Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 154 (2007) 1157–1169
Topology and its Applications
www.elsevier.com/locate/topol

Observations on quasi-uniform products

Şenol Dost [a,1], Hans-Peter A. Künzi [b,*,2]

[a] *Hacettepe University, Department of Secondary Science and Mathematics Education, Beytepe, Ankara, Turkey*
[b] *Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, South Africa*

Received 21 March 2006; received in revised form 14 November 2006; accepted 22 November 2006

Abstract

We prove that any product of quotient maps in the category of quasi-uniform spaces and quasi-uniformly continuous maps is a quotient map. We also show that a quasi-uniformly continuous map from a product of quasi-uniform spaces into a quasi-pseudometric T_0-space depends on countably many coordinates.

Furthermore we characterize those quasi-uniformities that are unique in their quasi-proximity class and prove that this property is preserved under arbitrary products in the category of quasi-uniform spaces.

MSC: 54E15; 54E05; 54B10; 54B15

Keywords: Quasi-uniformity; Quotient map; Quasi-proximity class; Unique quasi-uniformity; Quasi-proximally unique; Proximally unique; Totally bounded; Hereditarily precompact; Quasi-uniform product; Depending on countably many coordinates

1. Introduction

In [8] Hušek and Rice proved that any product of quotient maps in the category of uniform spaces and uniformly continuous maps is a quotient map. In this note we first show that the corresponding result also holds in the category of quasi-uniform spaces and quasi-uniformly continuous maps. Generalizing another result from the symmetric to the asymmetric setting, we then prove that each quasi-uniformly continuous map from a product of quasi-uniform spaces into a quasi-pseudometric T_0-space depends on countably many coordinates. This can be considered a variant of a result of Vidossich [19] about uniform and metric spaces.

In the main part of the article we shall deal with quasi-uniformities that are unique in their quasi-proximity class. We characterize this property and use the characterization to prove that the property is preserved under arbitrary products. This result should be compared with a result due to Isbell [10] and Hušek [7] who proved that the product of an arbitrary family of totally bounded proximally fine uniformities is (totally bounded and) proximally fine. It is

* Corresponding author.
E-mail addresses: dost@hacettepe.edu.tr (Ş. Dost), hans-peter.kunzi@uct.ac.za (H.-P.A. Künzi).

[1] The first author acknowledges post-doctoral support from the National Research Foundation of South Africa during his stay at the University of Cape Town in 2005.
[2] The second author thanks for partial financial research support under NRF-Grant 2053741.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.11.006

worth mentioning that our proof seems not to generalize the known techniques from the symmetric to the asymmetric setting. Instead we obtain the theorem from the truly asymmetric result that if for each $i \in I$ the quasi-proximity class of the totally bounded quasi-uniformity $\mathcal{U}_i$ contains only hereditarily precompact quasi-uniformities, then the quasi-proximity class of the product quasi-uniformity $\prod_{i\in I} \mathcal{U}_i$ contains only hereditarily precompact quasi-uniformities.

We shall assume that the reader is familiar with the basic theory and notation from the area of quasi-uniformities (see [4]). In particular given a quasi-uniformity $\mathcal{U}$, $\mathcal{U}_\omega$ will denote the finest totally bounded quasi-uniformity coarser than $\mathcal{U}$, and $\mathcal{U}^s$ the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$.

Throughout we shall use the well-known fact that $(\mathcal{U}_\omega)^{-1} = (\mathcal{U}^{-1})_\omega$ [4, Corollary 1.40].

We also recall that if $\mathcal{U}$ and $\mathcal{V}$ are two quasi-uniformities on a set X, then $(\mathcal{U} \vee \mathcal{V})_\omega = \mathcal{U} \vee \mathcal{V}_\omega$ provided that $\mathcal{U}$ is totally bounded (compare [4, Proposition 1.40]). Given a quasi-uniformity $\mathcal{U}$ on a set X, by $\ll_{\delta_\mathcal{U}}$ we shall denote the strong inclusion of the quasi-proximity $\delta_\mathcal{U}$ induced by the quasi-uniformity $\mathcal{U}$ on X. A quasi-uniform space $(X, \mathcal{U})$ is called *precompact* if for each entourage $U \in \mathcal{U}$ there is a finite subset F of X such that $U(F) = X$. It is called *hereditarily precompact* if each of its subspaces is precompact. As usual, we say that a quasi-uniform space $(X, \mathcal{U})$ is *totally bounded* if the uniformity $\mathcal{U}^s$ is precompact.

2. Products of quasi-uniform quotient maps

Qunif will denote the category of quasi-uniform spaces and quasi-uniformly continuous maps and we shall often use the following convention concerning notation (besides the one used in the book of Fletcher and Lindgren [4]). For a quasi-uniform space X the quasi-uniformity of the space X will be denoted by $\mathcal{U}_X$. Furthermore, for a quasi-uniform space X, DX will denote the underlying set of X equipped with the discrete uniformity.

As usual, an onto quasi-uniformly continuous map $q : X \to Y$ between quasi-uniform spaces X and Y is called *quotient* if the facts that $h : Y \to Z$ is any map into a quasi-uniform space Z and $h \circ q$ is quasi-uniformly continuous imply that h is quasi-uniformly continuous.

It is known that given a quasi-uniformly continuous surjection $f : X \to Y$ between quasi-uniform spaces X and Y, f is quotient if and only if Y carries the finest quasi-uniformity that makes f quasi-uniformly continuous (compare [1,5]).

In [8] Hušek and Rice generalize to infinite products the result of Isbell [9, Exercise 8, p. 53] that a finite product of quotient mappings between uniform spaces is quotient. The proof of Theorem 2.1 below imitates the argument developed in [8] for the category of uniform spaces and uniformly continuous maps, with the essential difference that we have to use entourages instead of uniform coverings (compare also [6]).

Lemma 2.1. *Let X and Y be two quasi-uniform spaces. Then the product quasi-uniformity $\mathcal{U}_{X\times Y}$ is the finest quasi-uniformity that is coarser than both $\mathcal{U}_{X\times DY}$ and $\mathcal{U}_{DX\times Y}$.*

Proof. In the notation of lattice theory, we have to show that $\mathcal{U}_{DX\times Y} \wedge \mathcal{U}_{X\times DY} = \mathcal{U}_{X\times Y}$. To prove the nontrivial inclusion, let $W, Z \in \mathcal{U}_{DX\times Y} \wedge \mathcal{U}_{X\times DY}$ be such that $W^2 \subseteq Z$. Then there are $U \in \mathcal{U}_X$ and $V \in \mathcal{U}_Y$ such that $U(x) \times \{y\} \subseteq W(x, y)$ and $\{x\} \times V(y) \subseteq W(x, y)$ whenever $x \in X$ and $y \in Y$. Fix $x \in X$ and $y \in Y$. Consider any $(a, b) \in U(x) \times V(y)$. Then $(a, b) \in U(x) \times \{b\} \subseteq W(x, b)$ and $(x, b) \in \{x\} \times V(y) \subseteq W(x, y)$. Thus $(a, b) \in W^2(x, y) \subseteq Z(x, y)$. Therefore $U(x) \times V(y) \subseteq Z(x, y)$ whenever $x \in X$ and $y \in Y$, and consequently $Z \in \mathcal{U}_{X\times Y}$. The stated equality has been established. □

Lemma 2.2. *Let $f : X \to Y$ be a quotient map between quasi-uniform spaces X and Y. Furthermore let $g : D \to E$ be any surjection, where D and E are uniform spaces carrying the discrete uniformity. Then $f \times g : X \times D \to Y \times E$ is a quotient map between quasi-uniform spaces.*

Proof. Let Z be a quasi-uniform space and let $h : Y \times E \to Z$ be any map between the underlying sets of $Y \times E$ and Z such that $h \circ (f \times g)$ is quasi-uniformly continuous.

So let $(Z_n)_{n\in\omega}$ be a sequence of entourages belonging to $\mathcal{U}_Z$ such that $Z_{n+1}^2 \subseteq Z_n$ whenever $n \in \omega$. For each $n \in \omega$, set $W_n = (h \times h)^{-1} Z_n$.

Then for each $n \in \omega$, W_n is reflexive and we have $W_{n+1}^2 \subseteq W_n$. Furthermore $((f \times g) \times (f \times g))^{-1} W_n \in \mathcal{U}_{X\times D}$ whenever $n \in \omega$, since $h \circ (f \times g)$ is quasi-uniformly continuous.

For each $n \in \omega$ set $V_n = \{(y_1, y_2) \in Y \times Y\colon ((y_1, e), (y_2, e)) \in W_n$ whenever $e \in E\}$.

Then for each $n \in \omega$, V_n is reflexive and $V_{n+1}^2 \subseteq V_n$ whenever $n \in \omega$: Fix $n \in \omega$. Suppose that $y \in Y$. For any $e \in E$ we have $((y, e), (y, e)) \in W_n$. So V_n is indeed reflexive. Suppose that $((y_1, e), (y_2, e)), ((y_2, e), (y_3, e)) \in W_{n+1}$ whenever $e \in E$; then $((y_1, e), (y_3, e)) \in W_n$ whenever $e \in E$. Thus $V_{n+1}^2 \subseteq V_n$.

