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ORIGINAL ARTICLE

On unsteady two-dimensional and axisymmetric squeezing flow between parallel plates

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Abstract Squeezing flow of a viscous fluid is considered. Two types of flows are discussed namely, the axisymmetric flow and two dimensional flow. Similarity transform proposed by Wang (1976) [13] has been used to reduce the Navier–Stokes equations to a highly non-linear ordinary differential equation which jointly describes both types of flows. Solution to aforementioned ordinary differential equation is obtained by using Variation of Parameters Method (VPM). VPM is free from the existence of small or large parameters and hence it can be applied to a large variety of problems as compared to the perturbation method applied by Wang (1976) [13]. Comparison among present and already existing solutions is also provided to show the efficiency of VPM. A convergence analysis is also carried out. Effects of different physical parameters on the flow field is discussed and demonstrated graphically with comprehensive discussions and explanations.

1. Introduction

Squeezing flow between parallel walls accrues in many industrial and biological systems. Moving pistons in engines, hydraulic brakes, chocolate filler and many other devices are based on the principle of flow between contracting domains. To develop these equipment and machines better understanding of such flow models which describe the squeezing flow between parallel walls is always needed. Classical work in this regard can be traced back to Stefan [1], who presented his work on squeezing flow by using lubrication assumption. Later in 1986 Reynolds [2] studied the case for elliptic plates, and Archibald [3] considered the squeezing flow between rectangular plates. After that several researchers have contributed their efforts to make squeezing flow model more understandable [4–8].

Earlier studies on squeezing flows are based on Reynolds equation however the scantiness of Reynolds equation for some cases has been shown by Jackson [9]. More flexible and useful similarity transforms are now available due to the efforts of Birkhoff [10], Yang [11] and Wang and Watson

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[12]. These similarity transforms reduce the Navier–Stokes equation into a fourth order nonlinear ordinary differential equation and have further been used in some other investigations as well [13–17].

Most of the real world problems are inherently in the form of nonlinearities. Over the years much attention has been devoted to develop new efficient analytical techniques that can cope up with such nonlinearities. Several approximation techniques have been developed to fulfill this purpose [18–27]. Nowadays, researchers prefer those techniques which are easy to implement, require less computational work and time to provide reliable results. One of these analytical techniques is Variation of Parameters Method (VPM) [28,29]. Main advantages of VPM are that it does not depend on existence of small or large parameters; it is free from round off errors, calculation of so called Adomian's polynomials, linearization or discretization. It uses initial conditions that are easier to be implemented and reduces the computational work while still maintaining a higher level of accuracy. One can easily access the recent applications of VPM in different available studies [30–33].

In this study one may clearly see that VPM can successfully be applied to solve the equations governing unsteady squeezing flows between parallel plates. Comparison of the results obtained by VPM to the numerical solution obtained by using Runge–Kutta order 4 is also provided to show the effectiveness of the technique. Obtained results are also compared with already existing studies. A convergence analysis is also carried out to check the computational cost benefits of VPM. It is evident from this article that VPM provides better results with less amount of laborious computational work.

2. Governing equations

Consider an incompressible flow of a viscous fluid between two parallel plates separated by a distance $z = \pm l(1-\alpha t)^{1/2} = \pm h(t)$, where l is the position at time $t = 0$. For $\alpha > 0$ plates are squeezed until they touch each other at $t = 1/\alpha$ for $\alpha < 0$ plates are separated. Let u, v and w be the velocity components in x, y and z directions respectively, shown in Fig. 1. Using transform introduced by Wang [13] for a two-dimensional flow:

$$u = \frac{\alpha x}{[2(1-\alpha t)]} F'(\eta), \tag{1}$$

$$w = \frac{-\alpha l}{[2(1-\alpha t)^{1/2}]} F(\eta), \tag{2}$$

where,

$$\eta = \frac{z}{\left[l(1-\alpha t)^{1/2} \right]}. \tag{3}$$

Substituting, Eqs. (1)–(3) in unsteady two-dimensional Navier–Stokes equations yield a non-linear ordinary differential equation of same form as discussed by [17],

