On approximation of continuous functions by trigonometric polynomials

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Abstract

We generalize the classical Jackson–Bernstein constructive description of Hölder classes of periodic functions on the interval \([-\pi, \pi]\). We approximate by trigonometric polynomials continuous functions defined on a compact set \(E \subset [-\pi, \pi]\). This set may consist of an infinite number of intervals.

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1. Introduction

For a compact set \(E \subset [-\pi, \pi] =: I\), denote by \(\tilde{C}(E)\) the class of continuous functions \(f : E \to \mathbb{R}\) which, in the case \(\pm \pi \in E\), satisfy the additional condition \(f(-\pi) = f(\pi)\). Let

\[
\omega_{f,E}(\delta) := \sup_{x_1, x_2 \in E \atop |x_1 - x_2| \leq \delta} |f(x_2) - f(x_1)|, \quad \delta > 0
\]

be the modulus of continuity of \(f \in \tilde{C}(E)\).

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Denote by $\Pi_n$, $n \in \mathbb{N} := \{1, 2, \ldots\}$ the class of trigonometric polynomials

$$t_n(x) = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \quad a_k, b_k \in \mathbb{R},$$

and let

$$E_n(f, E) := \inf_{t_n \in \Pi_n} \|f - t_n\|_E,$$

where $\| \cdot \|_S$ denotes the uniform norm over the set $S \subset \mathbb{R}$.

One of the basic statements in approximation theory, which follows immediately from the classical results of Jackson and Bernstein (see [24, pp. 257, 333] or [10, Chapter 7]), states that for $f \in \tilde{C}(I)$ and $0 < \alpha < 1$,

$$E_n(f, I) = O(n^{-\alpha}) \quad \text{as } n \to \infty \quad (1.1)$$

if and only if

$$\omega_{f,1}(\delta) = O(\delta^{\alpha}) \quad \text{as } \delta \to 0. \quad (1.2)$$

For a proper subset $E$ of $I$, the direct analogue of the equivalence $(1.1) \iff (1.2)$ (with $E$ instead of $I$) no longer holds (cf. Theorems 1 and 2 below). The main objective of this paper is to investigate the following two problems for such sets.

**Problem 1.** What are the smoothness properties of $f$ satisfying (1.1) with $E$ instead of $I$?

**Problem 2.** How closely can the function $f$ be approximated by trigonometric polynomials if $f$ satisfies the Hölder condition (1.2) with $E$ instead of $I$?

To the best of our knowledge, the only result in this direction is that of Shirokov [21, Theorem 3]. It deals with Problem 2 in the case $E = [-a, a]$, $0 < a < \pi$. A modification of the reasoning in [21] would allow us also to solve Problem 2 in the case where $E$ is the union of a finite number of disjoint closed intervals.

In this paper we focus our attention on the case where the number of components of $E$ may be infinite. We solve both problems by placing certain geometric restrictions on $E$. However, the solution methods that we have to use for the two problems are quite different.

The solution to Problem 1 follows directly relatively simply from the recent results given in [7] on the approximation of bounded continuous functions by entire functions of exponential type on subsets of the real line. The main idea is to introduce a new modulus of continuity by using a distance between the points on $E$ that is not Euclidean (cf. [11]). Since the corresponding theory on approximation of functions with a given modulus of continuity does not yet exist, we cannot use the same approach to solve Problem 2. Instead, we relate the trigonometric approximation on $E$ to the harmonic approximation on a subdomain of the unit disk involving the ideas and techniques from the theory of Dzyadyk-type polynomial approximation on compact subsets in $\mathbb{C}$ (see [12,23,8]).

2. **Main results**

We extend the set $E \neq I$ and the function $f \in \tilde{C}(E)$ $2\pi$-periodically (preserving the same notation $E$ and $f$) such that

$$x \in E \iff x + 2\pi \in E,$$

$$f(x) = f(x + 2\pi), \quad x \in E.$$
Let

\[ d(A, B) := \text{dist}(A, B) = \inf_{z \in A, \zeta \in B} |z - \zeta|, \quad A, B \subset \mathbb{C}, \]

and let \(|B|\) denote the one-dimensional Lebesgue measure (length) of \(B \subset \mathbb{C}\).

Unless otherwise stated, we denote by \(C, C_1, C_2, \ldots\) positive constants and by \(\varepsilon, \varepsilon_1, \varepsilon_2, \ldots\) sufficiently small positive constants that are either absolute or depend on \(E\) only.

The set \(E^* := \mathbb{R} \setminus E\) consists of an infinite number of disjoint open intervals \(J_j = (a_j, b_j)\). In 1960, Achieser and Levin [2] introduced a special conformal mapping of the upper half-plane \(\mathbb{H} := \{z : \text{Im} \, z > 0\}\) onto \(\mathbb{H}\) with vertical slits. Later, Levin [16] constructed the general theory of such mappings and used them to solve several extremal problems in classes of subharmonic functions. We are going to use the basic results of this theory adopted in [7] to investigate problems concerning approximation of continuous functions by entire functions of exponential type on subsets of the real line. Following [7, p. 92], we consider the case where \(E\) possesses the property

\[ \sup_j \sum_{k \neq j} \left( \frac{|J_k|}{d(J_k, J_j)} \right)^2 \leq C_1. \] (2.1)

Since \(E\) is \(2\pi\)-periodic, in (2.1) we can use the sum \(\sum_{k \neq j}'\) instead of the whole sum, which indicates that the summation is carried out only for such intervals \(J_k\) that the distance from them to \(J_j\) does not exceed some fixed constant \(\varepsilon_1\).

Let \(a \in E\) be a fixed point. Since by our assumption (2.1) we have \(|E \cap I| > 0\), according to Levin [16, Theorems 3.1, 3.2, and 3.8], there exist (uniquely determined) vertical intervals \(J'_j = [u_j, u_j + iv_j]\), \(u_j \in \mathbb{R}, v_j > 0\) and a conformal mapping

\[ \phi = \phi_a : \mathbb{H} \rightarrow \mathbb{H}_E := \mathbb{H} \setminus \left( \bigcup_j J'_j \right) \]

normalized by \(\phi(\infty) = \infty, \phi(a) = -\pi, \phi(a + 2\pi) = \pi\) such that \(\phi\) can be extended continuously to \(\mathbb{H}\) and \(\phi\) satisfies the boundary correspondence \(\phi(J_j) = J'_j\).

