P_\gamma)^\circ$ stabilizes the image of $\hat{\mathcal{O}}P$ and induces the group homomorphism (3.6.2); and there exists an $\mathcal{O}P$-interior algebra isomorphism

$$\eta : A_\gamma \xrightarrow{\cong} \hat{A}_{\hat{\gamma}} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ. \tag{3.8.2}$$

Moreover, by our choice of the group homomorphism (3.6.2), $\hat{E}_G(P_{\hat{\gamma}})^\circ$ stabilizes P and $\hat{A}_{\hat{\gamma}}$ also admits an $\hat{\mathcal{O}}(P \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ)$-interior $P \rtimes \hat{E}_G(P_\gamma)^\circ$-algebra structure, which extends the usual interior $\hat{\mathcal{O}}P$-algebra structure on $\hat{A}_{\hat{\gamma}}$; and the isomorphism (3.8.2) becomes an $\mathcal{O}(P \rtimes \hat{E}_G(P_\gamma)^\circ)$-interior algebra isomorphism. In particular, η^{-1} induces an injection

$$P \rtimes \hat{E}_G(P_\gamma)^\circ \to A_\gamma^*. \tag{3.8.3}$$

Theorem 3.9. *Notation as above. If D is a hyperfocal subalgebra of A_γ (i.e.,* (1.8.1) *holds for A_γ and D), then there are an $a \in i + J((A_\gamma^P))$, and a hyperfocal subalgebra $\hat{D}$ of $\hat{A}_{\hat{\gamma}}$ (i.e.,* (1.8.1) *holds for $\hat{A}_{\hat{\gamma}}$ and $\hat{D}$) which inherits from $\hat{A}_{\hat{\gamma}}$ an $\hat{\mathcal{O}}\hat{E}_G(P_{\hat{\gamma}})^\circ$-interior $\hat{E}_G(P_\gamma)^\circ$-algebra structure, and an $\mathcal{O}Q$-interior P-algebra isomorphism $\eta' : D^a \xrightarrow{\cong} \hat{D} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ$ such that the following diagram is commutative*:

$$\begin{array}{ccc} A_\gamma & \xrightarrow[\eta]{\cong} & \hat{A}_{\hat{\gamma}} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ \\ {\scriptstyle \mathrm{incl}}\uparrow & & \uparrow{\scriptstyle \mathrm{incl} \otimes \mathrm{id}} \\ D^a & \xrightarrow[\eta']{\cong} & \hat{D} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ \end{array} \tag{3.9.1}$$

where "incl" *and* "id" *denote the inclusion map and the identity map, respectively.*

Proof. We trace the construction of the isomorphism (3.8.2) in [6, 4.11–4.14].

Obviously the subgroup $\hat{k}^*$ of $\hat{E}_G(P_\gamma)$ determines a subgroup $\hat{k}^*$ of $(A_\gamma^P)^*$ through the isomorphism (3.8.2); now we fix the later subgroup $\hat{k}^*$. By [3, Lemma 2.3], we can assume without loss of the generality that D contains $\hat{k}^*$, thus by (3.7.4), we also can assume that D contains the image of $\hat{E}_G(P_\gamma)^\circ$ in A_γ and the homomorphism from $\hat{\mathcal{O}}$ to A_γ induces an injective unitary homomorphism of $\hat{E}_G(P_\gamma)^\circ$-algebras from $\hat{\mathcal{O}}$ to D.

Let Γ be the Galois group of $\hat{\mathcal{O}}$ over $\mathcal{O}$. We can regard $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D$ as an $\hat{\mathcal{O}}\hat{E}_G(P_\gamma)^\circ$-interior $\Gamma \times \hat{E}_G(P_\gamma)^\circ$-algebra (cf. [6, 1.6]). The formula (3.8.1) can be rewritten as

$$\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to \hat{\mathcal{O}} \otimes_{\mathcal{O}} D \subset \hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma, \tag{3.9.2}$$

which is a homomorphism of $\Gamma \times \hat{E}_G(P_\gamma)^\circ$-algebras over $\hat{\mathcal{O}}$.