$$F^{iv}(\eta) + S(-\eta F(\eta) - 3F''(\eta) - F'(\eta)F''(\eta) + F(\eta)F'''(\eta)) = 0, \tag{4}$$

where $S = \alpha l^2/2\nu$ is the non-dimensional Squeeze number, and ν is the kinematic viscosity. Boundary conditions for the problem are such that on plates the lateral velocities are zero and normal velocity is equal to velocity of the plate, that is

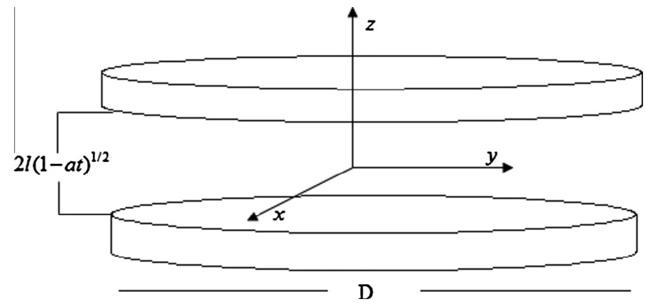

Figure 1 Schematic diagram of the problem.

$$F(0) = 0, \quad F''(0) = 0, \quad F(1) = 1, \quad F'(1) = 0. \tag{5}$$

Similarly for the axisymmetric case, transforms introduced by Wang [13] are

$$u = \frac{\alpha x}{[4(1-\alpha t)]} F'(\eta), \tag{6}$$

$$v = \frac{\alpha y}{[4(1-\alpha t)]} F'(\eta), \tag{7}$$

$$w = \frac{-\alpha l}{[2(1-\alpha t)]} F(\eta). \tag{8}$$

Using Eqs. (6)–(8) in unsteady axisymmetric Navier–Stokes equations we get a nonlinear ordinary differential equation of the form

$$F^{iv}(\eta) + S(-\eta F(\eta) - 3F''(\eta) + F(\eta)F'''(\eta)) = 0. \tag{9}$$

Thus, we have to solve non-linear ordinary differential equation of the form

$$F^{iv}(\eta) + S(-\eta F(\eta) - 3F''(\eta) - \beta F'(\eta)F''(\eta) + F(\eta)F'''(\eta)) = 0, \tag{10}$$

subject to the boundary conditions given in Eq. (5).

In Eq. (10), $\beta = 0$ corresponds to axisymmetric flow while $\beta = 1$ gives two-dimensional case.

3. Solution procedure

Following the standard procedure proposed for VPM [28–33], we can write Eq. (10) as

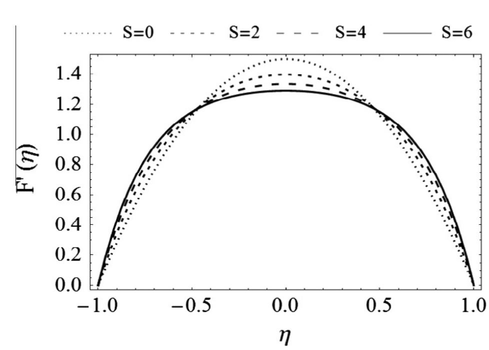

Figure 2 Effects of S on $F'(\eta)$ in expanding motion of plates (axisymmetric case).