Observe also that if each $V_n \in \mathcal{U}_Y$, then for all $n \in \omega$, $W_n \in \mathcal{U}_{Y\times E}$, since by the definition of V_n, $\{((y_1, e), (y_2, e))\colon (y_1, y_2) \in V_n,\ e \in E\} \in \mathcal{U}_{Y\times E}$ and $\{((y_1, e), (y_2, e))\colon (y_1, y_2) \in V_n,\ e \in E\} \subseteq W_n$.

So we shall conclude that h is quasi-uniformly continuous, and thus $f \times g$ is a quotient map, if we can show that $V_n \in \mathcal{U}_Y$ whenever $n \in \omega$.

In order to see that each V_n belongs to $\mathcal{U}_Y$, we only need to show that $(f \times f)^{-1}(V_n) \in \mathcal{U}_X$ whenever $n \in \omega$, because f is a quotient map.

Fix $n \in \omega$. Since $((f \times g) \times (f \times g))^{-1} W_n \in \mathcal{U}_{X\times D}$, there is $M_n \in \mathcal{U}_X$ such that $\{((f(m_1), g(d)), (f(m_2), g(d)))\colon (m_1, m_2) \in M_n$ and $d \in D\} \subseteq W_n$. Let $(m_1, m_2) \in M_n$. Consider any $e \in E$. Since g is surjective, there exists $d \in D$ such that $g(d) = e$. We deduce that $(f(m_1), f(m_2)) \in V_n$ by definition of V_n and hence $M_n \subseteq (f \times f)^{-1} V_n$.

So we finally conclude that each V_n belongs to $\mathcal{U}_Y$ and hence h is quasi-uniformly continuous. □

Lemma 2.3. *The product $f_1 \times f_2$ of two quotient maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ between quasi-uniform spaces X_1 and Y_1, and X_2 and Y_2, respectively, is a quotient map.*

Proof. Assume that $h \circ (f_1 \times f_2) : X_1 \times X_2 \to Z$ is quasi-uniformly continuous for a map $h : Y_1 \times Y_2 \to Z$ into an arbitrary quasi-uniform space Z. By Lemma 2.2, $f_1 \times f_2 : X_1 \times DX_2 \to Y_1 \times DY_2$ and $f_1 \times f_2 : DX_1 \times X_2 \to DY_1 \times Y_2$ are quotient maps. Therefore, considering the quasi-uniformly continuous maps $h \circ (f_1 \times f_2) : X_1 \times DX_2 \to Z$ and $h \circ (f_1 \times f_2) : DX_1 \times X_2 \to Z$, we conclude that $h : Y_1 \times DY_2 \to Z$ and $h : DY_1 \times Y_2 \to Z$ are quasi-uniformly continuous. Hence by Lemma 2.1, $h : Y_1 \times Y_2 \to Z$ is quasi-uniformly continuous. □

Theorem 2.1. *The product $\prod_I f_i$ of any family $(f_i)_{i\in I}$ of quotient maps $f_i : X_i \to Y_i$ in* **Qunif** *(where $i \in I$) is a quotient map.*

Proof. From Lemma 2.3 one obtains the result for finite I by induction.

For the case of infinite I, assume that $k = h \circ (\prod_I f_i) : \prod_I X_i \to Z$ is quasi-uniformly continuous where $h : \prod_I Y_i \to Z$ is any map into a quasi-uniform space Z.

Let $R \in \mathcal{U}_Z$. Because k is quasi-uniformly continuous, there exists a finite set $J \subseteq I$ such that if $x, y \in \prod_I X_i$ and $\pi_J^X(x) = \pi_J^X(y)$, then $(k(x), k(y)) \in R$. Here $\pi_J^X : \prod_I X_i \to \prod_J X_i$ denotes the obvious projection. Similarly we define $\pi_J^Y : \prod_I Y_i \to \prod_J Y_i$.

Suppose that a is a (fixed) point in $\prod_I X_i$. Let $b = (\prod_I f_i)(a)$. Define $e_a : \prod_J X_i \to \prod_I X_i$ by

$$\big(e_a(p)\big)_i = a_i \quad \text{if } i \notin J,$$
$$\big(e_a(p)\big)_j = p_j \quad \text{if } j \in J.$$

Similarly, define $e_b : \prod_J Y_i \to \prod_I Y_i$ by

$$\big(e_b(q)\big)_i = b_i \quad \text{if } i \notin J,$$
$$\big(e_b(q)\big)_j = q_j \quad \text{if } j \in J.$$

Note that e_a and e_b are quasi-uniformly continuous and that $(\prod_I f_i) \circ e_a = e_b \circ (\prod_J f_i)$.

By the first observation in this proof, $\prod_J f_i : \prod_J X_i \to \prod_J Y_i$ is a quotient map. Since the map $h \circ (\prod_I f_i) \circ e_a = (h \circ e_b) \circ \prod_J f_i$ is quasi-uniformly continuous, it then follows that $h \circ e_b$, and hence $h_e := h \circ e_b \circ \pi_J^Y : \prod_I Y_i \to Z$ are quasi-uniformly continuous.

Since h_e is quasi-uniformly continuous, there is $V \in \mathcal{U}_{\prod_I Y_i}$ such that $(h_e(p), h_e(q)) \in R$ whenever $(p, q) \in V$.

Consider now any $(p, q) \in V$. Choose $p', q' \in \prod_I X_i$ such that $(\prod_I f_i)(p') = p$ and $(\prod_I f_i)(q') = q$. We want to show that $(h(p), h(q)) \in R^3$.

First observe that $(h(p), h_e(p)) = ((h \circ (\prod_I f_i))(p'), (h \circ e_b \circ \pi_J^Y)(p)) = (k(p'), (k \circ e_a \circ \prod_J^X)(p')) \in R$ by the continuity property of k mentioned above and since $(\prod_I f_i) \circ e_a \circ \pi_J^X = e_b \circ \pi_J^Y \circ (\prod_I f_i)$.

We already know that $(h_e(p), h_e(q)) \in R$.

Finally, $(h_e(q), h(q)) = ((h \circ e_b \circ \pi_J^Y)(q), (h \circ (\prod_I f_i))(q')) = ((k \circ e_a \circ \prod_J^X)(q'), k(q')) \in R$.

Consequently, $(h(p), h(q)) \in R^3$ and we conclude that h is quasi-uniformly continuous. Hence we are done. □

3. Quasi-uniformly continuous maps on quasi-uniform products

Let us recall that a (real-valued) T_0-*quasi-pseudometric* d on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(1) $d(x, x) = 0$;
(2) $d(x, z) \leqslant d(x, y) + d(y, z)$;
(3) $d(x, y) = d(y, x) = 0$ implies that $x = y$.

The following result and its proof should be compared with a corresponding result of Vidossich [19, Theorem] about uniformly continuous maps from a subspace of a product of uniform spaces into a metric space.

Theorem 3.1. *Let Z be any subset of the product of an arbitrary family $(X_i)_{i\in I}$ of quasi-uniform spaces. Then every quasi-uniformly continuous map $f : Z \to (Y, d)$ where (Y, d) is a T_0-quasi-pseudometric space has the form $g \circ (\pi|_Z)$ with $\pi : \prod_{i\in I} X_i \to \prod_{i\in C} X_i$ being the projection with some countable $C \subseteq I$, and $g : \pi(Z) \to Y$ being a quasi-uniformly continuous map.* (*One says that f depends on the countably many coordinates in C* (*compare* [7]).)

Proof. By quasi-uniform continuity of f, for each $n \in \omega$ there are a finite subset I_n of I and entourages U_i of $\mathcal{U}_{X_i}$ where $i \in I_n$, such that

$$d\big(f(x), f(y)\big) < \frac{1}{n+1}$$

whenever $(x, y) \in [\bigcap_{i\in I_n} (\pi_i \times \pi_i)^{-1} U_i] \cap Z$. Put $C = \bigcup_{n\in\omega} I_n$ and let π be the projection $(x_i)_{i\in I} \mapsto (x_i)_{i\in C}$.

For every $x \in \pi(Z)$, let z_x be a point of $Z \cap \pi^{-1}(\{x\})$. Define the map $g : \pi(Z) \to Y$ by $x \mapsto f(z_x)$.

If $z', z'' \in Z$ have the same image under π, then $d(f(z'), f(z'')) < \frac{1}{n+1}$ whenever $n \in \omega$, since each $I_n \subseteq C$. This implies by symmetry that $d(f(z'), f(z'')) = 0 = d(f(z''), f(z'))$, that is, $f(z') = f(z'')$. Therefore g is well defined.

From the definition of g it follows that $f = g \circ (\pi|_Z)$.