Let $\hat{J}$ be the set of primitive idempotents of $\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}}$, and $\hat{j}$ be the element of $\hat{J}$ which does not vanish through the product map $\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \to \hat{\mathcal{O}}$. Through (3.9.2), by $\hat{I}$ and $\hat{i}$ we denote the image of $\hat{J}$ and $\hat{j}$ in $\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma^P$ respectively. Since the group $\Gamma \times \hat{E}_G(P_\gamma)^\circ$ stabilizes on $\hat{I}$, it also stabilizes $\hat{J}$. And both $\hat{j}$ and $\hat{i}$ have the same stabilizer, denoted by $\hat{H}$, in $\Gamma \times \hat{E}_G(P_\gamma)^\circ$. Since Γ acts regularly on $\hat{I}$ and $\hat{J}$, the second projection map

$$\Gamma \times \hat{E}_G(P_\gamma)^\circ \to \hat{E}_G(P_\gamma)^\circ$$

induces a group homomorphism

$$\varphi : \hat{H} \xrightarrow{\cong} \hat{E}_G(P_\gamma)^\circ. \tag{3.9.3}$$

Thus there is a suitable group homomorphism $\hat{\tau} : \hat{E}_G(P_\gamma)^\circ \to \Gamma$ such that

$$\hat{H} = \{(\hat{\tau}(\hat{x}), \hat{x})\}_{\hat{x} \in \hat{E}_G(P_\gamma)^\circ}.$$

It is easily checked that in $\hat{\mathcal{O}} \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ the action of $E_G(P_\gamma)$ on $\hat{\mathcal{O}}$ induced by $\hat{\tau}$ coincides with the action of $E_G(P_\gamma)$ in (3.8.2) (cf. [6, 4.12]), so the stabilizer of $\hat{j}$ and $\hat{i}$ in $\hat{E}_G(P_\gamma)^\circ$ (identified with $1 \times \hat{E}_G(P_\gamma)^\circ$) coincides with the converse image $\hat{K} \subseteq \hat{E}_G(P_\gamma)^\circ$ of the kernel K of the homomorphism $E_G(P_\gamma) \to \mathrm{Aut}_{\mathcal{O}}(\hat{\mathcal{O}})$.

Considering the corresponding action of Γ on $\hat{\mathcal{O}}G = \hat{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}G$, by [6, 4.13.2], we have that $\hat{i}$ belongs to a local point $\hat{\gamma}$ of P on $\hat{\mathcal{O}}G$, and $\hat{\gamma}^\sigma \neq \hat{\gamma}$ for any nontrivial element σ of Γ. In particular, $E_G(P_{\hat{\gamma}}) = K$. Let $\hat{\alpha} = \{\hat{b}\}$ be the point of G on $\hat{\mathcal{O}}G$ such that $P_{\hat{\gamma}} \subset G_{\hat{\alpha}}$; similarly to [6, 4.13.4], we have

The stabilizer $\Gamma^{\hat{\alpha}}$ of $\hat{\alpha}$ in Γ coincides with the image of $E_G(P_\gamma)$ in Γ,

and $\mathrm{Tr}_1^{\Gamma^{\hat{\alpha}}}(\hat{i})$ belongs to $Z(\hat{\mathcal{O}} \otimes_{\mathcal{O}} D)^P$. (3.9.4)

It is similar to [6, 4.14] that $\hat{D} = \hat{i}(\hat{\mathcal{O}} \otimes_{\mathcal{O}} D)\hat{i}$ inherits from $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D$ the $\hat{\mathcal{O}}\hat{K}$-interior $\hat{H}$-algebra structure, and $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D\hat{b}$ inherits the $\hat{\mathcal{O}}\hat{E}_G(P_\gamma)^\circ$-interior $\Gamma^{\hat{\alpha}} \times \hat{E}_G(P_\gamma)^\circ$-algebra structure. Hence the characterization [6, 2.7.4] applies to $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D\hat{b}$ in $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D$; whereas, since $\hat{E}_G(P_\gamma)^\circ$ is transitive on $\{\hat{i}^\sigma\}_{\sigma\in\Gamma^{\hat{\alpha}}}$ by (3.9.4), the characterization [6, 2.6.3] applies to $\hat{D}$ in $\hat{\mathcal{O}} \otimes_{\mathcal{O}} D\hat{b}$. Similar to the isomorphism [6, 4.14.1] which is written as ζ in the first row of diagram (3.9.5) below, we get an $\hat{\mathcal{O}}\hat{E}_G(P_\gamma)^\circ$-interior $\Gamma \times \hat{\mathcal{O}}\hat{E}_G(P_\gamma)^\circ$-algebra isomorphism ζ' shown in the second row of the diagram