Since trigonometric polynomials represent a relatively wide subclass of entire functions of exponential type, in order to solve Problem 1, it is natural to use the appropriate results [7, Theorems 2 and 3]. Following [7, p. 93], for \(x_1, x_2 \in E\) such that \(x_1 < x_2\), set

\[ \rho_E(x_1, x_2) = \rho_E(x_2, x_1) := \text{diam} \, \phi([x_1, x_2]), \]

where

\[ \text{diam} \, B := \sup_{z, \zeta \in B} |z - \zeta| \]

is the diameter of \(B \subset \mathbb{C}\).

In spite of its definition via the conformal mapping, the behavior of \(\rho_E\) can be characterized in purely geometrical terms as follows. By virtue of (2.1),

\[ d(J_j, E^* \setminus J_j) \geq C_2 |J_j|, \quad C_2 = C_1^{-1/2}. \] (2.2)

Let the constant \(\varepsilon\) be such that \(0 < \varepsilon < \min(1, C_2/2)\). For any component \(J_j\) of \(E^*\), denote by \(\tilde{J}_j\) the open interval with the same center as \(J_j\) and length \((1 + \varepsilon)|J_j|\). For \(x_1, x_2 \in E\) such that
$x_1 < x_2$, consider the function

$$
\tau_E(x_1, x_2) = \tau_E(x_2, x_1) = \tau_{E, \varepsilon}(x_1, x_2)
$$

and let

$$
(2.3)
$$

$$
\tau_{E, \varepsilon}(x_1, x_2) := \begin{cases} 
\left( \frac{|J_j|}{d([x_1, x_2], J_j)} \right)^{1/2} (x_2 - x_1) & \text{if } x_1, x_2 \in J_j \text{ for some } j \\
|J_j|^{1/2}(x_2 - x_1)^{1/2} & \text{if } x_2 - x_1 < d([x_1, x_2], J_j), \\
x_2 - x_1 & \text{if } x_1, x_2 \in J_j \text{ for some } j \text{ and } \nonumber \\
d([x_1, x_2], J_j) \leq x_2 - x_1 \leq \frac{\varepsilon}{2} |J_j|, \\
\end{cases}
$$

otherwise.

According to [7, Theorem 1], for $x_1, x_2 \in E$,

$$
\frac{1}{C_3} \leq \frac{\rho_E(x_1, x_2)}{\tau_E(x_1, x_2)} \leq C_3, \quad C_3 = C_3(\varepsilon) > 1.
$$

Let

$$
\omega_{f, E}^{\star}(\delta) := \sup_{x_1, x_2 \in E \atop \rho_E(x_1, x_2) \leq \delta} |f(x_2) - f(x_1)|, \quad \delta > 0
$$

and let $\omega(\delta), \delta > 0$ be a function of the modulus of continuity type, i.e., a positive nondecreasing function with $\omega(+0) = 0$ such that

$$
\omega(t \delta) \leq 2t \omega(\delta), \quad \delta > 0, t > 1.
$$

We assume that $\omega$ satisfies the condition

$$
\delta \int_{\delta}^{1} \frac{\omega(t)}{t^2} dt = O(\omega(\delta)) \quad \text{as } \delta \to 0. \tag{2.3}
$$

Let

$$
A_\omega(E) := \left\{ f \in \tilde{C}(E) : E_n(f, E) = O\left( \omega \left( \frac{1}{n} \right) \right) \text{ as } n \to \infty \right\},
$$

$$
C_\omega^*(E) := \left\{ f \in \tilde{C}(E) : \omega_{f, E}^{\star}(\delta) = O(\omega(\delta)) \text{ as } \delta \to 0 \right\}.
$$

If $\omega(\delta) = \delta^\alpha, 0 < \alpha < 1$, we denote the above defined classes as $A_\omega^\alpha(E)$ and $C_\omega^*(E)$, respectively. Our main result regarding Problem 1 can be formulated as follows.

**Theorem 1.** Let $E$ satisfy (2.1) and let $\omega$ satisfy (2.3). Then

$$
A_\omega(E) = C_\omega^*(E), \tag{2.4}
$$

and, consequently,

$$
A_\omega^\alpha(E) = C_\omega^*\alpha(E), \quad 0 < \alpha < 1. \tag{2.5}
$$

For the solution of Problem 2, we are going to use the estimates of the rate of approximation $|f(x) - t_n(x)|$ which depend not only on $E$, $f$, and $n$ but also on $x \in E$ in a way more typical for polynomial approximation on compact sets in $\mathbb{C}$ (see [12,23,8]).

For $\delta > 0$ and $x \in E$, let

$$
E_\delta := \{ z \in \mathbb{H} : \text{Im } \phi(z) = \delta \}, \quad \rho_\delta(x) := d(x, E_\delta).
$$
In spite of its definition via the conformal mapping, the behavior of $\rho_\delta$ can be characterized in purely geometrical terms as follows. Let

$$v_\delta(x) = v_{\delta,\varepsilon}(x) := \begin{cases} \frac{\delta^2}{|J_j|} & \text{if } d(x, J_j) \leq \frac{\delta^2}{|J_j|} < \varepsilon|J_j| \text{ for some } j, \\ \left(\frac{d(x, J_j)}{|J_j|}\right)^{1/2} & \text{if } \frac{\delta^2}{|J_j|} < d(x, J_j) < \varepsilon|J_j| \text{ for some } j, \\ \varepsilon^{1/2}\delta & \text{otherwise.} \end{cases}$$

According to [7, (3.4), (3.5) and Lemmas 12–14], for $0 < \delta \leq 1$ and $x \in E$ we have

$$\frac{1}{C_4} \leq \rho_\delta(x) \leq v_\delta(x) \leq C_4, \quad C_4 = C_4(\varepsilon) > 1.$$

Denote by $B_\omega(E)$ the class of functions $f \in \tilde{C}(E)$ such that for any $n \in \mathbb{N}$ there exists a polynomial $t_n \in \Pi_n$ satisfying

$$|f(x) - t_n(x)| \leq C_5 \omega(\rho_{1/n}(x)), \quad x \in E,$$

where $C_5 = C_5(f)$.

Let

$$C_\omega(E) := \{f \in \tilde{C}(E) : \omega_f, E(\delta) = O(\omega(\delta)) \text{ as } \delta \to 0\}.$$

If $\omega(\delta) = \delta^\alpha, 0 < \alpha < 1$, we denote the above defined classes as $B^\alpha(E)$ and $C^\alpha(E)$, respectively. Our main result regarding Problem 2 can be formulated as follows.

**Theorem 2.** Let $E$ satisfy (2.1) and let $\omega$ satisfy (2.3). Then

$$C_\omega(E) = B_\omega(E),$$

and, consequently,

$$C^\alpha(E) = B^\alpha(E), \quad 0 < \alpha < 1.$$

It turns out that the restriction (2.1) in Theorems 1 and 2 is essential, i.e., for the sets which are too sparse in the neighborhood of at least one of their points, (2.5) and (2.7) in general do not hold. We illustrate our claim by constructing an appropriate example.