Let $n \in \omega$. Suppose that $z, z' \in Z$ such that $(\pi(z)_i, \pi(z')_i) \in U_i$ whenever $i \in I_n$. Then $d(g\pi(z), g\pi(z')) = d(f(z), f(z')) < \frac{1}{n+1}$ by assumption. Thus g is quasi-uniformly continuous. □

The following question is motivated by the result of Hušek (see [7, Proposition 1]) who proved that a proximally continuous map $f : \prod_{i\in I} X_i \to Y$ from the (whole) product of a family $(X_i)_{i\in I}$ of uniform spaces into a metric space Y depends on countably many coordinates.

Problem 3.1. Let $f : \prod_{i\in I} X_i \to Y$ be a quasi-proximally continuous map from the product of a family $(X_i)_{i\in I}$ of quasi-uniform spaces into a T_0- quasi-pseudometric space Y. Does f depend on countably many coordinates (compare [7])?

4. Quasi-proximally unique quasi-uniformities

In this section we shall make use of the following terminology.

Definition 4.1. A quasi-uniformity is called *quasi-proximally fine* if it is the finest quasi-uniformity in its quasi-proximity class.

A quasi-uniformity is called *quasi-proximally unique* if it is unique in its quasi-proximity class.

A uniformity is called *proximally fine* if it is the finest uniformity in its proximity class.

A uniformity is called *proximally unique* if it is unique in its proximity class.

Note that if a quasi-uniformity $\mathcal{U}$ is quasi-proximally fine, then $\mathcal{U}^{-1}$ is quasi-proximally fine, too.

Similarly, if a quasi-uniformity $\mathcal{U}$ is quasi-proximally unique, then $\mathcal{U}^{-1}$ is quasi-proximally unique, too.

Observe also that each quasi-proximally unique quasi-uniformity and each proximally unique uniformity is totally bounded.

We illustrate these concepts by some examples. The following example is well known. In fact it is based on old results due to Efremovič [2] and Smirnov [18] about proximity spaces.

Example 4.1. Let $\mathcal{U}$ be a totally bounded uniformity with a countable base. Then $\mathcal{U}$ is proximally unique.

Proof. Recall that a uniformity having a countable base is proximally fine (see e.g. [4, Corollary 1.59]). Since $\mathcal{U}$ is totally bounded, it is also the coarsest member of its uniformity class (see e.g. [4, Corollary 1.34]). Hence $\mathcal{U}$ is proximally unique. □

We recall that Losonczi [15] has shown that the coarsest compatible (totally bounded) quasi-uniformity $\mathcal{Q}_0$ of a locally compact Hausdorff space X is quasi-proximally unique if and only if X is compact or non-Lindelöf. Furthermore he noted that the quasi-proximity class of $\mathcal{Q}_0$ always has a quasi-proximally fine member.

It is well known (see [17, p. 129], or e.g. [4, Theorem 1.46]) that a locally compact Hausdorff space X also admits a coarsest uniformity $\mathcal{U}_0$.

Note that $\mathcal{U}_0 = (\mathcal{Q}_0)^s$, since by definition $\mathcal{Q}_0 \subseteq \mathcal{U}_0$ and so $(\mathcal{Q}_0)^s$ is a compatible uniformity on X. It has been proved that $\mathcal{U}_0 = \mathcal{Q}_0$ if and only if X is compact (see e.g. [4, Proposition 1.47]).

Example 4.2. Consider the coarsest compatible uniformity $\mathcal{U}_0$ and the coarsest compatible quasi-uniformity $\mathcal{Q}_0$ on the countably infinite discrete topological space with underlying set ω.

Note that $\mathcal{Q}_0$ is generated by the subbasic entourages

$$\big[\big(\omega \setminus \{n\}\big) \times \omega\big] \cup \big[\omega \times \{n\}\big]$$

where $n \in \omega$.

Since $\mathcal{U}_0$ has a countable base, it is proximally unique by Example 4.1. However the quasi-uniformity generated on ω by the subbase $\{\leqslant\} \cup \mathcal{U}_0$, where $\leqslant$ denotes the usual order relation on ω, is strictly finer than $\mathcal{U}_0$, but clearly induces the same (quasi-)proximity as $\mathcal{U}_0$. Hence $\mathcal{U}_0$ is not quasi-proximally unique.

Note that the quasi-proximity class of $\mathcal{U}_0$ does not contain a finest member $\mathcal{F}$: Suppose otherwise. Then $\mathcal{F}_\omega = \mathcal{U}_0 = (\mathcal{F}^{-1})_\omega$, since $\mathcal{U}_0$ is a uniformity, and therefore $\mathcal{F} = \mathcal{F}^{-1}$, since $\mathcal{F}$ is quasi-proximally fine. Hence $\mathcal{F}$ is a uniformity and would agree with $\mathcal{U}_0$, because $\mathcal{U}_0$ is proximally unique—a contradiction to the fact that $\mathcal{U}_0$ is not quasi-proximally fine. Indeed our argument shows that the quasi-proximity class of a proximally unique uniformity that is not quasi-proximally unique does not have a quasi-proximally fine member.

Observe however that $\mathcal{Q}_0$ has a finest member $\mathcal{E}$ in its quasi-proximity class according to Losonczi. It can be described as the conjugate of the finest compatible quasi-uniformity of the cofinite topology on the set ω. Since the latter topology is hereditarily compact, it admits a unique compatible totally bounded quasi-uniformity [4, Theorem 2.36], namely $(\mathcal{Q}_0)^{-1}$, which is equal to the Pervin quasi-uniformity of the cofinite topology on ω.

For the following investigations it is also interesting to note that all quasi-uniformities in the quasi-proximity class of $(\mathcal{Q}_0)^{-1}$ are hereditarily precompact, because their induced topologies are hereditarily compact, while $\mathcal{E}$ is a member of the quasi-proximity class of $\mathcal{Q}_0$ that is not hereditarily precompact, since it contains the entourage $\geqslant$.

Proposition 4.1. *Let $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ be a quasi-proximally continuous surjection between quasi-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ and suppose that $\mathcal{U}$ is quasi-proximally unique. Then $\mathcal{V}$ is quasi-proximally unique.*

Proof. Let $\mathcal{V}'$ be any quasi-uniformity in the quasi-proximity class of $\mathcal{V}$. Since f is quasi-proximally continuous, there is a quasi-uniformity $\mathcal{U}'$ in the quasi-proximity class of $\mathcal{U}$ such that $f:(X,\mathcal{U}') \to (Y,\mathcal{V}')$ is quasi-uniformly continuous (see [4, Proposition 1.55]). Then $\mathcal{U}' = \mathcal{U}$, since $\mathcal{U}$ is quasi-proximally unique, so that $f:(X,\mathcal{U}) \to (Y,\mathcal{V}')$ is quasi-uniformly continuous. Because $\mathcal{U}$ is totally bounded and f quasi-uniformly continuous, we conclude that $\mathcal{V}'$ is totally bounded, too. Hence $\mathcal{V}$ is quasi-proximally unique. □

Corollary 4.1. *Suppose that $\mathcal{U}$ is a totally bounded quasi-uniformity on a set X such that $\mathcal{U}^s$ is quasi-proximally unique. Then $\mathcal{U}$ is quasi-proximally unique.*

Proof. Consider the identity map $i:(X,\mathcal{U}^s)\to(X,\mathcal{U})$ and note that i is quasi-uniformly continuous. Apply now Proposition 4.1. □

The converse of Corollary 4.1 does not hold, as Example 4.3 will show.

Proposition 4.2. *Let $\mathcal{U}$ be a totally bounded quasi-uniformity on a set X such that the topology $\tau(\mathcal{U}^s)$ is pseudocompact. Then $\mathcal{U}$ is quasi-proximally unique.*

Proof. Let $\mathcal{V}$ be a quasi-uniformity on X such that $\mathcal{V}_\omega=\mathcal{U}$. Since then $\tau(\mathcal{U})=\tau(\mathcal{V})$ and $\tau(\mathcal{U}^{-1})=\tau(\mathcal{V}^{-1})$, we have $\tau(\mathcal{U}^s)=\tau(\mathcal{V}^s)$. Because $\tau(\mathcal{U}^s)$ is pseudocompact, by [3, Problem 8.5.10] we conclude that the uniformity $\mathcal{V}^s$ and therefore the quasi-uniformity $\mathcal{V}$ are totally bounded. Thus $\mathcal{V}=\mathcal{U}$ and the quasi-uniformity $\mathcal{U}$ is quasi-proximally unique. □

Next we prove a technical proposition whose proof is rather involved. The difficult part is to verify that the constructed quasi-uniformity $\mathcal{V}$ belongs to the quasi-proximity class of the given quasi-uniformity $\mathcal{U}$. We should point out that the underlying construction becomes rather straightforward in the case that the quasi-uniformity $\mathcal{U}_\omega$ is transitive (compare Corollary 4.2 below).