$$\begin{array}{ccc} \hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma & \xrightarrow[\zeta]{\cong} & \hat{\mathcal{O}}\big(\Gamma \times \hat{E}_G(P_\gamma)^\circ\big) \otimes_{\hat{\mathcal{O}}(\Gamma^{\hat{\alpha}} \times \hat{E}_G(P_\gamma)^\circ)} \operatorname{Ind}_{\hat{K}}^{\hat{E}_G(P_\gamma)^\circ}(\hat{A}_{\hat{\gamma}}) \\ \uparrow{\scriptstyle \text{incl}} & & \uparrow{\scriptstyle \text{id}\otimes\text{incl}} \\ \hat{\mathcal{O}} \otimes_{\mathcal{O}} D & \xrightarrow[\zeta']{\cong} & \hat{\mathcal{O}}\big(\Gamma \times \hat{E}_G(P_\gamma)^\circ\big) \otimes_{\hat{\mathcal{O}}(\Gamma^{\hat{\alpha}} \times \hat{E}_G(P_\gamma)^\circ)} \operatorname{Ind}_{\hat{K}}^{\hat{E}_G(P_\gamma)^\circ}(\hat{D}) \end{array} \tag{3.9.5}$$

and $\zeta'(\hat{d}) = 1 \otimes (1 \otimes \hat{d} \otimes 1)$ for $\hat{d} \in \hat{D} = \hat{i}(\hat{\mathcal{O}} \otimes_{\mathcal{O}} D)\hat{i} \subseteq \hat{\mathcal{O}} \otimes_{\mathcal{O}} D$. Comparing with [6, 4.14], we see that the diagram (3.9.5) is commutative.

Since $(\hat{\mathcal{O}} \otimes_{\mathcal{O}} A_\gamma)^\Gamma = A_\gamma$ and $(\hat{\mathcal{O}} \otimes_{\mathcal{O}} D)^\Gamma = D$, by [6, 2.8 and 2.10] we have the following isomorphism (φ is the isomorphism (3.9.3)):

$$\operatorname{Res}_\varphi(A_\gamma) \cong \hat{A}_{\hat{\gamma}} \otimes_{\hat{K}} \hat{H} \quad \text{and} \quad \operatorname{Res}_\varphi(D) \cong \hat{D} \otimes_{\hat{K}} \hat{H}$$

where the first one is just [6, 4.14.2] and the second one is compatible with the first one. In addition, it is not difficult to check that $\hat{D} = \hat{i}(\hat{\mathcal{O}} \otimes_{\mathcal{O}} D)\hat{i}$ is a P-stable $\hat{\mathcal{O}}$-subalgebra of $\hat{A}_{\hat{\gamma}}$ satisfying that

$$\hat{D} \otimes_Q P = \hat{A}_{\hat{\gamma}} \quad \text{and} \quad \hat{D} \cap Pi = Qi.$$

In a word, taking the Γ-fixed algebras of the terms of the diagram (3.9.5), we get the desired commutative diagram (3.9.1). □

4. Hyperfocal subalgebras in the case that $\mathcal{O} < \hat{\mathcal{O}}$

4.1. Throughout this section we keep the notation in 1.4, 1.5 and 1.7, and always assume that $E_G(P_\gamma)$ is a p'-group, and fix the choice of θ in (3.6.2) and $\hat{i}$, $\hat{b}$ in 3.8. In particular, in (3.8.2) we have the isomorphism

$$\eta : A_\gamma \xrightarrow{\cong} \hat{A}_{\hat{\gamma}} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ. \tag{4.1.1}$$

Lemma 4.2. *Notation as above. Then there is a $P \rtimes \hat{E}_G(P_\gamma)$-stable subalgebra $\hat{D}$ of $\hat{A}_{\hat{\gamma}}$ such that*

$$\hat{D} \cap P\hat{i} = Q\hat{i} \quad \textit{and} \quad \hat{D} \otimes_Q P = \hat{A}_{\hat{\gamma}}, \tag{4.2.1}$$

and any two such subalgebras are conjugate by $\hat{i}+J(\hat{A}^P_{\hat{\gamma}})^{E_G(P_\gamma)}$. *Moreover, such a subalgebra* $\hat{D}$ *contains the image of* $\hat{E}_G(P_{\hat{\gamma}})^\circ$ *in* $\hat{A}_{\hat{\gamma}}$.

