Applied Mathematics Letters 25 (2012) 1959–1963

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Characterizing the image space of a shape-dependent operator for a potential flow problem

C. Leithäuser [a,b,*], R. Feßler [b]

[a] TU Kaiserslautern, Department of Mathematics, Gottlieb-Daimler-Straße, 67663 Kaiserslautern, Germany
[b] Fraunhofer ITWM, Transport Processes, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany

ARTICLE INFO

Article history:
Received 15 April 2011
Received in revised form 6 March 2012
Accepted 7 March 2012

Keywords:
Shape optimization
Inverse problem
Elliptic boundary value problem
Differential geometry
Conformal mapping

ABSTRACT

We study the reachability of a shape-dependent operator based on a potential flow and give a complete characterization of the image space. We draw a connection between the structure of the image space and the set of stagnation points, i.e. the set of surface points where the tangential velocity vanishes. We use conformal pull-back to a reference domain and reduce the problem to the question of whether there exists a diffeomorphism which pulls back one top-dimensional differential form to another. For volume forms this question has been answered by Moser 1965, but since we do allow singularities we have to prove a modified version. This leads to a volume condition, which must be fulfilled on every connected component of the nonzero set of the form.

1. Introduction

In this work we study a shape-dependent operator based on a potential flow problem and answer a controllability question by deriving a complete characterization of its image space. An approach to a similar controllability problem, but treated with different instruments, can be found in [1]: the authors prove approximate controllability for a shape design problem on the discrete level, by showing that the linearized operator has a dense image (cf. [2]) and applying the inverse function theorem. For our approach we stay on the continuous level and deal with the shape dependence through conformal pull-back to a reference domain. We can then derive an integral condition which characterizes the image. Our goal is to add insight to the theoretical background of shape optimization by identifying which states are reachable for a specific design problem.

For an introduction to the general theory of shape optimization we refer the reader to [3–5]. There are many ways to treat shape variations, e.g. using the speed method [4], level sets [6] or the pull-back to a reference domain. Free-form shape design uses a pull-back to derive a parameterization of the domain: see [7] for a classic approach and [8] for a state-of-the-art application which also utilizes model reduction techniques. We employ a pull-back by conformal mappings which have been used in shape and airfoil design for a long time. For an introduction to general techniques and applications to a wide range of shape design problems from engineering and applied science we refer the reader to [9]. See [10,11] for concepts of airfoil design and applications of conformal mappings. An advantage is that the conformal pull-back results in a simple formulation which is suitable for our analysis. Furthermore, the Riemann mapping theorem [9] assures that the use of conformal mappings is not a restriction to the space of admissible shapes.

The work is organized in the following way. In Section 2 we establish notation from differential geometry, which is necessary for applying a classic result from Moser [12]. A good introduction to differential geometry applied to boundary

* Corresponding author at: TU Kaiserslautern, Department of Mathematics, Gottlieb-Daimler-Straße, 67663 Kaiserslautern, Germany. Tel.: +49 631 31600 4411.
E-mail address: christian.leithaeuser@itwm.fraunhofer.de (C. Leithäuser).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2012.03.009

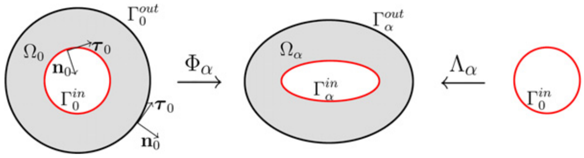

Fig. 1. Sketch of the reference domain Ω_0, the conformal map Φ_α to the conformal domain Ω_α and the boundary isometry Λ_α.

value problems can be found in [13]. In Section 3 we define our shape design problem and reformulate it on the reference domain by conformal pull-back: we consider an object placed in an external potential flow and the goal is to investigate which tangential velocities are reachable through deformations of that object. As we are going to see, the space of reachable velocities is essentially restricted by the immutable set of stagnation points, i.e. the set of points where the velocity on the reference geometry vanishes. In Section 4 the problem is reduced to a result from Moser [12]. However, we cannot use this result itself, and have to prove a modified version which accounts for singularities. Theorem 6 states our main result and gives a complete characterization of the image space of the shape problem.