For $k \in \mathbb{N} \cup \{0\}$ and $m = 0, 1, \ldots, 3^k$ consider the points

$$u_{k,m} := 2^{-k} - m3^{-k}2^{-k-1}$$

and vertical intervals

$$J'_{k,m} := [u_{k,m}, (1 + i)u_{k,m}].$$

Let $W_0 \subset \mathbb{H}$ be defined by the conditions

$$w \in W_0 \iff w + 2\pi \in W_0,$$

$$\Sigma_0 \cap W_0 = \Sigma_0 \setminus \bigcup_{k=0}^{\infty} \bigcup_{m=0}^{3^k} J'_{k,m},$$
where \( \Sigma_0 := \{ w = u + i v : |u| \leq \pi, v > 0 \} \). Let
\[
I_{k,m} := [u_{k,m}, u_{k,m-1}], \quad m = 1, \ldots, 3^k.
\]
Consider the conformal mapping \( \phi_0 : \mathbb{H} \to W_0 \) normalized by the conditions
\[
\phi_0(\infty) = \infty, \quad \phi_0(\pm \pi) = \pm \pi.
\]
We extend \( \phi_0 \) continuously to \( \mathbb{R} \) and define by \( J_{k,m} = [x'_{k,m}, x''_{k,m}] \subset \mathbb{R} \) and \( I_{k,m} \subset \mathbb{R} \) the intervals satisfying the boundary correspondence
\[
\phi_0(J_{k,m}) = J'_{k,m}, \quad \phi_0(I_{k,m}) = I'_{k,m}.
\]

Consider the set
\[
E_0 := \left( \bigcup_{k=0}^{\infty} \bigcup_{m=1}^{3^k} I_{k,m} \right) \cup [\phi_0^{-1}(0)] \cup [x''_{0,0}, \pi].
\]
The analysis carried out in Section 6 shows that \( E_0 \) is relatively sparse to the right of \( \phi_0^{-1}(0) \).

**Theorem 3.** For any \( 0 < \alpha < 1 \),
\[
C^{*\alpha}(E_0) \setminus A^{\alpha}(E_0) \neq \emptyset, \tag{2.8}
\]
\[
C^{\alpha}(E_0) \setminus B^{\alpha}(E_0) \neq \emptyset. \tag{2.9}
\]

Let us introduce the notation that we will be using throughout the paper. We continue to use the convention that \( C, C_1, \ldots, \varepsilon, \varepsilon_1, \ldots \) denote positive constants, different in different sections and depending only on inessential quantities. For \( a, b \geq 0 \) we write \( a \asymp b \) if \( a \leq b \) and \( b \leq a \) simultaneously.

Let for \( z \in \mathbb{C} \) and \( \delta > 0 \),
\[
D(z, \delta) := \{ \zeta : |\zeta - z| < \delta \}, \quad \mathbb{D} := D(0, 1),
\]
\[
\mathbb{D}^* := \overline{\mathbb{C}} \setminus \mathbb{D}, \quad \mathbb{T} := \{ z : |z| = 1 \}.
\]

### 3. Uniform approximation; proof of Theorem 1

The inclusion \( A_\omega(E) \subset C^*_\alpha(E) \) follows from [7, Theorem 3].

In order to prove the converse inclusion, we adapt the reasoning from [1, pp. 175–176] (see also [21, pp. 4938–4939]). Let \( f \in C^*_\alpha(E) \). According to [7, Theorem 2], for any \( n \in \mathbb{N} \) there exists an entire function \( g_n \) of exponential type at most \( n \) such that
\[
\| f - g_n \|_E \leq C_1 \omega \left( \frac{1}{n} \right), \quad C_1 = C_1(f, \omega, E).
\]
Therefore, by virtue of [20, p. 86, (28)] (see also [16, Theorem 3.8]) we have
\[
\| g_n \|_\mathbb{R} \leq e^{C_2 n}, \quad C_2 = C_2(C_1, f, \omega, E).
\]
For \( m \in \mathbb{N} \) consider the function
\[
h_{n,m}(x) := \frac{1}{2m+1} \sum_{j=-m}^{m} g_n(x + 2j\pi), \quad x \in \mathbb{R}.
\]
Since
\[ \|h_{n,m}\|_\mathbb{R} \leq \|g_n\|_\mathbb{R}, \]
according to the Fragemé–Lindelöf Theorem (see [14, p. 27]), for any fixed \( n \), \( \{h_{n,m}\}_{m \in \mathbb{N}} \) is a normal family. Therefore, there exists a subsequence \( \{h_{n,m_k}\}_{k \in \mathbb{N}} \) which uniformly converges on compact subsets in \( \mathbb{C} \) to an entire function \( t_n \) of exponential type at most \( n \).

Since for \( x \in \mathbb{R} \),
\[ |h_{n,m_k}(x + 2\pi) - h_{n,m_k}(x)| = \frac{1}{2m_k + 1} \left| g_n(x + 2(m_k + 1)\pi) - g_n(x - 2m_k\pi) \right| \leq \frac{2e^{C^2n}}{2m_k + 1}, \]
letting \( k \to \infty \), we have \( t_n(x + 2\pi) = t_n(x) \). Hence, by [1, p. 175] \( t_n \in \Pi_n \).

Moreover, since for \( x \in \mathbb{E} \),
\[ |f(x) - h_{n,m_k}(x)| = \frac{1}{2m_k + 1} \left| \sum_{j=-m_k}^{m_k} (f(x + 2j\pi) - g_n(x + 2j\pi)) \right| \leq C_1\omega \left( \frac{1}{n} \right), \]
letting \( k \to \infty \), we have
\[ \|f - t_n\|_E \leq C_1\omega \left( \frac{1}{n} \right), \]
i.e., \( f \in A_{\omega}(E) \) and consequently, \( C^*_{\omega}(E) \subset A_{\omega}(E) \). This completes the proof of (2.4).