Proposition 4.3. *The quasi-proximity class of a quasi-uniformity $\mathcal{U}$ on a set X contains a quasi-uniformity $\mathcal{V}$ that is not hereditarily precompact if and only if there are subsets P_n, P'_n of X with $n\in\omega$ and subsets P, P' of X such that*

(a) $P_n \ll_{\delta_\mathcal{U}} P'_n$ *whenever* $n\in\omega$,
(b) $P\ll_{\delta_\mathcal{U}} P'$,
(c) $P'\subseteq\bigcup_{n\in\omega}P_n$,
(d) *for each $n\in\omega$ there is a point $y_n\in P\cap P_n$ such that $y_n\notin P'_k$ whenever $k\in\omega$ and $k<n$.*

Proof. Suppose that $\mathcal{V}$ is a quasi-uniformity which is not hereditarily precompact and belongs to the quasi-proximity class of $\mathcal{U}$. Then there are $V\in\mathcal{V}$ and a sequence $(y_n)_{n\in\omega}$ of points in X such that $y_{n+1}\notin V^2(\{y_1,\dots,y_n\})$ whenever $n\in\omega$. For each $n\in\omega$ we put $P_n:=V(y_n)$ and $P'_n:=V^2(y_n)$. Furthermore we set $P:=\{y_n\colon n\in\omega\}$ and $P':=V(\{y_n\colon n\in\omega\})$. Consequently all conditions stated in the proposition are clearly satisfied for the defined sets.

For the converse, assume that for a given quasi-uniformity $\mathcal{U}$ on a set X there exist subsets P_n, P'_n, P, P' of X and points y_n with $n\in\omega$ as described in the proposition.

We want to construct a quasi-uniformity $\mathcal{V}$ on X that is not hereditarily precompact, but belongs to the quasi-proximity class of $\mathcal{U}$.

First we define the standard T_0-quasi-pseudometric m on the real unit interval $[0,1]$ as follows: $m(x,y)=x-y$ if $x\geqslant y$ and $m(x,y)=0$ otherwise. By $\mathcal{U}_m$ we shall denote the usual quasi-pseudometric quasi-uniformity induced by m on $[0,1]$. Recalling the well-known separation result of Urysohn type for quasi-proximities [4, Lemma 1.57], we see that for each $n\in\omega$ there exists a quasi-proximally continuous map $f_n:(X,\mathcal{U})\to([0,1],\mathcal{U}_m)$ such that $f_n(P_n)=1$ and $f_n(X\setminus P'_n)=0$. Similarly there is a quasi-proximally continuous map $f:(X,\mathcal{U})\to([0,1],\mathcal{U}_m)$ such that $f(P)=1$ and $f(X\setminus P')=0$.

For each $\varepsilon\in[\frac14,\frac34]$ and $n\in\omega$, set $Z_{n,\varepsilon}=f_n^{-1}[1-\varepsilon,1]$ and $Z_\varepsilon=f^{-1}[1-\varepsilon,1]$.

Moreover put $D_{n,\varepsilon}=Z_\varepsilon\cap\bigcup_{k\leqslant n}Z_{k,\varepsilon}$ whenever $n\in\omega$ and $\varepsilon\in[\frac14,\frac34]$.

Note that $\bigcup_{n\in\omega}D_{n,\varepsilon}=Z_\varepsilon\ll_{\delta_\mathcal{U}}Z_{\varepsilon'}=\bigcup_{n\in\omega}D_{n,\varepsilon'}$ whenever $\varepsilon,\varepsilon'\in[\frac14,\frac34]$ and $\varepsilon<\varepsilon'$, because f is quasi-proximally continuous and for each $\varepsilon\in[\frac14,\frac34]$, $Z_\varepsilon\subseteq P'\subseteq\bigcup_{n\in\omega}P_n\subseteq\bigcup_{n\in\omega}Z_{n,\varepsilon}$.

We also observe that for each $n\in\omega$, $y_n\in D_{n,\frac14}$, because $y_n\in P_n\cap P\subseteq Z_{n,\frac14}\cap Z_{\frac14}$, but $y_{n+1}\notin D_{n,\frac34}$, because $f_k(y_{n+1})=0$ and thus $y_{n+1}\notin Z_{k,\frac34}$ whenever $k\in\omega$ such that $k\leqslant n$.

Summarizing we remark that therefore for each $n\in\omega$ and $\varepsilon\in[\frac14,\frac34]$ there exists a subset $D_{n,\varepsilon}$ of X satisfying the following three conditions:

(i) $D_{n,\varepsilon} \subseteq D_{n+1,\varepsilon}$ and $D_{n,\varepsilon} \ll_{\delta_{\mathcal{U}}} D_{n,\varepsilon'}$ whenever $n \in \omega$ and $\varepsilon, \varepsilon' \in [\frac{1}{4}, \frac{3}{4}]$ with $\varepsilon < \varepsilon'$.
(ii) $\bigcup_{n\in\omega} D_{n,\varepsilon} \ll_{\delta_{\mathcal{U}}} \bigcup_{n\in\omega} D_{n,\varepsilon'}$ whenever $\varepsilon, \varepsilon' \in [\frac{1}{4}, \frac{3}{4}]$ with $\varepsilon < \varepsilon'$.
(iii) There is a sequence $(y_n)_{n\in\omega}$ of points in X such that for each $n \in \omega$, we have $y_n \in D_{n,\frac{1}{4}}$, but $y_{n+1} \notin D_{n,\frac{3}{4}}$.

For each $m \in \omega \setminus \{0\}$ we now set

$$W_m = \bigcap_{n\in\omega} \left(\bigcap_{\ell=0}^{2^{m-1}-1} H_{n,\ell,m} \right)$$

where

$$H_{n,\ell,m} = \left[(X \setminus D_{n,\frac{1}{4}+\ell 2^{-m}}) \times X \right] \cup [X \times D_{n,\frac{1}{4}+(\ell+1)2^{-m}}].$$

We see that $W_{m+1}^2 \subseteq W_m$ whenever $m \in \omega \setminus \{0\}$: For each $n \in \omega$, $m \in \omega \setminus \{0\}$ and $\ell = 0, \ldots, 2^{m-1} - 1$ it follows in the usual way that

$$(H_{n,2\ell,m+1} \cap H_{n,2\ell+1,m+1})^2 \subseteq H_{n,\ell,m}$$

(compare [4, Theorem 1.33]).

Hence $\{W_m\colon m \in \omega \setminus \{0\}\}$ is a (decreasing) base for a quasi-uniformity $\mathcal{W}$ on X.

Note that $\mathcal{W}$ is not hereditarily precompact, because $(y_k, y_n) \notin W_1 = \bigcap_{p\in\omega}([(X \setminus D_{p,\frac{1}{4}}) \times X] \cup [X \times D_{p,\frac{3}{4}}])$ whenever $n, k \in \omega$ and $n > k$.

We are now ready to start the proof that $\mathcal{W}_\omega \subseteq \mathcal{U}_\omega$.

Fix $m \in \omega \setminus \{0\}$. For convenience, also set $D_{\omega,\frac{1}{4}+\ell 2^{-m}} = X$ whenever $l = 0, \ldots, 2^{m-1}$. Next we define the concept of a type of a point $x \in X$ (relative to W_m).

For each $\ell = 0, \ldots, 2^{m-1} - 1$ let $n(\ell, x)$ be the minimal $n \in \omega$ such that $x \in D_{n,\frac{1}{4}+\ell 2^{-m}}$; set $n(\ell, x) = \omega$ if there is no such $n \in \omega$.

Let us observe that for each $x \in X$, the finite sequence $(n(\ell, x))_{\ell=0,\ldots,2^{m-1}-1}$ is non-increasing, since $D_{n,\varepsilon} \subseteq D_{n,\varepsilon'}$ whenever $n \in \omega$ and $\varepsilon, \varepsilon' \in [\frac{1}{4}, \frac{3}{4}]$ such that $\varepsilon < \varepsilon'$.

We shall call the sequence $(n(\ell, x))_{\ell=0,\ldots,2^{m-1}-1}$ the *type $t(x)$ of $x \in X$*.

Note that

$$x \in \bigcap_{\ell=0}^{2^{m-1}-1} D_{n(\ell,x),\frac{1}{4}+\ell 2^{-m}}.$$

In the light of condition (i) one also verifies that

$$W_m(x) = \bigcap_{\ell=0}^{2^{m-1}-1} D_{n(\ell,x),\frac{1}{4}+(\ell+1)2^{-m}}.$$

Consider now an arbitrary $C \subseteq X$. We want to show that $C \ll_{\delta_{\mathcal{U}}} W_m(C)$. The idea is to write C as the union of finitely many subsets each of which satisfies the latter condition; it then follows that C fulfils it. Those subsets are found using the types of the points of C.

Set $C_{2^{m-1}} = \{a \in C\colon n(\ell, a) = \omega$ whenever $\ell \leqslant 2^{m-1} - 1\}$.