2. Preliminaries

For any smooth and compact $\Omega \subset \mathbb{R}^2$, let the Euclidean metric be denoted by g_0. For a function $\alpha \in C^\infty(\Omega)$ the conformal metric g_α is defined such that $g_\alpha = e^{2\alpha} g_0$. The boundary Γ of Ω is a smooth and compact one-dimensional manifold. Let $\mathbf{n}_\alpha$ denote the outward pointing unit normal vector field on the boundary with respect to the metric g_α and let $\boldsymbol{\tau}_\alpha$ be the unit tangential vector field pointing left of $\mathbf{n}_\alpha$ (see Fig. 1). In the following we use concepts from differential geometry. A good introduction to this field and the application of Hodge decomposition to boundary value problems can be found in the book by Schwarz [13], upon which this brief introduction and the notation are based. Let $\boldsymbol{\Gamma}(T\Omega)$ denote the space of smooth vector fields on the n-dimensional manifold Ω. For $k \in 0, \dots, n$ let $\boldsymbol{\Omega}^k(\Omega)$ denote the space of k-forms, where 0-forms in $\boldsymbol{\Omega}^0(\Omega)$ can be directly identified with functions from $C^\infty(\Omega)$, and between vector fields and 1-forms a natural isomorphism G_α is given through the metric g_α by

$$G_\alpha : \boldsymbol{\Gamma}(T\Omega) \to \boldsymbol{\Omega}^1(\Omega) : X \mapsto G_\alpha(X) := g_\alpha(X, \cdot). \tag{1}$$

Furthermore, we need the Hodge operator $\star_\alpha : \boldsymbol{\Omega}^k(\Omega) \to \boldsymbol{\Omega}^{n-k}(\Omega)$ which maps a k-form to its dual, the exterior derivative $d : \boldsymbol{\Omega}^k(\Omega) \to \boldsymbol{\Omega}^{k+1}(\Omega)$ and the co-differential operator $\delta_\alpha : \boldsymbol{\Omega}^k(\Omega) \to \boldsymbol{\Omega}^{k-1}(\Omega)$. The exterior derivative is independent of the metric g_α while the other two operators are metric dependent. Let $\mu_\alpha \in \boldsymbol{\Omega}^n(\Omega)$ denote the Riemannian volume form corresponding to g_α, which is defined by $\mu_\alpha(X_1, \dots X_n) = \sqrt{\det(g_\alpha(X_i, X_j))}$. Then, for $f \in C^\infty(\Omega)$ the identity $\star_\alpha f = f\mu_\alpha$ holds. Only n-forms can be integrated, so to integrate a function $f \in C^\infty(\Omega)$ we have to take its Hodge dual first, $\star_\alpha f$, or compose it with a volume form $f\mu_\alpha$, which is the same thing. For the Euclidean metric g_0 the following identities hold between the k-form operators and vector field operators.

Proposition 1 (*[13]*).

(a) *Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain. For a vector field $X \in \boldsymbol{\Gamma}(T\Omega)$ let $\nabla \cdot X$ and $\nabla \times X$ be the well-known divergence and curl operators with respect to the Euclidean metric, then $\nabla \cdot X = \delta_0(G_0(X))$ and $\nabla \times X = \star_0 d(G_0(X))$ hold. Note that since $\dim(\Omega) = 2$, $\nabla \times X$ can be interpreted as a function in $C^\infty(\Omega)$.*

(b) *Let $\Gamma \subset \mathbb{R}^2$ be a one-dimensional boundary manifold with tangential vector field $\boldsymbol{\tau}_0$; then for $\omega \in C^\infty(\Gamma)$ the identity $\partial_{\boldsymbol{\tau}_0}\omega = \star_0 d\omega$ holds, where $\partial_{\boldsymbol{\tau}_0}$ denotes the derivative in direction $\boldsymbol{\tau}_0$.*

Proposition 2 (*[13]*).