4. Auxiliary conformal mappings

First, we discuss the interpretation of the Levin conformal mapping \( \phi \) that we use in our subsequent reasoning. Consider the functions \( \zeta = h(z) := e^{-iz}, z = h^{-1}(\zeta) := i \log \zeta, \)
\[ \alpha(\zeta) := \begin{cases} h \circ \phi \circ h^{-1}(\zeta) & \text{if } \zeta \in \mathbb{D}^* \setminus \{\infty\}, \\ \infty & \text{if } \zeta = \infty. \end{cases} \]
Note that even though \( h^{-1} \) is multi-valued, \( \alpha \) is single valued in view of the periodicity. Let
\[ \beta(\zeta) := \frac{1}{\alpha(1/\zeta)}, \quad \zeta \in \mathbb{D}. \]
According to the maximum principle for \( \zeta/\alpha(\zeta) \) in \( \mathbb{D}^* \), we obtain
\[ |\zeta| \leq |\alpha(\zeta)|, \quad \zeta \in \mathbb{D}^*. \quad (4.1) \]
Moreover, by [17, p. 210, Proposition 9.15] for \( \zeta \in \mathbb{D} \),
\[ \beta(\zeta) = e^{i\theta} \zeta (\operatorname{cap} E')^2 \exp \left( -2 \int \log(1 - i\xi) d\mu(t) \right), \]
where \( \theta \in I \) is a constant, \( \operatorname{cap} E' \) is the (logarithmic) capacity of \( E' := h(E) \), and \( \mu = \mu_{E'} \) is the equilibrium measure of \( E' \). We refer to the basic notions of potential theory (such as capacity,
potential, Green’s function, equilibrium measure, etc.) without explicit citations. The description of these notions and their properties can be found in [18,19].

Let \( g(\zeta) = g(\zeta, \infty, \overline{C \setminus E'}) \) be the Green function of \( \overline{C \setminus E'} \) with pole at \( \infty \). Since for \( \zeta \in \overline{C \setminus E'} \)

\[
U^\mu(\zeta) := \int \log \frac{1}{|\zeta - t|} d\mu(t) = -g(\zeta) - \log \text{cap } E',
\]

we have

\[
g(\zeta) = \begin{cases} 
\frac{1}{2} \log |\zeta| |\alpha(\zeta)|, & \text{if } \zeta \in \mathbb{D}^*, \\
\frac{1}{2} \log \left[ |\zeta| \left| \alpha \left( \frac{1}{\zeta} \right) \right| \right], & \text{if } \zeta \in \mathbb{D}.
\end{cases}
\]

(4.2)

Let for \( \delta > 0 \)

\[
E'_\delta := \{ \zeta : g(\zeta) = \delta \}, \quad \rho'_\delta(\zeta) := d(\zeta, E'_\delta).
\]

By virtue of (4.1) and (4.2), we have for \( \xi \in E' \) and \( 0 < \delta \leq 1 \)

\[
\rho'_\delta(\xi) \geq \min(d(\xi, h(E_\delta)), d(\xi, S_\delta)) = d(\xi, S_\delta) \geq d(\xi, h(E_\delta)),
\]

(4.3)

where

\[
S_\delta := \left\{ t \in \mathbb{D} : \frac{1}{i} \in h(E_\delta) \right\}.
\]

Indeed, if \( \zeta \in h(E_\delta) \), then

\[
g(\zeta) \leq \log |\alpha(\zeta)| = \delta
\]

which means that

\[
d(\xi, E'_\delta \cap \overline{\mathbb{D}^*}) \geq d(\xi, h(E_\delta)).
\]

(4.4)

At the same time, if \( \zeta \in S_\delta \), then

\[
g(\zeta) \leq \log \left| \alpha \left( \frac{1}{\zeta} \right) \right| = \delta
\]

which means that

\[
d(\xi, E'_\delta \cap \overline{\mathbb{D}}) \geq d(\xi, S_\delta).
\]

(4.5)

Comparing (4.4) and (4.5), we obtain the first part of (4.3). Since for any \( \zeta \in \mathbb{D}^* \setminus \{\infty\} \) and \( \xi \in \mathbb{T}, \)

\[
\left| \xi - \frac{1}{\zeta} \right| = \frac{1}{|\zeta|} |\xi - \zeta|,
\]

the second and third parts of (4.3) follow.

For \( x \in \mathbb{R} \) and \( z \in \mathbb{C} \) with \( |x - z| \leq 1 \), we have

\[
|x - z| \asymp |h(x) - h(z)|
\]

(4.6)
and consequently, for \( x \in E \) and \( 0 < \delta \leq 1 \),
\[
\rho_\delta(x) \asymp d(h(x), h(E_\delta))
\]
which together with (4.3) yield
\[
\rho_\delta(x) \leq \rho'_\delta(h(x)); \quad x \in E, \ 0 < \delta \leq 1.
\] (4.7)

Moreover, (4.1) and (4.2) imply also
\[
\rho_\delta(x) \geq \rho'_{\delta/2}(h(x)); \quad x \in E, \ 0 < \delta \leq 1.
\] (4.8)

Indeed, for \( \zeta \in h(E_\delta) \),
\[
g(\zeta) \geq \frac{1}{2} \log |\alpha(\zeta)| = \frac{\delta}{2}.
\]
Therefore, for \( \xi := h(x) \),
\[
\rho'_{\delta/2}(\xi) \leq d(\xi, E'_{\delta/2} \cap \overline{D}^*) \leq d(\xi, h(E_\delta)),
\]
which together with (4.6) yield (4.8).

Comparing (4.7) and (4.8), and [7, (3.5)] for \( \xi \in E' \) and \( 0 < \delta < \Delta \leq 1 \), we obtain
\[
\left( \frac{\delta}{\Delta} \right) ^{\varepsilon} \leq \frac{\rho'_{\delta}(\xi)}{\rho_{\delta}(\xi)} \leq \frac{\delta}{\Delta}.
\] (4.9)

Furthermore, the monotonicity of the Green function and the relation
\[
g(\xi, \infty, D^*) = \log |\zeta|, \quad \zeta \in D^*,
\]
imply that for \( 0 < \delta \leq 1 \) and \( \zeta \in E' \),
\[
\rho'_\delta(\zeta) \leq e^\delta - 1 \leq \delta.
\] (4.10)

Next, starting with \( E \) and \( E^* = \mathbb{R} \setminus E = \bigcup_j J_j = \bigcup_j (a_j, b_j) \) we construct auxiliary domains \( G \) and \( \Omega \) as follows. Since the functions in Theorems 1–3 are \( 2\pi \)-periodic, we assume without loss of generality that \( \pm \pi \in E \). Consider the curves
\[
S_j := [a_j - 2it_j, b_j - 2it_j] \bigcup \{z : |z - (a_j - it_j)| = t_j, \text{Re } z \leq a_j\} \\
\bigcup \{z : |z - (b_j - it_j)| = t_j, \text{Re } z \geq b_j\},
\]
where \( t_j := C_2 |J_j|/3 \) and \( C_2 \) is the constant from (2.2). Denote by \( \Omega \supset \mathbb{H} \) the Jordan domain bounded by
\[
L = \partial \Omega = E \bigcup \bigcup_j S_j
\]
and let \( G := \mathbb{C} \setminus \overline{\Omega} \). Denote by \( \varphi : \Omega \rightarrow \mathbb{H} \) the conformal mapping normalized by
\[
\varphi(\infty) = \infty, \quad \varphi(\pm \pi) = \pm \pi.
\]