For each $q \in \omega$ such that $q \leqslant 2^{m-1} - 1$ we set $C_q = \{a \in C\colon n(q, a)$ is the first term of $t(a)$ distinct from $\omega\}$. Obviously $C = \bigcup_{q=0}^{2^{m-1}} C_q$. Hence it will suffice to prove that $C_q \ll_{\delta_{\mathcal{U}}} W_m(C_q)$ whenever $q \in \{0, \ldots, 2^{m-1}\}$.

For the next part of the proof we need some additional notation.

The empty sequence is considered a finite sequence. If $n \in \omega$ and $s = (s_f, \ldots, s_{2^{m-1}-1})$ is a finite (possibly empty) sequence with terms s_ℓ in $\omega + 1$ having as its domain the finite interval $[f, 2^{m-1} - 1]$ (where $f \in \omega$) of ω and where the terms s_ℓ are listed in the order determined by increasing indices, we shall denote the sequence obtained from n and s (in this order) with the help of concatenation by $n\|s$, that is, $n\|s = (n, s_f, \ldots, s_{2^{m-1}-1})$. (In particular s is the empty sequence if $f > 2^{m-1} - 1$ so that the interval domain is empty.)

Fix $q \in \{0, \ldots, 2^{m-1}\}$. In a next step we want to write C_q as the union of finitely many subsets.

Let M be the set of the finite non-increasing sequences $s = (s_f, \dots, s_{2^{m-1}-1})$ with terms s_ℓ in $\omega + 1$ that satisfy simultaneously the following two conditions:

(1a) *for infinitely many $p \in \omega$ there is some point $a_p \in C_q$ such that $p \| s$ is a final segment of $t(a_p)$, or*
(1b) *$s = t(a)$ for some point $a \in C_q$; and*
(2) *there is no shorter (that is, the length of the interval domain is strictly smaller) final segment r of s such that for infinitely many $p \in \omega$ there is some point $a_p \in C_q$ such that $p \| r$ is a final segment of $t(a_p)$.*

Note next that the set M is finite:

Assume the contrary. Obviously each sequence $s = (s_f, \dots, s_{2^{m-1}-1})$ has at most 2^{m-1} terms. Thus our assumption that M is infinite implies that there is an index $\ell \leqslant 2^{m-1} - 1$ such that $\{s_\ell \colon s \in M\}$ is infinite. Without loss of generality we assume that ℓ is chosen maximal with respect to the latter property.

But then $E := \{(s_{\ell+1}, \dots, s_{2^{m-1}-1}) \colon s \in M\}$ is finite. (Recall that here $(s_{2^{m-1}}, \dots, s_{2^{m-1}-1})$ means the empty sequence.) Therefore there is $(t_{\ell+1}, \dots, t_{2^{m-1}-1}) \in E$ so that for infinitely many $s_\ell \in \omega$ there is a point $a(s_\ell) \in C_q$ such that $s_\ell \| (t_{\ell+1}, \dots, t_{2^{m-1}-1})$ is a final segment of $t(a(s_\ell))$. However by condition (2) used in the definition of M, we then see that the corresponding sequences s do not belong to M, because $(t_{\ell+1}, \dots, t_{2^{m-1}-1})$ is shorter than each such s. We have reached a contradiction and conclude that M is finite.

For each $s \in M$, set $C_{q,s} = \{a \in C_q \colon s \text{ is a final segment of } t(a)\}$.

We claim that $C_q = \bigcup_{s \in M} C_{q,s}$.

Indeed consider any $a \in C_q$. We distinguish two cases:

Case 1: There is a shortest (possibly empty) final segment s of $t(a)$ such that $\{p \in \omega \colon \text{there is } x \in C_q \text{ such that } p \| s$ is a final segment of $t(x)\}$ is infinite. Then by definition it follows that $s \in M$ and $a \in C_{q,s}$.

Case 2: There is no such final segment, in which case $t(a) \in M$ and $a \in C_{q,t(a)}$.

Hence we have verified that claim.

It remains to check that $C_{q,s} \ll_{\delta_{\mathcal{U}}} W_m(C_{q,s})$ whenever $s \in M$.

First let $s \in M$ be such that there is $a_0 \in C_q$ with $s = t(a_0)$ (see condition (1b) above). We have $C_{q,s} = \{a \in C_q \colon t(a) = t(a_0)\}$. Note that $C_{q,s} \subseteq \bigcap_{\ell=0}^{2^{m-1}-1} D_{n(\ell,a_0), \frac{1}{4}+\ell 2^{-m}}$. Furthermore $W_m(C_{q,s}) = \bigcap_{\ell=0}^{2^{m-1}-1} D_{n(\ell,a_0), \frac{1}{4}+(\ell+1)2^{-m}}$. Hence $C_{q,s} \ll_{\delta_{\mathcal{U}}} W_m(C_{q,s})$.

Consider now any $s = (s_{f+1}, \dots, s_{2^{m-1}-1}) \in M$ with fewer than 2^m terms (see condition (1a)). Since types are non-increasing, by condition (1a) used in the definition of M, we see that $q \leqslant f$ and all terms of s are distinct from ω. Observe that for any $a \in C_{q,s}$ and $\ell < q$, we have $D_{n(\ell,a), \frac{1}{4}+\ell 2^{-m}} = X$, because $n(\ell, a) = \omega$.

Then

$$C_{q,s} \subseteq \bigcup_{a \in C_{q,s}} \left[\left(\bigcap_{\ell=q}^{f} D_{n(\ell,a), \frac{1}{4}+\ell 2^{-m}} \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell, \frac{1}{4}+\ell 2^{-m}} \right) \right]$$
$$= \left(\bigcap_{\ell=q}^{f} \left(\bigcup_{n \in \omega} D_{n, \frac{1}{4}+\ell 2^{-m}} \right) \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell, \frac{1}{4}+\ell 2^{-m}} \right).$$

The nontrivial inclusion of the last equality, namely

$$\left(\bigcap_{\ell=q}^{f} \left(\bigcup_{n \in \omega} D_{n, \frac{1}{4}+\ell 2^{-m}} \right) \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell, \frac{1}{4}+\ell 2^{-m}} \right) \subseteq \bigcup_{a \in C_{q,s}} \left[\left(\bigcap_{\ell=q}^{f} D_{n(\ell,a), \frac{1}{4}+\ell 2^{-m}} \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell, \frac{1}{4}+\ell 2^{-m}} \right) \right],$$

is a consequence of the facts that by condition (1a) used in the definition of s the set $\{n(f, a) \colon a \in C_{q,s}\}$ is unbounded in ω, types are non-increasing and $D_{n,\varepsilon} \subseteq D_{n+1,\varepsilon}$ whenever $n \in \omega$ and $\varepsilon \in [\frac{1}{4}, \frac{3}{4}]$: Indeed for any sequence $(n_q, \dots, n_f)$ with terms in ω we can find $a \in C_{q,s}$ such that $\max\{n_q, \dots, n_f\} \leqslant n(f, a)$, and thus $n_\ell \leqslant n(\ell, a)$ whenever $\ell = q, \dots, f$.

Furthermore analogously we get

$$W_m(C_{q,s}) = \bigcup_{a\in C_{q,s}} \left[\left(\bigcap_{\ell=q}^{f} D_{n(\ell,a),\frac{1}{4}+(\ell+1)2^{-m}} \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell,\frac{1}{4}+(\ell+1)2^{-m}} \right) \right]$$
$$= \left(\bigcap_{\ell=q}^{f} \left(\bigcup_{n\in\omega} D_{n,\frac{1}{4}+(\ell+1)2^{-m}} \right) \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell,\frac{1}{4}+(\ell+1)2^{-m}} \right).$$

Observe finally that by conditions (i) and (ii) mentioned above

$$\left(\bigcap_{\ell=q}^{f} \left(\bigcup_{n\in\omega} D_{n,\frac{1}{4}+\ell 2^{-m}} \right) \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell,\frac{1}{4}+\ell 2^{-m}} \right)$$
$$\ll_{\delta_\mathcal{U}} \left(\bigcap_{\ell=q}^{f} \left(\bigcup_{n\in\omega} D_{n,\frac{1}{4}+(\ell+1)2^{-m}} \right) \right) \cap \left(\bigcap_{\ell=f+1}^{2^{m-1}-1} D_{s_\ell,\frac{1}{4}+(\ell+1)2^{-m}} \right).$$

It follows that $C_{q,s} \ll_{\delta_\mathcal{U}} W_m(C_{q,s})$.

Thus altogether we deduce that $C \ll_{\delta_\mathcal{U}} W_m(C)$. Hence $\mathcal{W}_\omega \subseteq \mathcal{U}_\omega$.