(a) *Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional domain. For $\omega \in \boldsymbol{\Omega}^k(\Omega)$ the following relations hold between the operators corresponding to the metrics g_α and g_0:*

$$\star_\alpha \omega = e^{(2-2k)\alpha} \star_0 \omega \qquad \delta_\alpha\omega = e^{2(k-2)\alpha}\delta_0 e^{-2(k-1)\alpha}\omega \qquad \mathbf{n}_0 = e^\alpha \mathbf{n}_\alpha \qquad \boldsymbol{\tau}_0 = e^\alpha \boldsymbol{\tau}_\alpha. \tag{2}$$

(b) *Let $\Gamma \subset \mathbb{R}^2$ be the one-dimensional boundary manifold; then for $\omega \in \Omega^k(\Gamma)$ the identity $\star_\alpha \omega = e^{(1-2k)\alpha} \star_0 \omega$ holds.*

3. Definition of the shape operator

In the following let $\Omega_0 \subset \mathbb{R}^2$ be a smooth and compact fixed reference domain with holes. For simplicity of notation we consider only the case with one hole, but the generalization to multiple holes is straightforward. Let the outer and inner boundaries be denoted by Γ_0^{out} and Γ_0^{in}, respectively (compare Fig. 1).

Proposition 3 (*See [9]*). *For Δ being the Euclidean Laplace operator, let $\alpha \in C^\infty(\Omega_0)$ with $\Delta\alpha = 0$; then there exists a domain $\Omega_\alpha \subset \mathbb{R}^2$ and a diffeomorphism $\Phi_\alpha : \Omega_0 \to \Omega_\alpha$ with $\Phi_\alpha^* g_0 = g_\alpha$, where Φ_α^* denotes the pull-back operator. The diffeomorphism is called a conformal map and Ω_α is conformal to Ω_0 with respect to the conformal parameter α.*

Note that Φ_α and Ω_α are only uniquely defined up to global translations and rotations of the domain. Thus, in the following Ω_α stands for the whole equivalence class of domains which are conformal to Ω_0 with respect to the conformal parameter α, but the operations that we are considering are well-defined for these equivalence classes.

We want to study a potential flow problem and investigate which tangential velocities are reachable on the inner boundary. Therefore, let $\mathbf{v}_0 \in \boldsymbol{\Gamma}(T\Omega_0)|_{\Gamma_0^{out}}$ be a smooth vector field defining the outer inflow boundary condition. This vector field cannot be arbitrary and must be chosen compatible in such a way that a solution to the potential flow problem exists. In the following setup, if $\mathbf{v}_0$ is compatible for the domain Ω_0 it is also compatible on every $\Omega_\alpha \in \mathcal{D}$, for $\mathcal{D}$ defined below. Then, let $\mathbf{u}_\alpha \in \boldsymbol{\Gamma}(T\Omega_\alpha)$ be the unique solution (see Eq. (6)) of the potential flow problem

$$\begin{aligned} &\nabla \times \mathbf{u}_\alpha = 0 \qquad \nabla \cdot \mathbf{u}_\alpha = 0 \quad \text{on } \Omega_\alpha \\ &\mathbf{u}_\alpha|_{\Gamma_\alpha^{out}} = \Phi_{\alpha *}\mathbf{v}_0 \qquad \mathbf{n} \cdot \mathbf{u}_\alpha|_{\Gamma_\alpha^{in}} = 0 \end{aligned} \tag{3}$$

where $\Phi_{\alpha *}$ denotes the push-forward operator induced by Φ_α. We want to define on the basis of the flow problem (3) a shape operator $\mathbf{S}$ which maps any domain $\Omega_\alpha \in \mathcal{D}$ to the tangential velocity on the inner boundary given by $\boldsymbol{\tau}_0 \cdot \mathbf{u}_\alpha|_{\Gamma_\alpha^{in}}$. A fundamental question is how to define the observation space for shape-dependent problems, because not only does $\boldsymbol{\tau}_0 \cdot \mathbf{u}_\alpha|_{\Gamma_\alpha^{in}}$ depend on $\Omega_\alpha \in \mathcal{D}$ but so also does the solution space $C^\infty(\Gamma_\alpha^{in})$ itself, whereas the observation space must be independent of Ω_α. A natural choice is to define it on the fixed reference domain, i.e. $C^\infty(\Gamma_0^{in})$. However, we still have to define a map pulling the solution from $C^\infty(\Gamma_\alpha^{in})$ to $C^\infty(\Gamma_0^{in})$. Using the pull-back by Φ_α would be possible, but here we use instead an isometry $\Lambda_\alpha : \Gamma_0^{in} \to \Gamma_\alpha^{in}$ (see Fig. 1), that is a conformal map on the boundary with conformal parameter 0. The advantage of using the isometry is that it preserves the length ratio and thus we get a result closer to our expectation. An isometry can only exist if both boundaries have the same length and is only unique if we introduce the following equivalence relation on $C^\infty(\Gamma_0^{in})$:

