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# A Relationship between Certain Colored Generalized Frobenius Partitions and Ordinary Partitions

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Ramanujan's congruence $p(5n+4) \equiv 0 \pmod 5$ for ordinary partitions is well-known. This congruence is just the first in a family of congruences modulo 5; namely, $p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}$ for $\alpha \geqq 1$ where $\delta_\alpha$ represents the reciprocal of 24 modulo $5^\alpha$. A similar family of congruences exists for ordinary partitions modulo 7. In this paper we prove the corresponding congruences for generalized Frobenius partitions with 5 and 7 colors modulo 5 and 7, respectively, by establishing an equality between these two classes of generalized Frobenius partitions and certain ordinary partitions. The proofs are based on some elegant identities of Ramanujan. 

In 1984 Andrews [1] introduced the idea of generalized Frobenius partitions, F-partitions for short. These are two-lined arrays

$$\begin{pmatrix} a_1 & a_2 \cdots a_r \\ b_1 & b_2 \cdots b_r \end{pmatrix},$$

where the entries in each row are nonnegative integers arranged in nonincreasing order. The number being partitioned by such an array is $n = \sum_{i=1}^{r} (a_i + b_i + 1)$. We will consider F-partitions with $k$ colors where the entries in each row are distinct and are taken from $k$ copies of the nonnegative integers distinguished by color and in each row the entries are ordered according to the rule that $x_i < y_j$ if $x < y$ or if $x = y$ and $i < j$ where $i$ and $j$ are integers in the interval $[1, k]$ indicating the color of the nonnegative integer. The number of such F-partitions of $n$ with $k$ colors is denoted by $c\phi_k(n)$ and we note that $c\phi_1(n) = p(n)$, the number of ordinary partitions of $n$. We denote the number of F-partitions of $n$ using $k$ colors whose order is $k$ under cyclic permutation of the $k$ colors by $\overline{c\phi_k}(n) = \sum_{d \mid (k,n)} \mu(d)\, c\phi_{k/d}(n/d)$.

In this paper we will prove the following theorem which establishes a relationship between colored F-partitions with 5 and 7 colors and ordinary partitions.



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THEOREM. $\overline{c\phi_5}(n) = 5p(5n-1)$ *and* $\overline{c\phi_7}(n) = 7p(7n-2)$ *for* $n$ *a positive integer.*

As an immediate corollary we will have

COROLLARY. *For* $\alpha$ *a positive integer,* $c\phi_5(5^{\alpha-1}n + (\delta_\alpha + 1)/5) \equiv 0 \pmod{5^{\alpha+1}}$ *and* $c\phi_7(7^{\alpha-1}n + (\lambda_\alpha + 2)/7) \equiv 0 \pmod{7^{[(\alpha+4)/2]}}$ *where* $\delta_\alpha$ *and* $\lambda_\alpha$ *are the reciprocals of* 24 *modulo* $5^\alpha$ *and* $7^\alpha$, *respectively.*

The corollary follows from the fact that $p(5^\alpha n + \delta_\alpha) \equiv 0 \pmod{5^\alpha}$ and $p(7^\alpha n + \lambda_\alpha) \equiv 0 \pmod{7^{[(\alpha+2)/2]}}$. The former congruence was conjectured by Ramanujan in 1919 and was proved by G. N. Watson [6] in 1938. The latter congruence is a minor variation of a conjecture by Ramanujan and was also proved by Watson [6].

The theorem is based on the following two lemmas

LEMMA 1.

$$(q; q)^5 \sum_{n=0}^{\infty} c\phi_5(n)\, q^n = 1 + 25 \sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{q^r}{(1-q^r)^2} - 5 \sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{rq^r}{1-q^r}$$

LEMMA 2.

$$(q; q)^7 \sum_{n=0}^{\infty} c\phi_7(n)\, q^n = 1 + \frac{343}{8} \sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{q^r + q^{2r}}{(1-q^r)^3} - \frac{7}{8} \sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{r^2 q^r}{1-q^r}$$

*and the following identities due to Ramanujan* [2, 4]

$$\sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{q^r}{(1-q^r)^2} = q \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty}$$

$$1 - 5 \sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{rq^r}{1-q^r} = \frac{(q; q)_\infty^5}{(q^5; q^5)_\infty}$$

$$\sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{q^r + q^{2r}}{(1-q^r)^3} = q(q; q)_\infty^3\, (q^7; q^7)_\infty^3 + 8q^2 \frac{(q^7; q^7)_\infty^7}{(q; q)_\infty}$$

$$7 \sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{r^2 q^r}{1-q^r} = 8 - 8 \frac{(q; q)_\infty^7}{(q^7; q^7)_\infty} - 49q(q; q)_\infty^3\, (q^7; q^7)_\infty^3.$$