We conclude that the quasi-uniformity $\mathcal{V} := \mathcal{W} \vee \mathcal{U}_\omega$ is not hereditarily precompact, but belongs to the quasi-proximity class of $\mathcal{U}$, since $(\mathcal{W} \vee \mathcal{U}_\omega)_\omega = \mathcal{W}_\omega \vee \mathcal{U}_\omega = \mathcal{U}_\omega$. □

Lemma 4.1. *Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{U}$ is quasi-proximally unique if and only if both the quasi-proximity classes of $\mathcal{U}$ and of $\mathcal{U}^{-1}$ contain only hereditarily precompact quasi-uniformities.*

Proof. Note first that a quasi-uniformity is totally bounded if and only if its conjugate is totally bounded. Furthermore if a quasi-uniformity $\mathcal{V}$ belongs to the quasi-proximity class of a quasi-uniformity $\mathcal{U}$, then $\mathcal{V}^{-1}$ belongs to the quasi-proximity class of $\mathcal{U}^{-1}$.

Assume now that $\mathcal{U}$ is quasi-proximally unique. Then each member of the quasi-proximity class of $\mathcal{U}$ is totally bounded and thus hereditarily precompact. Hence also each member of the quasi-proximity class of $\mathcal{U}^{-1}$ is totally bounded and thus hereditarily precompact.

On the other hand, suppose that $\mathcal{U}$ is not quasi-proximally unique. Then the quasi-proximity class of $\mathcal{U}$ contains a member $\mathcal{V}$ that is not totally bounded. Therefore $\mathcal{V}$ or $\mathcal{V}^{-1}$ is not hereditarily precompact by [12, Lemma 1.1]. Hence the quasi-proximity class of $\mathcal{U}$ or that of $\mathcal{U}^{-1}$ contains a member that is not hereditarily precompact. □

The preceding lemma together with Proposition 4.3 can be considered a characterization of quasi-proximally unique quasi-uniformities. We next show that the latter proposition and the second part of its proof can be considerably simplified if $\mathcal{U}_\omega$ is transitive.

Corollary 4.2. *The quasi-proximity class of a transitive quasi-uniformity $\mathcal{U}$ on a set X contains a member $\mathcal{V}$ that is not hereditarily precompact if and only if there is a sequence $(D_n)_{n\in\omega}$ of subsets of X such that*

(i) *$(D_n)_{n\in\omega}$ is strictly increasing,*
(ii) *$D_n \ll_{\delta_\mathcal{U}} D_n$ whenever $n \in \omega$, and*
(iii) *$\bigcup_{n\in\omega} D_n \ll_{\delta_\mathcal{U}} \bigcup_{n\in\omega} D_n$.*

Proof. Let us first recall [4, Lemma 6.3] that $\mathcal{U}_\omega$ is transitive, since the quasi-uniformity $\mathcal{U}$ is transitive.

Suppose that the quasi-proximity class of $\mathcal{U}$ contains a (possibly non-transitive) quasi-uniformity $\mathcal{V}$ that is not hereditarily precompact. Then using transitivity of $\mathcal{U}_\omega$ and the statement and notation of Proposition 4.3, we can find sets Q_n ($n \in \omega$) and Q such that $Q_n \ll_{\delta_\mathcal{U}} Q_n$ whenever $n \in \omega$ and $Q \ll_{\delta_\mathcal{U}} Q$, satisfying $P_n \subseteq Q_n \subseteq P_n'$ and $P \subseteq Q \subseteq P'$. Set $D_n = Q \cap \bigcup_{k\leqslant n} Q_k$ whenever $n \in \omega$. Then $D_n \ll_{\delta_\mathcal{U}} D_n$ whenever $n \in \omega$. Furthermore $\bigcup_{n\in\omega} D_n \ll_{\delta_\mathcal{U}} \bigcup_{n\in\omega} D_n$, since $Q \subseteq P' \subseteq \bigcup_{n\in\omega} P_n \subseteq \bigcup_{n\in\omega} Q_n$ and thus $\bigcup_{n\in\omega} D_n = Q \cap \bigcup_{k\in\omega} Q_k = Q$.

Finally for each $n \in \omega$, $y_n \in P \cap P_n \subseteq Q \cap Q_n \subseteq D_n$ and $y_{n+1} \notin D_n$, because for each $k \leqslant n$, $y_{n+1} \notin Q_k$, so that $(D_n)_{n\in\omega}$ is strictly increasing. Therefore all conditions stated in the corollary are satisfied.

On the other hand, the existence of a sequence of subsets of X as described in Corollary 4.2 leads to a transitive quasi-uniformity $\mathcal{W}$ generated by the preorder $T := \bigcap_{n\in\omega}([(X \setminus D_n) \times X] \cup [X \times D_n])$ on X. (More precisely, $\{T\}$ is a base for $\mathcal{W}$.) Note that $T(x) = D_n$, where $n \in \omega$ is minimal such that $x \in D_n$, and $T(x) = X$ if there is no $n \in \omega$ such that $x \in D_n$. Considering as in the proof of Proposition 4.3 an arbitrary $C \subseteq X$, one shows now easily that $C \ll_{\delta_\mathcal{U}} T(C)$ by distinguishing three cases: (1) $C \subseteq D_n$ for some (minimal) $n \in \omega$, which yields $T(C) = D_n$, (2) $C \nsubseteq \bigcup_{n\in\omega} D_n$ which yields $T(C) = X$, and (3) otherwise, which yields $T(C) = \bigcup_{n\in\omega} D_n$. Thus $\mathcal{W}_\omega \subseteq \mathcal{U}_\omega$. Because of $(\mathcal{U}_\omega \vee \mathcal{W})_\omega = \mathcal{U}_\omega \vee \mathcal{W}_\omega = \mathcal{U}_\omega$, we conclude that the quasi-proximity class of $\mathcal{U}$ contains the (transitive) member $\mathcal{U}_\omega \vee \mathcal{W}$. That quasi-uniformity is clearly not hereditarily precompact, since T belongs to it. □

Corollary 4.3. *If the quasi-proximity class of a transitive quasi-uniformity contains a member that is not hereditarily precompact, then it contains a transitive quasi-uniformity that is not hereditarily precompact.*

Proof. The assertion is a consequence of the preceding argument. □

Given a transitive quasi-uniform space $(X,\mathcal{U})$ we set $\mathcal{B}_\mathcal{U} = \{G \subseteq X\colon G \ll_{\delta_\mathcal{U}} G\}$. Note that $\mathcal{B}_{\mathcal{U}^{-1}} = \{F \subseteq X\colon X \setminus F \in \mathcal{B}_\mathcal{U}\}$.

Corollary 4.4. *Let $\mathcal{U}$ be a transitive quasi-uniformity. Then the quasi-proximity class of $\mathcal{U}$ contains a unique quasi-uniformity if and only if $\mathcal{B}_\mathcal{U}$ satisfies the following two conditions*:

(1) *$\mathcal{B}_\mathcal{U}$ does not contain any strictly increasing sequence $(B_n)_{n\in\omega}$ such that $\bigcup_{n\in\omega} B_n \in \mathcal{B}_\mathcal{U}$, and*
(2) *$\mathcal{B}_\mathcal{U}$ does not contain any strictly decreasing sequence $(B_n)_{n\in\omega}$ such that $\bigcap_{n\in\omega} B_n \in \mathcal{B}_\mathcal{U}$.*

Proof. If $\mathcal{B}_\mathcal{U}$ does not satisfy condition (1) respectively condition (2), then the quasi-proximity class of $\mathcal{U}$ respectively of $\mathcal{U}^{-1}$ contains a (transitive) quasi-uniformity that is not hereditarily precompact by (the proof of) Corollary 4.2. Hence if $\mathcal{U}$ and thus $\mathcal{U}^{-1}$ are quasi-proximally unique, then both stated conditions are satisfied.

On the other hand, if $\mathcal{B}_\mathcal{U}$ fulfils both conditions, then by Corollary 4.2 any quasi-uniformity belonging to the quasi-proximity class of $\mathcal{U}$ respectively of $\mathcal{U}^{-1}$ is hereditarily precompact.

We conclude by Lemma 4.1 that $\mathcal{U}$ is quasi-proximally unique. □

It is known that a topological space admits a unique quasi-uniformity if and only if its Pervin quasi-proximity class contains a unique member [14]. Hence the preceding corollary generalizes the fact that a topological space X admits a unique quasi-uniformity if and only if X does neither possess a strictly increasing sequence $(G_n)_{n\in\omega}$ of open sets (that is, if X is hereditarily compact), nor a strictly decreasing sequence $(H_n)_{n\in\omega}$ of open sets with an open intersection (compare [11]).

Example 4.3. The supremum of two quasi-proximally unique conjugate quasi-uniformities need not be quasi-proximally unique.

Proof. Let $X = \omega$ and for each $n \in \omega$ set $A_n = \{2k + 1\colon k \leqslant n,\ k \in \omega\}$ and $B_n = \{2k\colon k \leqslant n,\ k \in \omega\}$. Put $\mathcal{B} = \{\emptyset, X\} \cup \{A_n\colon n \in \omega\} \cup \{X \setminus B_n\colon n \in \omega\}$. Note that the collection $\mathcal{B}$ is closed under finite intersections and finite unions. Let $\mathcal{U}$ be the transitive quasi-uniformity on X generated by the subbase $\{[(X \setminus G) \times X] \cup [X \times G]\colon G \in \mathcal{B}\}$. It is well known and easy to see, for instance also from Losonczi's theory of l-bases (see e.g. [16] or [13, p. 274]), that $\mathcal{B} = \{G \subseteq X\colon G \ll_{\delta_\mathcal{U}} G\} = \mathcal{B}_\mathcal{U}$.