$$f \sim g \Leftrightarrow \exists \text{ isometry } \Psi \in \mathrm{Diff}(\Gamma_0^{in}) : \Psi^* f = g \tag{4}$$

for $f, g \in C^\infty(\Gamma_0^{in})$. This means that f and g are equivalent if there exists an isometry $\Psi : \Gamma_0^{in} \to \Gamma_0^{in}$ such that f is pulled back to g. Using this equivalence relation we define the observation space by $\mathcal{O} := C^\infty(\Gamma_0^{in})/\sim$. This assures that the shape operator which we are going to introduce is well-defined and independent of the actual choice of Λ_α.

We define $\mathcal{D} = \{\Omega_\alpha = \Phi_\alpha(\Omega_0) | \alpha \in \mathcal{A}, |\Gamma_0^{in}| = |\Gamma_\alpha^{in}|\}$ to be the space of all domains which are conformal to Ω_0 and where the inner boundaries have the same length as the reference boundary Γ_0^{in}. Here $\mathcal{A} := \{\alpha \in C^\infty(\Omega_0) | \Delta\alpha = 0, \alpha|_{\Gamma_0^{out}} = 0\}$ denotes the space of conformal parameters where the condition $\Delta\alpha = 0$ justifies through Proposition 3 that a conformal mapping exists and the condition $\alpha|_{\Gamma_0^{out}} = 0$ prevents the outer boundary from being scaled. However, the shape of the outer boundary is not preserved. Then, we can define the shape operator by

$$\begin{aligned} &\mathbf{S} : \mathcal{D} \to \mathcal{O} = C^\infty(\Gamma_0^{in})/\sim \\ &\Omega_\alpha \mapsto \Lambda_\alpha^*(\boldsymbol{\tau}_0 \cdot \mathbf{u}_\alpha)|_{\Gamma_\alpha^{in}}. \end{aligned} \tag{5}$$

In order to characterize the image space of this operator, let us translate the potential flow equation into a problem on the fixed reference domain Ω_0. Therefore, let $\omega_\alpha^0 = G_0(\mathbf{u}_\alpha) \in \boldsymbol{\Omega}^1(\Omega_\alpha)$ and let $\eta_0 = G_0(\mathbf{v}_0) \in \boldsymbol{\Omega}^1(\Omega_0)|_{\Gamma_0^{out}}$ be the outer boundary condition corresponding to $\mathbf{v}_0$. From the identities of Proposition 1 we see that Eq. (3) is equivalent to the problem formulated on 1-forms

$$\begin{aligned} &d\omega_\alpha^0 = 0 \qquad \delta_0\omega_\alpha^0 = 0 \quad \text{on } \Omega_\alpha \\ &\omega_\alpha^0|_{\Gamma_\alpha^{out}} = \Phi_{\alpha *}\eta_0 \qquad \omega_\alpha^0(\mathbf{n}_0)|_{\Gamma_\alpha^{in}} = 0 \end{aligned} \tag{6}$$

where the existence of a unique solution ω_α^0 for compatible outer boundary conditions is shown in [13]. According to our naming convention, the lower index indicates that ω_α^0 is the solution on the domain Ω_α and the upper index shows that the corresponding metric is g_0. Using the pull-back operator Φ_α^* and defining $\omega_0^\alpha = \Phi_\alpha^*\omega_\alpha^0$, this problem is equivalent to the following one on the fixed computational domain Ω_0:

$$\begin{aligned} &d\omega_0^\alpha = 0 \qquad \delta_\alpha\omega_0^\alpha = 0 \quad \text{on } \Omega_0 \\ &\omega_0^\alpha|_{\Gamma_0^{out}} = \eta_0 \qquad \omega_0^\alpha(\mathbf{n}_\alpha)|_{\Gamma_0^{in}} = 0. \end{aligned} \tag{7}$$