Let \( \Omega' := h(\Omega) \cup \{\infty\}, G' := h(G) \cup \{0\}, L' := h(L) \), and let \( \Phi : \Omega' \rightarrow \mathbb{D}^* \) be the conformal mapping normalized by
\[
\Phi(\infty) = \infty, \quad \Phi'(\infty) := \lim_{\zeta \rightarrow \infty} \frac{\Phi(\zeta)}{\zeta} > 0.
\]
Let $\Psi := \Phi^{-1}$. By the periodicity of $\varphi$, the function $h \circ \varphi^{-1} \circ h^{-1}$ is a conformal mapping of $\mathbb{D}^*$ onto $\Omega'$ with $\infty$ as a fixed point. Thus,

$$
\Phi \circ h \circ \varphi^{-1} \circ h^{-1}(\zeta) = e^{i\theta} \zeta, \quad \zeta \in \mathbb{D}^*,
$$

where $\theta \in I$ is a constant, i.e.,

$$
\Psi(\zeta) = h \circ \varphi^{-1} \circ h^{-1}(e^{-i\theta} \zeta), \quad \zeta \in \mathbb{D}^*.
$$

(4.11)

Therefore, some metric properties of $\Phi$ and $\Psi$ can be derived from the properties of $\varphi$ proved in [7, Section 4]. For the convenience of the reader we repeat the relevant material from [4,6,7] without proofs, thus making our exposition self-contained.

Let for $\delta > 0$ and $\zeta \in \mathbb{C}$

$$
L_\delta := \{ \zeta \in \Omega' : |\Phi(\zeta)| = 1 + \delta \}, \quad d'_\delta(\zeta) := d(\zeta, L'_\delta).
$$

Since (4.6) and (4.11) imply for $x \in E$ and $0 < \delta \leq 1$

$$
d'_\delta(h(x)) \simeq d(x, \{ z \in \Omega : \text{Im} \varphi(z) = \log(1 + \delta) \})
$$

and [7, (3.4), (3.5), Lemma 19, and (4.36)] yield

$$
d(x, \{ z \in \Omega : \text{Im} \varphi(z) = \log(1 + \delta) \}) \leq \rho_\delta(x),
$$

we have

$$
d'_\delta(h(x)) \leq \rho_\delta(x).
$$

(4.12)

Our further analysis of the behavior of the function $\Phi$ is based on the application of the notion of the module $m(I')$ of a family of curves $I'$. We refer to the basic properties of the module (such as conformal invariance, comparison principle, composition laws, etc.) without explicit citations. All these facts can be found in [3,15].

Special families of separating curves (cf. [9,8]) play an important role in our reasoning. Let $U \subset \overline{\mathbb{C}}$ be a simply connected Jordan domain. For $z_1, \ldots, z_m \in \overline{U}$ and $\zeta_1, \ldots, \zeta_k \in \overline{U}$ we denote by $F(z_1, \ldots, z_m; \zeta_1, \ldots, \zeta_k; U)$ the family of all cross-cuts of $U$ (i.e., locally rectifiable Jordan arcs $\gamma \subset U$ with end points on $\partial U$) which separate the points $z_1, \ldots, z_m$ from $\zeta_1, \ldots, \zeta_k$ in $U$.

Let $U \subset \mathbb{C}$ be a bounded domain whose boundary is a Jordan curve $J = \partial U$; $V := \overline{\mathbb{C}} \setminus \overline{U}$. We say that $\overline{U} \in H$ if every pair of points $z, \zeta \in J$ can be joined by an arc $\gamma(z, \zeta) \subset \overline{U}$ such that its length satisfies

$$
|\gamma(z, \zeta)| \leq C_1|z - \zeta|, \quad C_1 = C_1(U) \geq 1.
$$

Let $\overline{U} \in H$ and let $z \in J, 0 < r < \frac{1}{2} \text{diam } U$. Consider any circular cross-cut $\gamma \subset \{ \zeta : |\zeta - z| = r \}$ (with end points $\zeta', \zeta'' \in J$) which separates $z$ from $\infty$ in $V$. We have

$$
|\gamma| \geq r.
$$

(4.13)

Indeed, since we can connect $\zeta'$ and $\zeta''$ by an arc $l = l(\zeta', \zeta'') \subset \overline{U}$ with the property

$$
|l| \leq |\zeta' - \zeta''| \leq |\gamma|,
$$

(4.14)

comparing the diameter and the length of the boundary of the domain bounded by a Jordan curve $l \cup \gamma$ we obtain

$$
r \leq |l \cup \gamma| = |l| + |\gamma| \leq |\gamma|,
$$

which means (4.13).
For $z \in J$ and $r > 0$, denote by $\gamma_z(r) \subset V$ a subarc of the circle \{ $\zeta : |\zeta - z| = r$ \} that separates $z$ from $\infty$ in $V$. If $\gamma_z(r)$ is not uniquely determined (due to (4.13) there could be only a finite number of such arcs), we agree to choose it such that, in the division of $V$ by $\gamma_z(r)$ into two subdomains, the unbounded domain is as large as possible for given $z$ and $r$.

If $0 < r < R < \frac{1}{2} \text{diam } U$, then $\gamma_z(r)$ and $\gamma_z(R)$ are the sides of a quadrilateral $Q_z(r, R) \subset V$ whose other two sides are parts of $J$. We denote by $m_z(r, R)$ the module of this quadrilateral, i.e., the module of the family of arcs that separate the sides $\gamma_z(r)$ and $\gamma_z(R)$ in $Q_z(r, R)$ (see [3,15]).

**Lemma 1** ([6, Theorem 2]). Suppose that $\overline{U} \in H$, $z \in J$, $0 < r_1 < r_2 < r_3 < \frac{1}{2} \text{diam } \overline{U}$. Then

$$0 \leq m_z(r_1, r_3) - (m_z(r_1, r_2) + m_z(r_2, r_3)) \leq C_2,$$

$$\frac{1}{2\pi} \log \frac{r_2}{r_1} \leq m_z(r_1, r_2) \leq C_3 \log \frac{r_2}{r_1} + C_4.$$

We say that $\overline{U} \in H^*$ if $\overline{U} \in H$ and there exist positive numbers $C_5 = C_5(U)$ and $\varepsilon = \varepsilon(U) < \frac{1}{2} \text{diam } U$ such that

$$|m_z(|z - \zeta|, \varepsilon) - m_z(|z - \zeta|, \varepsilon)| \leq C_5$$

for all points $z, \zeta \in J$ with the property $|z - \zeta| < \varepsilon$.