One readily checks that $\mathcal{B}$ does neither possess any strictly increasing sequence $(G_n)_{n\in\omega}$ such that $\bigcup_{n\in\omega} G_n \in \mathcal{B}$, nor any strictly decreasing sequence $(H_n)_{n\in\omega}$ such that $\bigcap_{n\in\omega} H_n \in \mathcal{B}$.

By Corollary 4.4, $\mathcal{U}$ is quasi-proximally unique and thus $\mathcal{U}^{-1}$ is quasi-proximally unique. On the other hand, it is straightforward to check that for each $n \in \omega$ the collection of sets $[[0, n] \times [0, n]] \cup [(X \setminus [0, n]) \times (X \setminus [0, n])]$ yields a subbase for the uniformity $\mathcal{U}^s$. Hence $\mathcal{U}^s$ is the coarsest uniformity $\mathcal{U}_0$ on ω that induces the discrete topology. As we have observed in Example 4.2, that uniformity is not quasi-proximally unique. □

Isbell [10] and later Hušek [7] gave different arguments to show that the product of any family of proximally unique uniformities is proximally unique. Next we want to establish the analogue for quasi-proximally unique quasi-uniformities using our asymmetric approach via hereditary precompactness.

Definition 4.2. A subset $\prod_{i\in I} A_i$ of the product set $\prod_{i\in I} X_i$ is called a *box* if $A_i \subseteq X_i$ whenever $i \in I$. A box is called a *topological box* if $A_i = X_i$ for all but finitely many coordinates $i \in I$.

Note that the complement of any topological box $\prod_{i\in I} A_i$ in $\prod_{i\in I} X_i$ is the union of finitely many topological boxes. Indeed it is equal to $\bigcup_{i\in F} \pi_i^{-1}(X_i \setminus A_i)$ where $F = \{i \in I\colon\ A_i \neq X_i\}$ and $\pi_j : \prod_{i\in I} X_i \to X_j$ $(j \in F)$ is the projection map.

It is known that the product of a finite number of topological spaces which admit a unique quasi-uniformity also admits a unique quasi-uniformity [11]. One half of the proof of this result consists of verifying that the topological property of hereditary compactness is preserved under finite products. While for that result the restriction to a finite number of factor spaces is crucial, our next theorem does not need this additional condition. Let us note however that some steps in the arguments to establish the two results are quite similar.

Theorem 4.1. *Suppose that $(X_i)_{i\in I}$ is a nonempty family of (nonempty) totally bounded quasi-uniform spaces.*

Then the quasi-proximity class of $\mathcal{U}_{\prod_I X_i}$ contains only hereditarily precompact quasi-uniformities if and only if for each $i \in I$ the quasi-proximity class of $\mathcal{U}_{X_i}$ contains only hereditarily precompact quasi-uniformities.

Proof. Let $i_0 \in I$ and suppose that $\mathcal{V}$ is a quasi-uniformity in the quasi-proximity class of $\mathcal{U}_{X_{i_0}}$ that is not hereditarily precompact. Then the product quasi-uniformity $\prod_{i\in I} \mathcal{H}_{X_i}$ with $\mathcal{H}_{X_{i_0}} = \mathcal{V}$, and $\mathcal{H}_{X_j} = \mathcal{U}_{X_j}$ if $j \in I$ and $j \neq i_0$, is not hereditarily precompact, but belongs to the quasi-proximity class of $\mathcal{U}_{\prod_I X_i}$ according to [4, Proposition 1.53]. Hence we conclude that if the quasi-proximity class of $\mathcal{U}_{\prod_I X_i}$ contains only hereditarily precompact quasi-uniformities, then for each $i \in I$ the quasi-proximity class of $\mathcal{U}_{X_i}$ contains only hereditarily precompact quasi-uniformities.

For the converse suppose that for each $i \in I$ the quasi-proximity class of $\mathcal{U}_{X_i}$ contains only hereditarily precompact quasi-uniformities.

Assume that the statement does not hold and let $\mathcal{V}$ be a quasi-uniformity belonging to the quasi-proximity class of $\mathcal{U}_{\prod_I X_i}$ that is not hereditarily precompact.

Then there exist a sequence $(x_n)_{n\in\omega}$ in $\prod_{i\in I} X_i$ and $V \in \mathcal{V}$ such that $x_{n+1} \notin V^2(\{x_1, \ldots, x_n\})$ whenever $n \in \omega$.

Then thinking in terms of quasi-proximities, we see that for each $n \in \omega$ there is a finite set F_n such that

$$V(x_n) \subseteq \bigcup_{f\in F_n} \prod_{i\in I} A^f_{i,n} \subseteq \bigcup_{f\in F_n} \prod_{i\in I} B^f_{i,n} \subseteq V^2(x_n),$$

where for each $n \in \omega$, $i \in I$ and $f \in F_n$ we have $A^f_{i,n} \ll_{\delta_{\mathcal{U}_{X_i}}} B^f_{i,n}$. Furthermore for each $n \in \omega$ and $f \in F_n$, $\prod_{i\in I} A^f_{i,n}$ and $\prod_{i\in I} B^f_{i,n}$ are topological boxes.

Indeed this follows from the facts that for each $U \in \mathcal{U}_{\prod_I X_i}$ there are a finite set H and sets $P_{i_h}, Q_{i_h} \subseteq X_{i_h}$ with $P_{i_h} \ll_{\delta_{\mathcal{U}_{X_{i_h}}}} Q_{i_h}$ whenever $h \in H$ such that

$$\bigcap_{h\in H} (\pi_{i_h} \times \pi_{i_h})^{-1}\big(\big[(X_{i_h} \setminus P_{i_h}) \times X_{i_h}\big] \cup \big[X_{i_h} \times Q_{i_h}\big]\big) \subseteq U$$

and that $V(x_n) \ll_{\delta_{\mathcal{U}_{\prod_I X_i}}} V^2(x_n)$ whenever $n \in \omega$.

Similarly there is a finite set E such that

$$\{x_n\colon\ n \in \omega\} \subseteq \bigcup_{e\in E} \prod_{i\in I} A^e_i \subseteq \bigcup_{e\in E} \prod_{i\in I} B^e_i \subseteq V\big(\{x_n\colon\ n \in \omega\}\big)$$

where $A^e_i \ll_{\delta_{\mathcal{U}_{X_i}}} B^e_i$ whenever $i \in I$ and $e \in E$, and where for each $e \in E$, $A^e := \prod_{i\in I} A^e_i$ and $B^e := \prod_{i\in I} B^e_i$ are topological boxes.

Consider now the cover $\mathcal{C} := \{\prod_{i\in I} A^f_{i,n}\colon f \in F_n,\ n \in \omega\}$ of $\bigcup_{e\in E} \prod_{i\in I} B^e_i$. Furthermore consider the corresponding cover $\mathcal{C}' := \{\prod_{i\in I} B^f_{i,n}\colon f \in F_n,\ n \in \omega\}$. For convenience we shall enumerate the elements of $\mathcal{C}$ by B_n $(n \in \omega)$ and denote the box corresponding to B_n in $\mathcal{C}'$ by B'_n.

Since E is finite, there is $e_0 \in E$ such that $S := \{x_n\colon n \in \omega\} \cap A^{e_0}$ is infinite. Note also that each box B'_n belonging to $\mathcal{C}'$ contains only finitely many points of S by definition of the sequence $(x_n)_{n\in\omega}$.

Next we establish the following claim:

Claim. *There are $i \in I$ and an infinite subset J of ω such that $B^{e_0}_i \subseteq \bigcup_{n\in J} \pi_i B_n$ and no finite subcollection of $\{\pi_i B'_n\colon n \in J\}$ covers $\pi_i(S)$.*

Proof. Set $M_0 = B^{e_0}$ and $S_0 = S$.

Inductively we define a strictly increasing sequence $(k_n)_{n\in\omega}$ in ω, a sequence $(i_n)_{n\in\omega}$ in I, a decreasing sequence $(M_n)_{n\in\omega}$ of boxes in $\prod_{i\in I} X_i$ and a decreasing sequence $(S_n)_{n\in\omega}$ of infinite subsets of S. In particular for each $n \in \omega$ the following two conditions will be satisfied:

(1) *The box M_n contains the infinite set S_n.*
(2) *for each $p \leqslant k_{n-1}$, B_p does not intersect M_n.* (*This condition falls away for $n = 0$.*)

Let $n \in \omega$. Suppose that k_p and i_p have been defined for any $p < n$, and assume that M_p and S_p have been defined for any $p \leqslant n$, such that conditions (1) and (2) hold. Note that the induction can start, since M_0 and S_0 satisfy condition (1).