By Proposition 2 we can write this in terms of the Euclidean operators

$$\begin{aligned} &d\omega_0^\alpha = 0 \qquad \delta_0\omega_0^\alpha = 0 \quad \text{on } \Omega_0 \\ &\omega_0^\alpha|_{\Gamma_0^{out}} = \eta_0 \qquad \omega_0^\alpha(\mathbf{n}_0)|_{\Gamma_0^{in}} = 0. \end{aligned} \tag{8}$$

This shows that ω_0^α is actually independent of α, i.e. $\omega_0^\alpha = \omega_0^0$ for all $\alpha \in \mathcal{A}$. This enables us to write $\mathbf{S}$ in a local form, i.e. the dependence on the conformal parameter α is only local and not global through the PDE:

$$\mathbf{S}(\Omega_\alpha) = \Lambda_\alpha^*(\boldsymbol{\tau}_0 \cdot \mathbf{u}_\alpha)|_{\Gamma_\alpha^{in}} = \Lambda_\alpha^*(\omega_\alpha^0(\boldsymbol{\tau}_0)|_{\Gamma_\alpha^{in}}) = \Lambda_\alpha^*\Phi_{\alpha *}\omega_0^\alpha(\boldsymbol{\tau}_\alpha)|_{\Gamma_0^{in}} = \Lambda_\alpha^*\Phi_{\alpha *}(e^{-\alpha}(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0)|_{\Gamma_0^{in}}). \tag{9}$$

In the first place this local property is why it is possible to give an explicit characterization of the image space of the shape-dependent operator $\mathbf{S}$. Many constructive airfoil design algorithms are based on this property.

4. Characterizing the image of S

We now prove our modification of Moser's result [12] to general top-dimensional forms, including forms with singularities. In this case a mapping can only exist if the two forms have similar zero sets. For simplicity and with our application to the shape operator in view we restrict ourselves to one-dimensional manifolds, i.e. $n = 1$, but it should be possible to extend the result to arbitrary dimensions. Moser's theorem requires the total volume on the whole manifold to be invariant. We get a similar volume condition on every connected component of the nonzero sets of the forms.

Definition 4 (*Nonzero Components*)**.** For $\varrho \in \boldsymbol{\Omega}^1(\Gamma)$ let $\mathcal{C}(\varrho) = \{\Gamma_i \subset \Gamma$ compact and connected, $|\varrho|_{\Gamma_i \setminus \partial\Gamma_i} \neq 0, \varrho|_{\partial\Gamma_i} = 0\}$ where $\partial\Gamma_i$ denotes the boundary points of Γ_i. Note that by this definition, $\Gamma_i \in \mathcal{C}(\varrho)$ itself is a manifold with boundary. This set consists of all subsets $\Gamma_i \subset \Gamma$ which connect two adjacent zeros of ϱ.

Theorem 5. *Let Γ be a one-dimensional boundary manifold and let $\boldsymbol{\tau}_0$ denote its tangential vector field. Then, let $\varrho_0 \in \boldsymbol{\Omega}^1(\Gamma)$ with transversal zeros only, i.e. if $\varrho_0(x)$ is zero, then its derivative in direction $\boldsymbol{\tau}_0$ is nonzero in x. Let $\varrho_1 = e^{\beta}\varrho_0$ for $\beta \in C^\infty(\Gamma)$, such that $\int_{\Gamma_i} \varrho_0 = \int_{\Gamma_i} \varrho_1$ hold for every $\Gamma_i \in \mathcal{C}(\varrho_0)$. Then there exists a conformal map $\Theta : \Gamma \to \Gamma$ with $\Theta^*\varrho_1 = \varrho_0$.*