For $z, \zeta \in L$ we denote by $r_z(\zeta)$ the supremum of those $r > 0$ for which the arc $\gamma_z(r)$ separates $z$ from $\zeta$ in $V$. If $\overline{U} \in H$, then directly from the definition, we have

$$|z - \zeta| \leq r_z(\zeta).$$

The following two results are useful in testing whether condition (4.14) is satisfied.

**Lemma 2** ([6, Theorem 3]). Let $\overline{U} \in H$ and let $z, \zeta \in J$ be such that $0 < |z - \zeta| < \varepsilon < \frac{1}{2} \text{diam } U$. Then:

(i) if $r_z(\zeta) \leq C_5|z - \zeta|$, then (4.14) is satisfied;

(ii) if $r_z(\zeta) > |z - \zeta|$, then (4.14) is equivalent to the condition

$$|m_z(|z - \zeta|, r_z(\zeta)) - m_z(|z - \zeta|, r_z(\zeta))| \leq C_7.$$

**Lemma 3** ([13, (1) and (3)]). For $0 < r_1 < r_2 < \infty$, let

$$Q = Q(r_1, r_2) := \{ re^{i\theta} : r_1 < r < r_2, -\theta_1(r) < \theta < \theta_2(r) \},$$

where the functions $\theta_j$, $j = 1, 2$, have finite total variation $V_j$ on $[r_1, r_2]$ and satisfy

$$0 < \theta_0 \leq \theta_j(r) \leq 2\pi.$$

Then for the module of $Q$, i.e., the module of the family $\Gamma = \Gamma(Q)$ of arcs separating its boundary circular components in $Q$, we have

$$\int_{r_1}^{r_2} \frac{dr}{(\theta_1(r) + \theta_2(r))r} \leq m(\Gamma) \leq \int_{r_1}^{r_2} \frac{dr}{(\theta_1(r) + \theta_2(r))r} + \frac{\pi}{\theta_0}(V_1 + V_2).$$
It is clear that $G' \in H$. Moreover, we claim that
\[
G' \in H^*.
\]
(4.16)

Indeed, consider two particular cases for the location of points $z, \zeta \in L'$.

(i) Let $\zeta = h(x), x \in \tilde{J}_j \setminus J_j, z = h(\tau), \tau \in S_j$ satisfy
\[
\text{Im } \tau \geq t_j \asymp |J_j|, \quad |z - \zeta| \leq \frac{1}{2} d(x, h(J_j)),
\]
where $t_j$ is from the definition of $S_j$. In this case, $r_z(\zeta) = d(z, h(J_j))$ and by Lemmas 1 and 3 we have
\[
\|m_z(|z - \zeta|, r_z(\zeta)) - \frac{1}{\pi} \log \frac{d(z, h(J_j))}{|z - \zeta|}\| \leq 1,
\]
(4.17)
\[
\|m_\zeta(|z - \zeta|, r_z(\zeta)) - \frac{1}{\pi} \log \frac{d(z, h(J_j))}{|z - \zeta|}\| \leq 1.
\]
(4.18)

Comparing (4.17) and (4.18) we obtain (4.15) which, by Lemma 2, implies (4.14).

(ii) In all other cases we have $r_z(\zeta) \leq |z - \zeta|$ and (4.14) follows from Lemma 2.

We complete this section with the result which is essentially due to Belyi [9]. Let $z_1 \in \mathbb{R}$ and $z_2, z_3 \in \mathbb{H}$ be distinct points and let $\Gamma = \Gamma(z_1, z_2; z_3, \infty; \mathbb{H})$. If $|z_1 - z_2| \leq |z_1 - z_3|$ then
\[
\frac{1}{\pi} \log \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \leq m(\Gamma) \leq \frac{1}{\pi} \log \left| \frac{z_1 - z_3}{z_1 - z_2} \right| + 2.
\]
(4.19)

If $|z_1 - z_2| \geq |z_1 - z_3|$ then
\[
m(\Gamma) \leq 2
\]
(4.20)
(cf. [9, Lemma 4.1] or [8, p. 35]).

5. Pointwise approximation; proof of Theorem 2

First, we assume that $f \in B_\varnothing(E)$. According to (4.7), this means that for any $n \in \mathbb{N}$ there exists a rational function of the form
\[
r_n(\zeta) = \sum_{k=-n}^{n} c_k \zeta^k, \quad c_k \in \mathbb{C},
\]
(5.1)
such that
\[
|\tilde{f}(\zeta) - r_n(\zeta)| \leq \omega(\rho'_{1/n}(\zeta)), \quad \zeta \in E' = h(E),
\]
where
\[
\tilde{f}(\zeta) := f(h^{-1}(\zeta)), \quad \zeta \in E'.
\]
By virtue of (2.2), the set $E'$ is $c$-dense in the terminology of [23], i.e., for any $\zeta \in E'$ and $0 < \delta < 1$,
\[
\text{cap } E' \cap \overline{D(z, \delta)} \geq \delta.
\]
Below, we use a straightforward modification of the standard technique for the proof of inverse theorems concerning polynomial approximation (see [23, Chapter 4], [12, Chapter IX, Section 10]), to obtain for \( \zeta_1, \zeta_2 \in E' \)

\[
|\tilde{f}(\zeta_1) - \tilde{f}(\zeta_2)| \leq \omega(|\zeta_1 - \zeta_2|),
\]

that is, \( f \in C_\omega(E) \) and, consequently,

\[
B_\omega(E) \subset C_\omega(E).
\]

The first key point of the above mentioned modification is the following analogue of the classical Markov–Bernstein polynomial inequality.

**Lemma 4.** If the rational function \( r_n \) of the form (5.1) satisfies

\[
|r_n(\zeta)| \leq \omega(\rho_{1/n}'(\zeta)), \quad \zeta \in E',
\]

then for \( \zeta \in E' \) and \( \tau \in \mathbb{T} \) with \( |\tau - \zeta| \leq \rho_{1/n}'(\zeta)/C_1 \) we have

\[
|r_n'(\tau)| \leq C_2 \frac{\omega(\rho_{1/n}'(\zeta))}{\rho_{1/n}'(\zeta)},
\]

where \( C_j = C_j(E', \omega) \geq 1, j = 1, 2. \)

**Proof.** Let \( \zeta \in E' \). By our assumption and [23, p. 161, Corollary to Theorem 4.6.1] for the polynomial \( p_n(\zeta) = \xi^n r_n(\zeta) \) and \( \tau \in \mathbb{T} \) satisfying \( |\tau - \zeta| \leq \rho_{1/n}'(\zeta)/C_1 \) we have

\[
|p_n(\tau)| \leq \omega(\rho_{1/n}'(\zeta)),
\]

\[
|p_n'(\tau)| \leq \frac{\omega(\rho_{1/n}'(\zeta))}{\rho_{1/n}'(\zeta)}.
\]