Proof. Since we have proved in Section 2 that Theorem 1.8 holds for $\hat{A}_{\hat{\gamma}}$, there exists a P-stable $\hat{\mathcal{O}}$-subalgebra $\hat{D}$ satisfying (4.2.1), and $\hat{i}+J(\hat{A}^P_{\hat{\gamma}})$ acts transitively on the set $\hat{\mathcal{D}}$ of all the P-stable $\hat{\mathcal{O}}$-subalgebras $\hat{D}$ satisfying (4.2.1). By (3.6.4), $\hat{E}_G(P_\gamma)^\circ$ not only stabilizes P, and stabilizes Q as well; so $\hat{E}_G(P_\gamma)$ also acts on $\hat{\mathcal{D}}$. Thus $(\hat{i}+J(\hat{A}^P_{\hat{\gamma}}))E_G(P_\gamma)$ acts on $\hat{\mathcal{D}}$. For any $\hat{D}\in\hat{\mathcal{D}}$, by Lemma 2.4, we have

$$N_{\hat{i}+J(\hat{A}^P_{\hat{\gamma}})}\big(\hat{D}\big)=\big(\hat{i}+J\big(Z\big(\hat{A}_{\hat{\gamma}}\big)\big)\big)\big(\hat{i}+J\big(\hat{D}^P\big)\big),$$

which is a $E_G(P_\gamma)$-acted group. By [6, 4.3 and 3.11], $N_{\hat{i}+J(\hat{A}^P_{\hat{\gamma}})}(\hat{D})$ is a uniquely split $E_G(P_\gamma)$-acted group; moreover by [10, 4.6], $\hat{i}+J(\hat{A}^P_{\hat{\gamma}})$ is a uniquely split $E_G(P_\gamma)$-acted group. So, by [6, 3.3], $\hat{\mathcal{D}}^{E_G(P_\gamma)}$ is nonempty and $(\hat{i}+J(\hat{A}^P_{\hat{\gamma}}))^{E_G(P_\gamma)}$ acts transitively on $\hat{\mathcal{D}}^{E_G(P_\gamma)}$.

Let $\hat{D}$ be a $P\rtimes\hat{E}_G(P_\gamma)^\circ$-stable $\hat{\mathcal{O}}$-subalgebra of $\hat{A}_{\hat{\gamma}}$ such that (4.2.1) holds. Then Proposition 3.7 applies to the case $\mathcal{O}=\hat{\mathcal{O}}$, and we get a subgroup $\hat{F}$ of D^* such that $\hat{k}^*\subseteq\hat{F}\subseteq N_{D^*}(P\hat{i})$ and $\hat{F}\cong\hat{E}_G(P_{\hat{\gamma}})^\circ$. Let $\hat{\mathcal{F}}$ be the set of all such subgroups $\hat{F}$ of D^*, then $N_{\hat{D}^*}(P)\cap((\hat{i}+J(\hat{A}_{\hat{\gamma}^P}))P)$ acts by conjugation on $\hat{\mathcal{F}}$ transitively. Hence $(N_{\hat{D}^*}(P)\cap((\hat{i}+J(\hat{A}_{\hat{\gamma}^P}))P))\rtimes E_G(P_\gamma)$ acts on $\hat{\mathcal{F}}$ transitively. However, by (3.7.3), $N_{\hat{D}^*}(P)\cap((\hat{i}+J(\hat{A}_{\hat{\gamma}^P}))P)$ is a uniquely split $E_G(P_\gamma)$-acted group; hence, by [6, 3.3], $\hat{\mathcal{F}}^{E_G(P_\gamma)}\neq\emptyset$. That is, $E_G(P_\gamma)$ stabilizes a subgroup F of $N_{\hat{D}^*}(P\hat{i})$ with a group isomorphism $\sigma:\hat{E}_G(P_{\hat{\gamma}})^\circ\cong F$.