Proof. Define $\varrho_t = (1-t)\varrho_0 + t\varrho_1$ for all $t \in [0, 1]$ and let $\rho_t = \star_0 \varrho_t \in C^\infty(\Gamma)$ be the corresponding Hodge dual. For every $\Gamma_i \in \mathcal{C}(\varrho_0)$ let $\omega_{\Gamma_i} \in C^\infty(\Gamma_i)$ be the solution of

$$d\omega_{\Gamma_i} = -(\varrho_1 - \varrho_0) \quad \text{on } \Gamma_i \tag{10}$$

$$\omega_{\Gamma_i}|_{\partial\Gamma_i} = 0. \tag{11}$$

Because of the volume condition $\int_{\Gamma_i} \varrho_0 = \int_{\Gamma_i} \varrho_1$, this solution exists and is unique due to a result from [13]. Let ω be the composition of all partial solutions, i.e. $\omega|_{\Gamma_i} = \omega_{\Gamma_i}$ for $\Gamma_i \in \mathcal{C}(\varrho_0)$. Then, ω is continuous on Γ due to Eq. (11) and since the derivative of ω, that is the right hand side of Eq. (10), is smooth, ω is smooth on the whole boundary Γ, i.e. $\omega \in C^\infty(\Gamma)$. Let $x_0 \in \Gamma$ be a zero of ρ_0. Then, by construction $\omega(x_0) = 0$ holds and the following limit exists:

$$\lim_{x\to x_0} \frac{\partial_{\boldsymbol{\tau}_0}\omega}{\partial_{\boldsymbol{\tau}_0}\rho_t}(x) = \lim_{x\to x_0} \frac{\star_0\, d\omega}{\partial_{\boldsymbol{\tau}_0}\rho_t}(x) = \lim_{x\to x_0} \frac{-(e^\beta\rho_0 - \rho_0)}{((1-t) + te^\beta)\partial_{\boldsymbol{\tau}_0}\rho_0 + t\partial_{\boldsymbol{\tau}_0}(e^\beta)\rho_0}(x) = 0 \tag{12}$$

because $\lim_{x\to x_0} \rho_0 = 0$ and $\lim_{x\to x_0} \partial_{\boldsymbol{\tau}_0}\rho_0 \neq 0$. Since this holds for every zero of ρ_0 we can define $\xi_t = \omega/\rho_t \in C^\infty(\Gamma)$ by L'Hôpital's rule (see [14]). Since ξ_t is also smooth in the variable t it is smooth on the compact space $[0, 1] \times \Gamma$. Then, its first derivative is bounded and therefore ξ_t is Lipschitz continuous in t and x (see [15]). This of course means that ξ_t is Lipschitz continuous in x with a uniform constant in t. Define $X_t := \xi_t\boldsymbol{\tau}_0 \in \boldsymbol{\Gamma}(T\Gamma)$ and by the Lipschitz continuity there exists a unique flow Θ_t such that $\Theta_0 = id$ and $\partial_t\Theta_t = X_t$, due to the Picard–Lindelöf Theorem (see [15]). Then, the following holds:

$$\varrho_t(X_t) = \rho_t\mu_0(\xi_t\boldsymbol{\tau}_0) = \rho_t\sqrt{g_0(\xi_t\boldsymbol{\tau}_0, \xi_t\boldsymbol{\tau}_0)} = \rho_t\xi_t = \omega. \tag{13}$$

And finally,

$$\begin{aligned}\frac{d}{dt}\Theta_t^*\varrho_t &= \Theta_t^*(\mathcal{L}_{X_t}\varrho_t + \partial_t\varrho_t) = \Theta_t^*(d\mathbf{i}_{X_t}\varrho_t + (\varrho_1 - \varrho_0)) \\ &= \Theta_t^*(d\varrho_t(X_t) + (\varrho_1 - \varrho_0)) = \Theta_t^*(d\omega + (\varrho_1 - \varrho_0)) = 0\end{aligned} \tag{14}$$

where $\mathcal{L}_{X_t}$ denotes the Lie derivative. Thus, we have created a diffeomorphism with $\Theta_t^*\varrho_t = \varrho_0$ and in particular $\Theta_1^*\varrho_1 = \varrho_0$. Since every diffeomorphism on a one-dimensional manifold is conformal, this completes the proof. □

We have prepared everything and can prove our main result characterizing the image space of **S**.

Theorem 6. *Assume that $\mathbf{S}(\Omega_0) = \boldsymbol{\tau}_0 \cdot \mathbf{u}_0|_{\Gamma_0^{in}}$ has only transversal zeros. Then, $\nu \in \mathrm{im}(\mathbf{S})$ if and only if:*

(a) $\nu = e^\beta\theta^*(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0) \in C^\infty(\Gamma_0^{in})$ *for some* $\beta \in C^\infty(\Gamma_0^{in})$, $\theta \in \mathrm{Diff}(\Gamma_0^{in})$;
(b) $\int_{\theta(\Gamma)}(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0)\mu_0 = \int_\Gamma \nu\mu_0$ *for all* $\Gamma \in \mathcal{C}(\nu)$.