Therefore, (4.10) implies

\[
|r_n'(\tau)| \leq n|p_n(\tau)| + |p_n'(\tau)| \leq \frac{\omega(\rho_{1/n}'(\zeta))}{\rho_{1/n}'(\zeta)}. \quad \square.
\]

The second key point of our modification is the existence of \( k = k(E') \in \mathbb{N} \setminus \{1\} \) with the property

\[
1 \leq \frac{\rho_{k-j-1}'(\zeta)}{\rho_{k-j}'(\zeta)} \leq \frac{1}{2}; \quad \zeta \in E', \ j \in \mathbb{N},
\]

which follows from (4.9). This means that in our case the sequence \( \{k^{-j}\}_{j \in \mathbb{N}} \) can be used instead of a so called supporting subsequence which is defined in much more complicated way (see [23, Chapter 4, Section 7]).

**Proof of (5.2).** Without loss of generality we assume that \( \delta := |\zeta_1 - \zeta_2| \) is sufficiently small, i.e., \( C_1 \delta < d_1 \), where we use the notation

\[
d_j := \rho_{k-j}'(\zeta_1), \quad j \in \mathbb{N}.
\]
We have
\[ \tilde{f}(\zeta_2) - \tilde{f}(\zeta_1) = r_k(\zeta_2) - r_k(\zeta_1) + \sum_{j=1}^{N+1} [R_j(\zeta_2) - R_j(\zeta_1)] \]
\[ + [\tilde{f}(\zeta_2) - r_k(\zeta_2)] - [\tilde{f}(\zeta_1) - r_k(\zeta_1)]. \]
where
\[ R_j(\zeta) := r_{kj+1}(\zeta) - r_{kj}(\zeta), \quad j = 1, \ldots, N - 1, \]
and \( N \in \mathbb{N} \) is chosen such that \( d_{N+1} \leq C \delta < d_N \).

Denoting by \( \mathbb{T}(\zeta_1, \zeta_2) \) the shortest circular subarc of \( \mathbb{T} \) joining \( \zeta_1 \) and \( \zeta_2 \) we obtain
\[ |\tilde{f}(\zeta_2) - \tilde{f}(\zeta_1)| \leq \int_{\mathbb{T}(\zeta_1, \zeta_2)} |r_k'(\zeta)| \, d\zeta | + \sum_{j=1}^{N-1} \int_{\mathbb{T}(\zeta_1, \zeta_2)} |R'_j(\zeta)| \, d\zeta | \]
\[ + |\tilde{f}(\zeta_2) - r_k(\zeta_2)| + |\tilde{f}(\zeta_1) - r_k(\zeta_1)|. \]

Therefore, according to (2.3), Lemma 4, and (5.4) we have
\[ |\tilde{f}(\zeta_2) - \tilde{f}(\zeta_1)| \leq |\zeta_2 - \zeta_1| \left( 1 + \sum_{j=1}^{N-1} \frac{\omega(d_j)}{d_j} \right) + \omega(d_N) \]
\[ \leq \delta \int_\delta^1 \frac{\omega(t)}{t^2} \, dt + \omega(\delta) \asymp \omega(\delta), \]
which proves (5.2). \( \square \)

The proof of the inclusion converse to (5.3) is based on the results concerning the approximation of functions by harmonic polynomials of the form
\[ h_n(\zeta) = \text{Re} \sum_{k=0}^n a_k \zeta^k, \quad a_k \in \mathbb{C}. \] (5.5)

Let \( f \in C_\omega(E) \), i.e., \( \tilde{f} \) satisfies (5.2) on \( E' \). Applying the procedure described in [22, Chapter VI] (see [8, p. 15, Theorem 1.11]) we extend \( \tilde{f} \) continuously to \( \mathbb{T} \) such that \( \tilde{f} \) satisfies (5.2) on \( \mathbb{T} \). Furthermore, we extend \( \tilde{f} \) harmonically to \( \mathbb{D} \) using the Poisson integral
\[ \tilde{f}(\zeta) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |\zeta|^2}{|e^{it} - \zeta|^2} \tilde{f}(e^{it}) \, dt, \quad \zeta \in \mathbb{D}. \]

Note that \( \tilde{f} \) satisfies (5.2) for any \( \zeta_1, \zeta_2 \in \overline{\mathbb{D}} \). Indeed, it can be easily shown (for example, by a discussion similar to that of [23, pp. 52–53, Lemma 1.3.1]) that it is sufficient to demonstrate the validity of (5.2) only for \( \zeta_1 = (1 - \delta)e^{i\theta} \) and \( \zeta_2 = e^{i\theta} \), where \( 0 \leq \theta < 2\pi \) and \( 0 < \delta \leq 1 \). In this case, by (2.3) we obtain
\[ |\tilde{f}((1 - \delta)e^{i\theta}) - \tilde{f}(e^{i\theta})| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\delta(2 - \delta)}{|e^{it} - (1 - \delta)e^{i\theta}|^2} (\tilde{f}(e^{it}) - \tilde{f}(e^{i\theta})) \right| \, dt \]
\[ = \frac{1}{2\pi} \left( \int_{|t - \theta| \leq \delta} \text{idem} + \int_{\delta < |t - \theta| \leq \pi} \text{idem} \right) \]
\[ \approx = \omega(\delta) + \delta \int_{\delta}^{\pi} \frac{\omega(u)}{u^2} du \leq \omega(\delta). \]

Next, we restrict \( \tilde{f} \) to the closed domain \( \overline{G} \) defined in Section 4. By virtue of (4.16) and [5, p. 3], for any \( n \in \mathbb{N} \) there exists a harmonic polynomial of the form (5.5) such that
\[ |\tilde{f}(\zeta) - h_n(\zeta)| \leq \omega(d_{n/1}(\zeta)), \quad \zeta \in L'. \]
Therefore, (4.12) yields \( f \in B_\omega(E) \), i.e.,
\[ C_\omega(E) \subset B_\omega(E). \quad (5.6) \]
Comparing (5.3) and (5.6) we have (2.6). This completes the proof of Theorem 2.