For convenience, we identify the image of $\hat{E}_G(P_{\hat{\gamma}})^\circ$ in $\hat{A}^*_{\hat{\gamma}}$ with $\hat{E}_G(P_{\hat{\gamma}})^\circ$. Then it is easily checked that the set $\{\sigma(\hat{x})\hat{x}^{-1}\mid\hat{x}\in\hat{E}_G(P_{\hat{\gamma}})^\circ\}$ is a p'-subgroup of $(\hat{A}^P_{\hat{\gamma}})^*$; however, $(\hat{A}^P_{\hat{\gamma}})^*\cong\hat{k}^*\times(\hat{i}+J(\hat{A}_{\hat{\gamma}}))$ by [13, Chapter II, Proposition 8] and $\hat{i}+J(\hat{A}_{\hat{\gamma}})$ is a p'-divisible group, $\{\sigma(\hat{x})\hat{x}^{-1}\mid\hat{x}\in\hat{E}_G(P_{\hat{\gamma}})^\circ\}\subset\hat{k}^*$. That is, we have proved the equality $F=\hat{E}_G(P_{\hat{\gamma}})^\circ$. □

4.3. *A proof of the existence of Theorem 1.8*

By Lemma 4.2, there exists a P-stable $\hat{\mathcal{O}}$-subalgebra $\hat{D}$ of $\hat{A}_{\hat{\gamma}}$ which satisfies (4.2.1) and contains the image of $\hat{E}_G(P_{\hat{\gamma}})^\circ$ in $\hat{A}_{\hat{\gamma}}$ and is stabilized by $\hat{E}_G(P_\gamma)^\circ$. Then we have the following $P\rtimes\hat{E}_G(P_\gamma)^\circ$-interior algebra isomorphisms

$$\begin{aligned}A_\gamma&\cong\hat{A}_{\hat{\gamma}}\otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ}\hat{E}_G(P_\gamma)^\circ\\&\cong\hat{A}_{\hat{\gamma}}\otimes_{P\rtimes\hat{E}_G(P_{\hat{\gamma}})^\circ}\big(P\rtimes\hat{E}_G(P_\gamma)^\circ\big)\end{aligned}$$

$$
\begin{aligned}
&\cong \big(\hat{D} \otimes_Q P\big) \otimes_{P \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ} \big(P \rtimes \hat{E}_G(P_\gamma)^\circ\big) \\
&\cong \big(\hat{D} \otimes_{Q \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ} \big(P \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ\big)\big) \otimes_{P \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ} \big(P \rtimes \hat{E}_G(P_\gamma)^\circ\big) \\
&\cong \hat{D} \otimes_{Q \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ} \big(P \rtimes \hat{E}_G(P_\gamma)^\circ\big) \\
&\cong \big(\hat{D} \otimes_{Q \rtimes \hat{E}_G(P_{\hat{\gamma}})^\circ} \big(Q \rtimes \hat{E}_G(P_\gamma)^\circ\big)\big) \otimes_{Q \rtimes \hat{E}_G(P_\gamma)^\circ} \big(P \rtimes \hat{E}_G(P_\gamma)^\circ\big) \\
&\cong \big(\hat{D} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ\big) \otimes_Q P.
\end{aligned}
$$

Thus, set D to be the image in A_γ of the crossed product $\hat{D} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ$ through the isomorphism (4.1.1); then D is a P-stable unitary $\mathcal{O}$-subalgebra D of A_γ and satisfies the condition

$$D \cap Pi = Qi \quad \text{and} \quad D \otimes_Q P = A_\gamma.$$

4.4. A proof of the uniqueness of Theorem 1.8

Let D be as above, and assume that D' is also a P-stable $\mathcal{O}$-subalgebra of A_γ which satisfies

$$D' \cap Pi = Qi \quad \text{and} \quad D' \otimes_Q P = A_\gamma.$$

By Theorem 3.9, there are an $a \in i + J(A_\gamma^P)$ and a hyperfocal subalgebra $\hat{D}'$ of $\hat{A}_{\hat{\gamma}}$ such that D'^a is the image in A_γ of $\hat{D}' \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ$ through the isomorphism (4.1.1). Since it is proved in Section 2 that Theorem 1.8 holds for $\hat{A}_{\hat{\gamma}}$, there is an $\hat{a} \in \hat{i} + J(\hat{A}_{\hat{\gamma}}^P)^{\hat{E}_G(P_\gamma)^\circ}$ such that $\hat{D}'^{\hat{a}} = \hat{D}$; therefore, there exists $a' \in 1 + J(A_\gamma^P)$ such that $D'^{aa'}$ is the image in A_γ of $\hat{D} \otimes_{\hat{E}_G(P_{\hat{\gamma}})^\circ} \hat{E}_G(P_\gamma)^\circ$ through the isomorphism (4.1.1); that is, $D'^{aa'} = D$.

Acknowledgment

Both authors sincerely thank Professor Luis Puig in Paris University 7 for many useful discussions.

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