Interpretation of Theorem 6: (a) assures that ν and $\boldsymbol{\tau}_0 \cdot \mathbf{u}_0$ have similar zero sets and θ pulls every zero of $\boldsymbol{\tau}_0 \cdot \mathbf{u}_0$ back to a zero of ν; (b) guarantees that $\boldsymbol{\tau}_0 \cdot \mathbf{u}_0$ and ν have the same volume between two adjacent zeros. From this theorem we see that $\mathrm{im}(\mathbf{S})$ is essentially restricted by the zero set of $\boldsymbol{\tau}_0 \cdot \mathbf{u}_0$, i.e. the set of stagnation points.

Proof. Let $\nu \in C^\infty(\Gamma_0^{in})$ fulfilling (a) and (b) be given. We have to show that $\nu \in \mathrm{im}(\mathbf{S})$. Therefore, define $\varrho_0 = \theta^* \star_0(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0|_{\Gamma_0^{in}})$ and $\varrho_1 = \star_0\, \nu$. Then, (a) leads to

$$\varrho_1 = \star_0\, \nu = \star_0\, e^\beta\theta^*(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0|_{\Gamma_0^{in}}) = e^\beta e^{-\alpha_\theta}\theta^* \star_0(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0|_{\Gamma_0^{in}}) = e^{\beta-\alpha_\theta}\varrho_0 \tag{15}$$

for $\alpha_\theta \in C^\infty(\Gamma_0^{in})$ being the conformal parameter of $\theta \in \mathrm{Diff}(\Gamma_0^{in})$, and (b) yields

$$\int_\Gamma \varrho_0 = \int_\Gamma \theta^* \star_0 (\boldsymbol{\tau}_0 \cdot \mathbf{u}_0|_{\Gamma_0^{in}}) = \int_{\theta(\Gamma)} (\boldsymbol{\tau}_0 \cdot \mathbf{u}_0)\mu_0 = \int_\Gamma \nu\mu_0 = \int_\Gamma \varrho_1 \tag{16}$$

for all $\Gamma \in \mathcal{C}(\varrho_0)$. Thus, by Theorem 5 there exists a smooth map $\Theta \in \mathrm{Diff}(\Gamma_0^{in})$ with $\Theta^*\varrho_1 = \varrho_0$. Then, $\Psi := \theta^{-1} \circ \Theta \in \mathrm{Diff}(\Gamma_0^{in})$ implies $\Psi^* \star_0 \nu = \star_0(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0)$ and, with α_0 being the conformal parameter of the boundary map Ψ, this yields on Γ_0^{in}

$$\star_0(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0) = \Psi^* \star_0 \nu = \star_{\alpha_0} \Psi^*\nu = \star_0 e^{\alpha_0}\Psi^*\nu \Rightarrow \Psi_*(e^{-\alpha_0}\boldsymbol{\tau}_0 \cdot \mathbf{u}_0) = \nu. \tag{17}$$

We have to split Ψ into a global conformal map Φ_α and a boundary isometry Λ_α. Let $\alpha \in C^\infty(\Omega_0)$ be the solution of

$$\Delta\alpha = 0 \qquad \alpha|_{\Gamma_0^{out}} = 0 \qquad \alpha|_{\Gamma_0^{in}} = \alpha_0. \tag{18}$$

Then, $\alpha \in \mathcal{A}$ and by Proposition 3 there exists a corresponding conformal map $\Phi_\alpha : \Omega_0 \to \Omega_\alpha$ and conformal domain Ω_α. Define $\Lambda_\alpha := \Phi_\alpha \circ (\Psi|_{\Gamma_0^{in}})^{-1} : \Gamma_0^{in} \to \Gamma_\alpha^{in}$, which is an isometry since for all $x \in \Gamma_0^{in}$,

$$\frac{d}{d\boldsymbol{\tau}_0}\Lambda_\alpha(x) = \partial_{\boldsymbol{\tau}_0}\Phi_\alpha(\Psi^{-1}(x))\left(\partial_{\boldsymbol{\tau}_0}\Psi(\Psi^{-1}(x))\right)^{-1} = e^\alpha(\Psi^{-1}(x))e^{-\alpha}(\Psi^{-1}(x)) = 1. \tag{19}$$