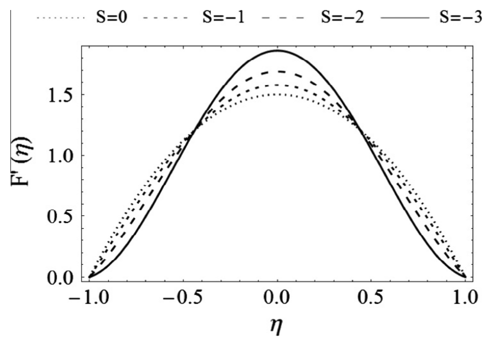

Figure 3 Effects of S on $F'(\eta)$ in contracting motion of plates (axisymmetric case).

$$F_{n+1}(\eta)=A_1+A_2\eta+A_3\frac{\eta^2}{2}+A_4\frac{\eta^3}{6}+\int_0^\eta\left(\frac{\eta^3}{3!}-\frac{\eta^2 s}{2!}+\frac{\eta s^2}{2!}+\frac{s^3}{3!}\right)(S(-sF(s)-3F''(s)-\beta F'(s)F''(s)+F(s)F'''(s)))ds.$$

Using boundary conditions given in Eq. (5), above equation can be written as

$$F_{n+1}(\eta)=A_2\eta+A_4\frac{\eta^3}{6}+\int_0^\eta\left(\frac{\eta^3}{3!}-\frac{\eta^2 s}{2!}+\frac{\eta s^2}{2!}+\frac{s^3}{3!}\right)(S(-\eta F(s)-3F''(s)-\beta F'(s)F''(s)+F(s)F'''(s)))ds,$$

with $n = 0, 1, 2, \ldots$,

where A_2 and A_4 are constants which can be computed by using boundary conditions $F(1) = 1$ and $F'(1) = 0$, respectively.

First few terms of the solution are given as

$$F_0(\eta)=A_2\eta+A_4\frac{\eta^3}{6},$$

$$F_1(\eta)=A_2\eta+A_4\frac{\eta^3}{6}-\left(\frac{1}{30}SA_4-\frac{1}{120}SA_2A_4+\frac{1}{120}S\beta A_2A_4\right)\eta^5-\left(\frac{1}{5040}SA_4^2-\frac{1}{1680}S\beta A_4^2\right)\eta^7,$$

$$\begin{aligned}F_2(\eta)=&A_2\eta+A_4\frac{\eta^3}{6}-\left(\frac{1}{30}SA_4-\frac{1}{120}SA_2A_4+\frac{1}{120}S\beta A_2A_4\right)\eta^5\\&-\left(\frac{1}{5040}SA_4^2-\frac{1}{1680}S\beta A_4^2-\frac{1}{5040}S^2\beta^2A_2^2A_4\right.\\&\left.-\frac{1}{504}S^2\beta A_2A_4+\frac{1}{280}S^2A_2A_4-\frac{1}{1680}S^2A_2^2A_4-\frac{1}{120}S^2A_4\right)\eta^7\\&+\left(\frac{1}{20,160}S^2\beta^2A_2A_4^2-\frac{1}{8640}S^2\beta A_2A_4^2+\frac{1}{22,680}S^2A_2A_4^2\right.\\&\left.+\frac{1}{4320}S^2\beta A_4^2-\frac{13}{90,720}S^2A_4^2\right)\eta^9+\ldots\end{aligned}\quad(11)$$

Similarly, other iterations of the solution can also be computed.

4. Results and discussions

It is important to note that for $\beta = 0$, the series solution presented in Eq. (11) reduces to provide the solution for an axisymmetric case while for $\beta = 1$ we have solution to two dimensional squeezing flows.