6. Approximation on sparse sets; proof of Theorem 3

Proof of (2.8). For \( 0 < \alpha < 1 \) and \( x \in E_0 \), consider the function
\[ f_\alpha^*(x) := \begin{cases} (-1)^m 2^{-km} & \text{if } x \in \bigcup_{k=0}^{\infty} \bigcup_{m=1}^{k} I_{k,m}, \\ 0 & \text{otherwise.} \end{cases} \]
Since \( f_\alpha^* \in C^{*\alpha}(E_0) \), to complete the proof it is sufficient to show that \( f_\alpha^* \not\in A^\alpha(E_0) \).
Assume, to the contrary, that \( f_\alpha^* \in A^\alpha(E_0) \), i.e., for any \( n \in \mathbb{N} \) there exists \( t_n \in \Pi_n \) such that
\[ |f_\alpha^*(x) - t_n(x)| \leq C_1 n^{-\alpha}, \quad x \in E_0 \quad (6.1) \]
with \( C_1 = C_1(\alpha) \).
For sufficiently large \( n \in \mathbb{N} \), we find \( k \in \mathbb{N} \) with the property
\[ 2^k < \frac{n}{C_1^{1/\alpha}} \leq 2^{k+1}. \]
Since the trigonometric polynomial \( t_n \) (of degree \( \leq C_1^{1/\alpha} 2^{k+1} \)) changes its sign between intervals \( I_{k,m} \) and \( I_{k,m+1} \), it has at least \( 3^k - 1 \) zeros. This means that \( t_n \equiv 0 \), which contradicts (6.1). \( \square \)

Next, we establish some metric properties of \( E_0 \) which we need in order to prove (2.9).
First, we claim that
\[ |I_{k,m}| \leq |J_{k,m}| \asymp |J_{k,m-1}|. \quad (6.2) \]
Indeed, without loss of generality we assume that \( k > 0 \) is sufficiently large and consider the families of cross-cuts
\[ \Gamma = \Gamma(x'_{k,m}, x''_{k,m}; x''_{k,m-1}, \infty; \mathbb{H}), \quad \Gamma' = \phi_0(\Gamma). \]
By the left-hand side of (4.19),
\[ m(\Gamma) \geq \frac{1}{\pi} \log \frac{|J_{k,m}| + |I_{k,m}| + |J_{k,m-1}|}{|I_{k,m}|}. \quad (6.3) \]
Meanwhile,
\[ m(\Gamma') \leq 1. \quad (6.4) \]
Indeed, consider the metric
\[
\rho(x + iy) = \begin{cases} 
2 \cdot 6^k & \text{if } u_{k,m} \leq x \leq u_{k,m-2}, \\
 u_{k,m} - 2^{-1} \cdot 6^{-k} \leq y \leq u_{k,m-2} + 2^{-1} \cdot 6^{-k}, \\
0 & \text{otherwise},
\end{cases}
\]
where we set
\[
u_{k,-1} := u_{k-1,3^{k-1}-1}.
\]
Since for any \(\gamma \in \Gamma'\),
\[
\gamma \cap [(1 + i)u_{k,m}, (1 + i)u_{k,m-2}] \neq \emptyset,
\]
we obtain
\[
\inf_{\gamma' \in \Gamma'} \int_{\gamma} \rho(z) |dz| \geq 1.
\]
Therefore, by the definition of the module,
\[
m(\Gamma') \leq \int \int \rho(x + iy)^2 dx dy = 4 \cdot 3^k (u_{k,m-2} - u_{k,m})(u_{k,m-2} - u_{k,m} + 6^{-k}) \leq 1,
\]
which implies (6.4).
Comparing (6.3) and (6.4) we have
\[
\frac{|J_{k,m}| + |I_{k,m}| + |J_{k,m-1}|}{|J_{k,m}|} \leq 1. \tag{6.5}
\]
The same calculation, involving the families of cross-cuts
\[
\Gamma = \Gamma(x'_{k,m-1}, x'_{k,m}; x'_{k,m}, \infty; \mathbb{H}), \quad \Gamma' = \phi_0(\Gamma),
\]
implies
\[
\frac{|J_{k,m}| + |I_{k,m}| + |J_{k,m-1}|}{|J_{k,m-1}|} \leq 1. \tag{6.6}
\]
The inequalities (6.5) and (6.6) yield (6.2).

Next, we claim that for \(n \geq 2^{k+2}\) and \(x \in I_{k,m}\),
\[
\rho_{1/n}(x) \leq |J_{k,m}| \exp \left( -\frac{\pi 3^k}{2} \right). \tag{6.7}
\]
Indeed, let \(z \in (E_0)_{1/n} = \phi_0^{-1}(\{\text{Im } w = 1/n\})\) satisfy \(|x - z| = \rho_{1/n}(x)\) and let \(u := \phi_0(x), w := \phi_0(z)\). Consider families of cross-cuts
\[
\Gamma = \Gamma(x, z; x'_{k,m}, \infty; \mathbb{H}), \quad \Gamma' = \phi_0(\Gamma),
\]
\[
\Gamma'_1 = \left\{ \gamma_y := [u_{k,m} + iy, u_{k,m-1} + iy] : \frac{1}{n} < y < 2^{-k-1} \right\}.
\]
According to the right-hand side of (4.19) and (4.20),
\[ m(\Gamma) \leq \frac{1}{\pi} \log \left( 1 + \frac{|J_{k,m}| + |I_{k,m}|}{\rho_{1/n}(x)} \right) + 2. \]  
(6.8)

On the other hand,
\[ m(\Gamma') \geq m(\Gamma'_1) = \frac{2^{-k-1} - \frac{n}{2^{k-1}}}{3^{-k}} \geq \frac{3^k}{2}. \]  
(6.9)

Therefore, (6.2), (6.8) and (6.9) imply (6.7).

**Proof of (2.9).** For 0 < \( \alpha < 1 \) and \( x \in E_0 \), consider the function
\[ f_\alpha(x) := \begin{cases} (-1)^m |J_{k,m-1}|^\alpha & \text{if } x \in \bigcup_{k=0}^\infty \cup_{m=1}^{3^k} I_{k,m}, \\ 0 & \text{otherwise} \end{cases} \]

By virtue of (6.2), \( f_\alpha \in C^\alpha(E_0) \). In order to prove (2.9) it is sufficient to show that \( f_\alpha \notin B^\alpha(E_0) \).

We assume, to the contrary, that \( f_\alpha \in B^\alpha(E_0) \), i.e., for any \( n \in \mathbb{N} \) there exists \( t_n \in \Pi_n \) such that
\[ |f_\alpha(x) - t_n(x)| \leq \rho_{1/n}(x)^\alpha, \quad x \in E_0. \]  
(6.10)

Let \( n = 2^{k+2} \) be sufficiently large. Since by (6.7) and (6.10) the polynomial \( t_n \) changes its sign between \( I_{k,m} \) and \( I_{k,m-1} \), it has at least \( 3^k - 1 \) zeros. Hence, \( t_n \equiv 0 \), which contradicts (6.10). \( \square \)

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**References**