Because there exists an isometry between Γ_0^{in} and Γ_α^{in}, these boundaries have the same length and, together with $\alpha \in \mathcal{A}$, this shows that $\Omega_\alpha \in \mathcal{D}$. To prove $\nu \in \mathrm{im}(\mathbf{S})$ it remains to show that the constructed Ω_α is mapped to ν:

$$\mathbf{S}(\Omega_\alpha) = {\Lambda_\alpha}^*\Phi_{\alpha*}(e^{-\alpha}(\boldsymbol{\tau}_0 \cdot \mathbf{u}_0)|_{\Gamma_0^{in}}) = \Psi_*(e^{-\alpha_0}\boldsymbol{\tau}_0 \cdot \mathbf{u}_0) = \nu. \tag{20}$$

The "only if" part can easily be seen by setting $\theta := (\Phi_\alpha|_{\Gamma_0^{in}})^{-1} \circ \Lambda_\alpha$ and $\beta := \theta^*(-\alpha)$. □

5. Conclusion

The focus of this work is on contributing to a better understanding of shape design problems by analyzing the reachability of a shape-dependent operator. We have done this by drawing a connection between $\mathrm{im}(\mathbf{S})$ and the set of stagnation points. Basically more stagnation points lead to more restrictions and therefore a smaller image space. However, such an explicit characterization may be challenging in a general setting because our approach does strongly rely on the fact that we are using potential flow and can take advantage of the local property (cf. Eq. (9)). With the application in view, this approach can be used in a constructive way to design objects with a specific surface velocity: For a given velocity fulfilling the condition from Theorem 6 one can compute the corresponding conformal parameter α, reconstruct the conformal map Φ_α and derive the domain Ω_α which realizes the desired velocity. This can be done in a single step without iterations.

Acknowledgment

This work was supported by the German Federal Ministry of Education and Research (BMBF) grant no. 03MS606F.

References

[1] D. Chenais, E. Zuazua, Controllability of an elliptic equation and its finite difference approximation by the shape of the domain, Numerische Mathematik 95 (2003) 63–99.
[2] A. Osses, J. Puel, On the controllability of the Laplace equation observed on an interior curve, Revista Matemática Complutense 11 (1998) 403–441.
[3] O. Pironneau, Optimal Shape Design for Elliptic Systems, Springer, 1984.
[4] J. Sokolowski, J. Zolesio, Introduction to Shape Optimization: Shape Sensitivity Analysis, vol. 16, Springer-Verlag, 1992.
[5] B. Mohammadi, O. Pironneau, Applied Shape Optimization for Fluids, Oxford University Press, USA, 2001.
[6] G. Allaire, F. Jouve, A. Toader, Structural optimization using sensitivity analysis and a level-set method, Journal of Computational Physics 194 (2004) 363–393.
[7] T. Sederberg, S. Parry, Free-form deformation of solid geometric models, ACM Siggraph Computer Graphics 20 (1986) 151–160.
[8] T. Lassila, G. Rozza, Parametric free-form shape design with pde models and reduced basis method, Computer Methods in Applied Mechanics and Engineering 199 (2010) 1583–1592.
[9] R. Schinzinger, P. Laura, Conformal Mapping: Methods and Applications, Dover Pubns, 2003.
[10] I. Abbott, A. Von Doenhoff, Theory of Wing Sections: Including a Summary of Airfoil Data, Dover Pubns, 1959.
[11] R. Eppler, Airfoil Design and Data, Springer, Berlin, 1990.
[12] J. Moser, On the volume elements on a manifold, Transactions of the American Mathematical Society 120 (1965) 286–294.
[13] G. Schwarz, Hodge Decomposition—A Method for Solving Boundary Value Problems, Springer, 1995.
[14] H. Heuser, Lehrbuch der Analysis. Teil 1, in: Mathematische Leitfäden, Vieweg+Teubner Verlag, 2009.
[15] P. Hartman, Ordinary Differential Equations, in: Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, 2002.