Influence of squeeze number S over axisymmetric flow is shown in Figs. 2 and 3. It is worth mentioning that $S < 0$ corresponds to the squeezing flow of plates while $S > 0$ describes receding motion of the plates. Fig. 2 depicts the influence of nonnegative S on $F'(\eta)$. Increase in S increases $F'(\eta)$ near the plates while in center a delayed streamline flow is observed. When plates leave each other a vacant space is created and fluid near that portion fills that empty region. This phenomenon is perhaps responsible for an accelerated flow near the plates. While for contracting motion ($S < 0$) influence of S on flow is shown in Fig. 3. It can be seen that with absolute increase in S, $F'(\eta)$ decreases near plates while in center it appears to be an increasing function of absolute S. It can also be observed from Fig. 3 that for negative S increasing absolute S results in back flow and there might be separation. Further, the problem is also solved numerically by using well known RK-4 method. A comparison between VPM, numerical and the solution obtained by Wang [13] is carried out. Numerical values for velocity $F''(1)$ are tabulated in Table 1 for this purpose. An excellent agreement can clearly be seen in both the solutions for low values of S. The problem with Wang's solution is that it is only valid for very small values of perturbation parameter where its higher powers vanish or else perturbation method cannot be applied. Variation of Parameters Method removes this restriction and is free of existence of small or large parameters. Table 2 is drawn to discuss the convergence of VPM solution and a comparison with HAM [17] is also carried out. It can be observed that for axisymmetric case, VPM converges quite rapidly. Only fourth order approximations are enough to obtain a convergent solution. On the other hand, HAM [17] requires six iterations of the solution to converge. Numerical values for $F(\eta)$ are tabulated for different values of S to check the convergence efficiency.

For two-dimensional case, effects of S on $F'(\eta)$ are shown in Figs. 4 and 5. It is clear that the effects are similar as compared to axisymmetric flow but variation in two dimensional case is more prominent.

Table 3 presents the value of $F''(1)$ for different values of S for two-dimensional case. A comparison is made with the solutions obtained by Wang [13]. Again an excellent agreement

Table 1 Comparison of VPM and numerical solutions for axisymmetric ($\beta = 0$) with existing results.

$S\downarrow$	$F''(1)$ present results (VPM)	$F''(1)$ present results (RK-4)	$F''(1)$ Wang [13]
−0.9952	−2.401	−2.401	−2.410
−0.4997	−2.7151	−2.7151	−2.7161
−0.1	−2.9254	−2.9254	−2.9252
0	−3.000	−3.000	−3.000
0.11576	−3.0622	−3.0622	−3.0622
0.4138	−3.2165	−3.2165	−3.2160
2.081	−3.9610	−3.9610	−3.9610

Table 2 Convergence of VPM solution, numerical values of $F(\eta)$ for axisymmetric case ($\beta = 0$) and comparison with HAM solution.

S	η	VPM solution (4th order approximation)	Numerical (RK-4)	HAM [17] (6th order approximation)
−1.5	0.2	0.319526	0.319526	0.319526
	0.4	0.603830	0.603830	0.603830
	0.6	0.822876	0.822876	0.822875
	0.8	0.956801	0.956801	0.956800
−0.5	0.2	0.302582	0.302582	0.302582
	0.4	0.578082	0.578082	0.578082
	0.6	0.800780	0.800780	0.800780
	0.8	0.947702	0.947702	0.947702
0.5	0.2	0.290322	0.290322	0.290322
	0.4	0.559252	0.559252	0.559252
	0.6	0.784303	0.784303	0.784303
	0.8	0.940703	0.940703	0.940703
1.5	0.2	0.281010	0.319526	0.319526
	0.4	0.544779	0.603830	0.603830
	0.6	0.771371	0.822876	0.822875
	0.8	0.935936	0.956801	0.956800
2.5	0.2	0.273682	0.273682	0.273682
	0.4	0.533246	0.533246	0.533247
	0.6	0.760847	0.760847	0.760848
	0.8	0.930280	0.930280	0.930281

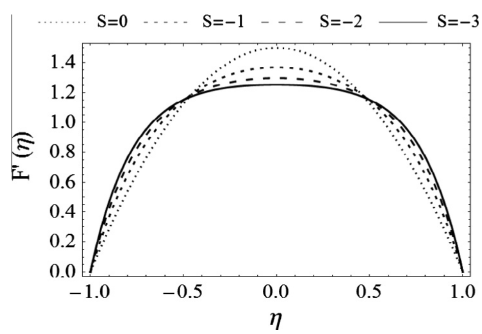

Figure 4 Effects of S on $F'(\eta)$ in expanding motion of plates (two-dimensional case).