Let k_n be the smallest element $t \in \omega$ such that

$$B_t \cap M_n \neq \emptyset.$$

Recall that the topological box B'_{k_n} contains only finitely many points of S. Since M_n satisfies condition (1), by considering the complement of B'_{k_n} in M_n, we see that there exists $i \in I$ such that

$$M_n \cap \pi_i^{-1}\big(B^{e_0}_i \setminus \pi_i(B'_{k_n})\big)$$

contains infinitely many points of S_n. Denote this i by i_n and choose the latter points of S_n to be the elements of the set S_{n+1}.

In particular $M_{n+1} := M_n \cap \pi_{i_n}^{-1}(B^{e_0}_{i_n} \setminus \pi_{i_n}(B_{k_n}))$ contains S_{n+1}. So condition (1) is fulfilled at stage $n + 1$.

Note that $B_{k_n} \cap M_{n+1} = \emptyset$. Hence condition (2) is satisfied at stage $n + 1$.

Furthermore the sequence $(k_n)_{n\in\omega}$ is seen to be strictly increasing by the construction of k_{n+1} and condition (2). This completes the induction.

Suppose now that there is a point $(x_i)_{i\in I} \in \bigcap_{n\in\omega} M_n$. It follows that there is $\ell \in \omega$ such that $(x_i)_{i\in I} \in B_\ell$. Then by condition (2) $B_\ell \cap M_{\ell+1} = \emptyset$, since $\ell \leqslant k_\ell$ and $B_p \cap M_{\ell+1} = \emptyset$ whenever $p \leqslant k_\ell$.

We have reached a contradiction and conclude that $\bigcap_{n\in\omega} M_n = \emptyset$. Therefore there is $i \in I$ such that $\bigcap_{n\in\omega} \pi_i(M_n) = \emptyset$. Set $J = \{k_n \in \omega\colon i_n = i\}$. Consequently $\bigcap_{s\in J}(B^{e_0}_i \setminus \pi_i B_s) = \emptyset$. We also note that no finite subcollection of $\{\pi_i B'_s\colon s \in J\}$ contains all points of $\pi_i(S)$, since for each $n \in \omega$, $\pi_{i_n}(S_{n+1}) \cap \pi_{i_n} B'_{k_n} = \emptyset$ and the sets of the decreasing sequence $(S_n)_{n\in\omega}$ are nonempty at each stage $n \in \omega$ of the construction. This completes the proof of the claim. □

The established claim means for the proof of Theorem 4.1 that we have $\pi_i S \subseteq A^{e_0}_i \ll_{\delta\mathcal{U}_{X_i}} B^{e_0}_i$, $B^{e_0}_i \subseteq \bigcup_{s\in J} \pi_i B_s$ and $\pi_i B_s \ll_{\delta\mathcal{U}_{X_i}} \pi_i B'_s$ whenever $s \in J$; moreover $\pi_i S$ is not covered by finitely many members of the collection $\{\pi_i B'_s\colon s \in J\}$.

We now define inductively a sequence $(s_n)_{n\in\omega}$ of points in S and a sequence $(m_n)_{n\in\omega}$ in J as follows. Set m_0 equal to the minimal element of J. Let $n \in \omega$. Suppose that s_k $(k < n)$ and m_k $(k \leqslant n)$ have been chosen. Find $s \in S$ such that $\pi_i(s) \notin \bigcup_{k\leqslant m_n, k\in J} \pi_i B'_k$. Set $s_n = s$. Then choose $p \in J$ such that $\pi_i(s_n) \in \pi_i B_p$ and let $m_{n+1} = p$. Note that $m_{n+1} > m_n$. Hence the induction can be completed.

Set $P_r = \bigcup\{\pi_i B_k\colon k \leqslant m_{r+1},\ k \in J\}$ and $P'_r = \bigcup\{\pi_i B'_k\colon k \leqslant m_{r+1},\ k \in J\}$ whenever $r \in \omega$.

Observe that the sequences $(P_r)_{r\in\omega}$ and $(P'_r)_{r\in\omega}$ of subsets of X_i are increasing. Furthermore $P_r \ll_{\delta_{\mathcal{U}_{X_i}}} P'_r$ whenever $r \in \omega$. Finally set $P := A_i^{e_0}$ and $P' := B_i^{e_0}$. Then $P \ll_{\delta_{\mathcal{U}_{X_i}}} P' \subseteq \bigcup_{k\in J} \pi_i B_k = \bigcup_{r\in\omega} P_r$. Furthermore set $y_r := \pi_i(s_r)$ whenever $r \in \omega$.

Then $y_r \in P_r \cap P$, but $y_{r+1} = \pi_i(s_{r+1}) \notin P'_r$ whenever $r \in \omega$.

Applying Proposition 4.3 to $\mathcal{U}_{X_i}$, we conclude that the quasi-proximity class of $\mathcal{U}_{X_i}$ contains a quasi-uniformity that is not hereditarily precompact—a contradiction. Hence the proof is complete. □

Corollary 4.5. *Let $(X_i)_{i\in I}$ be a nonempty family of (nonempty) quasi-uniform spaces. Then the product quasi-uniformity $\mathcal{U}_{\prod_I X_i}$ is quasi-proximally unique if and only if for each $i \in I$, $\mathcal{U}_{X_i}$ is quasi-proximally unique.*

Proof. Assume first that $\mathcal{U}_{X_i}$ is quasi-proximally unique whenever $i \in I$.

Note that the quasi-uniformities $\mathcal{U}_{\prod_I X_i}$ and $\prod_I (\mathcal{U}_{X_i})^{-1}$ are conjugate. Since for each $i \in I$, by Lemma 4.1 the quasi-proximity classes of $\mathcal{U}_{X_i}$ and of $(\mathcal{U}_{X_i})^{-1}$ contain only hereditarily precompact quasi-uniformities, by Theorem 4.1 the same holds for the quasi-proximity classes of $\mathcal{U}_{\prod_I X_i}$ and of $\prod_I (\mathcal{U}_{X_i})^{-1}$. Hence by Lemma 4.1 we have shown that $\mathcal{U}_{\prod_I X_i}$ is quasi-proximally unique.

The converse is a consequence of Proposition 4.1 applied to the quasi-uniformly continuous projection maps of the product under consideration. □

References

[1] J.W. Carlson, Quotient structures for quasi-uniform spaces, Colloq. Math. 36 (1976) 63–68.
[2] V.A. Efremovič, The geometry of proximity, I, Mat. Sbornik N.S. 31 (73) (1952) 189–200 (in Russian).
[3] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[4] P. Fletcher, W.F. Lindgren, Quasi-Uniform Spaces, Lecture Notes Pure Appl. Math., vol. 77, Dekker, New York, 1982.
[5] D.T. Hogan, Quotient quasi-uniformities, Math. Chronicle 4 (1976) 108–111.
[6] M. Hušek, Generalized proximity and uniform spaces, I, Comment. Math. Univ. Carolinae 5 (1964) 247–266;
M. Hušek, Generalized proximity and uniform spaces, II, Comment. Math. Univ. Carolinae 6 (1965) 119–139.
[7] M. Hušek, Factorization of mappings (products of proximally fine spaces), in: Seminar Uniform Spaces (Prague, 1973–1974, Mat. Ústav Československě Akad. Véd, Prague, 1975), pp. 173–190.
[8] M. Hušek, M.D. Rice, Productivity of coreflective subcategories of uniform spaces, General Topology Appl. 9 (1978) 295–306.
[9] J.R. Isbell, Uniform Spaces, Amer. Math. Soc., Providence, RI, 1964.
[10] J.R. Isbell, Spaces without large projective subspaces, Math. Scand. 17 (1965) 89–105;
J.R. Isbell, Math. Scand. 22 (1968) 310, Correction.
[11] H.-P.A. Künzi, Some remarks on quasi-uniform spaces, Glasgow Math. J. 31 (1989) 309–320.
[12] H.-P.A. Künzi, Functorial admissible quasi-uniformities on topological spaces, Topology Appl. 43 (1992) 27–36.
[13] H.-P.A. Künzi, A. Losonczi, Semilattices of totally bounded quasi-uniformities, Topology Appl. 114 (2001) 273–284.
[14] W.F. Lindgren, Topological spaces with a unique compatible quasi-uniformity, Canad. Math. Bull. 14 (1971) 369–372.
[15] A. Losonczi, The cardinality of the coarsest quasi-proximity class of locally compact T_2 spaces, Topology Proc. 23 (1998) 245–262.
[16] A. Losonczi, Topological spaces with a coarsest compatible quasi-uniformity, Quaestiones Math. 23 (2000) 67–75.
[17] P. Samuel, Ultrafilters and compactification of uniform spaces, Trans. Amer. Math. Soc. 64 (1948) 100–132.
[18] Yu.M. Smirnov, On proximity spaces, Mat. Sbornik N.S. 31 (73) (1952) 543–574 (in Russian).
[19] G. Vidossich, Two remarks on A. Gleason's factorization theorem, Bull. Amer. Math. Soc. 76 (1970) 370–371.