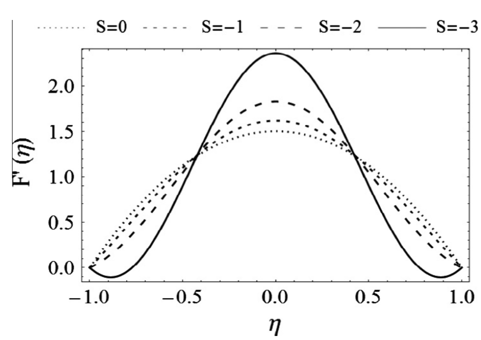

Figure 5 Effects of S on $F'(\eta)$ in contracting motion of plates (two-dimensional case).

is found as expected for smaller values of S. However for larger values of S Wang's solution is more likely to be divergent due to restriction necessary for validity of perturbation solution. In Table 4 convergence of VPM solution for two-dimensional case is discussed and a comparison with HAM [17] is also carried out. It can be observed that for two-dimensional case, VPM converges at fifth order approximations. On the other hand, HAM [17] requires seven iterations for a convergent solution. Again, numerical values for $F(\eta)$ are tabulated for different values of S to check the convergence efficiency.

Table 3 Comparison of VPM and numerical solutions for two-dimensional ($\beta = 1$) case with existing results.

$S\downarrow$	Present results (VPM)	Present results (RK-4)	Wang [13]
−0.9780	−2.1915	−2.1915	−2.235
−0.4977	−2.6193	−2.6193	−2.6272
−0.09998	−2.9277	−2.9277	−2.9279
0	−3.000	−3.000	−3.000
0.09403	−3.0663	−3.0663	−3.0665
0.4341	−3.2943	−3.2943	−3.2969
1.1224	−3.708	−3.708	−3.714

5. Conclusions

In this article, a relatively novel analytical technique called the Variation of Parameters Method has been employed to solve squeezing flow problem for axisymmetric and two-dimensional flows. Convergence analysis is carried out to check the computational efficiency of VPM. Comparison is also carried out between current and existing solutions. It can be concluded from the tables and discussions that the VPM can easily and efficiently be applied to solve higher order non-linear equations for real world problems. Unlike other analytical techniques, VPM do not require existence of small or large parameters, calculation of any kind of polynomials and attains the convergence at fewer number of iterations which reduces the computational cost. Graphs are plotted to discuss the behavior of squeeze number S on velocity profile.

Table 4 Convergence of VPM solution, numerical values of $F(\eta)$ for two dimensional case ($\beta = 1$) and comparison with HAM solution.

S	η	VPM solution (5th order approximation)	Numerical (RK-4)	HAM [17] (6th order approximation)
−1.5	0.2	0.333618	0.333618	0.333617
	0.4	0.624358	0.624358	0.624358
	0.6	0.839325	0.839325	0.839324
	0.8	0.962984	0.962984	0.962983
−0.5	0.2	0.305545	0.305545	0.305545
	0.4	0.582470	0.582470	0.582470
	0.6	0.804392	0.804392	0.804392
	0.8	0.949108	0.949108	0.949108
0.5	0.2	0.288260	0.288260	0.288260
	0.4	0.556143	0.556143	0.556143
	0.6	0.781671	0.781671	0.781671
	0.8	0.939640	0.939640	0.939640
1.5	0.2	0.276432	0.276432	0.276432
	0.4	0.537752	0.537752	0.537752
	0.6	0.765249	0.765249	0.765249
	0.8	0.932471	0.932471	0.932471
2.5	0.2	0.267791	0.267791	0.267791
	0.4	0.524045	0.524045	0.524045
	0.6	0.752605	0.752605	0.752605
	0.8	0.926703	0.926703	0.926704

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