To prove Lemmas 1 and 2 we begin by looking at the generating functions for $c\phi_5(n)$ and $c\phi_7(n)$ in the form

$$\sum_{n=0}^{\infty} c\phi_k(n)\, q^n = \frac{\sum_{n=0}^{\infty} r_k(2n, 0)\, q^n}{(q; q)_\infty^k},$$

where $r_k(2n, 0)$ = the number of solutions of $n = \sum_{1 \leqslant i \leqslant j \leqslant k-1} x_i x_j$ [1]. For $n > 0$, $r_5(2n, 0) = 25n(1 - (n_0/5)\, 5^{-\beta}) \sum_{d \mid n} (d/5)(1/d)$ where $n = 5^{\beta-1} n_0$ with $(n_0, 5) = 1$ and $r_7(2n, 0) = \frac{343}{8} n^2 (1 - (n_0/7)\, 7^{-2\beta}) \sum_{d \mid n} (d/7)(1/d^2)$ where $n = 7^{\beta-1} n_0$ with $(n_0, 7) = 1$ [4].

Hence

$$\begin{aligned}
(q;q)_\infty^5 \sum_{n=0}^{\infty} c\phi_5(n)\, q^n
&= 1 + \sum_{\beta=1}^{\infty} \sum_{\substack{m=1 \\ (m,5)=1}}^{\infty} q^{5^{\beta-1} m} 5^{\beta+1} m \left(1 - \left(\frac{m}{5}\right) 5^{-\beta}\right) \sum_{d \mid 5^{\beta-1} m} \left(\frac{d}{5}\right) \frac{1}{d} \\
&= 1 + 25 \sum_{n=1}^{\infty} n q^n \sum_{d \mid n} \left(\frac{d}{5}\right) \frac{1}{d} - 5 \sum_{\beta=1}^{\infty} \sum_{m=1}^{\infty} q^{5^{\beta-1} m} m \left(\frac{m}{5}\right) \sum_{d \mid m} \left(\frac{d}{5}\right) \frac{1}{d} \\
&= 1 + 25 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} r s q^{rs} \left(\frac{r}{5}\right) \frac{1}{r} - 5 \sum_{\beta=1}^{\infty} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} q^{5^{\beta-1} rs} rs \left(\frac{rs}{5}\right)\left(\frac{s}{5}\right) \frac{1}{s} \\
&= 1 + 25 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} s q^{rs} \left(\frac{r}{5}\right) - 5 \sum_{\beta=1}^{\infty} \sum_{\substack{s=1 \\ (s,5)=1}}^{\infty} \sum_{r=1}^{\infty} q^{5^{\beta-1} rs}\, r \left(\frac{r}{5}\right) \\
&= 1 + 25 \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} s q^{rs} \left(\frac{r}{5}\right) - 5 \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} q^{nr}\, r \left(\frac{r}{5}\right) \\
&= 1 + 25 \sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{q^r}{(1-q^r)^2} - 5 \sum_{r=1}^{\infty} \left(\frac{r}{5}\right) \frac{r q^r}{1-q^r}.
\end{aligned}$$

Similarly we can show that

$$(q;q)_\infty^7 \sum_{n=0}^{\infty} c\phi_7(n)\, q^n = 1 + \frac{343}{8} \sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{q^r + q^{2r}}{(1-q^r)^3} - \frac{7}{8} \sum_{r=1}^{\infty} \left(\frac{r}{7}\right) \frac{r^2 q^r}{1-q^r}.$$

Furthermore some additional results [5]

$$\frac{1}{(q^5;q^5)_\infty} = \sum_{n=0}^{\infty} p(n)\, q^{5n} \qquad \text{and} \qquad 25q \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} = \sum_{n=1}^{\infty} 5p(5n-1)\, q^n$$

$$\frac{1}{(q^7;q^7)_\infty} = \sum_{n=0}^{\infty} p(n)\, q^{7n}$$

and

$$49q \frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^4} + 343q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8} = \sum_{n=1}^{\infty} 7p(7n-2)\, q^n$$

provide the connection with the ordinary partition functions necessary to establish our theorem.

Combining Lemmas 1 and 2, the four identities due to Ramanujan, and the four results just stated we have

$$\sum_{n=1}^{\infty} \left( c\phi_5(n) - p\left(\frac{n}{5}\right)\right) q^n = \sum_{n=1}^{\infty} 5p(5n-1)\, q^n$$

and

$$\sum_{n=1}^{\infty} \left( c\phi_7(n) - p\left(\frac{n}{7}\right)\right) q^n = \sum_{n=1}^{\infty} 7p(7n-2)\, q^n,$$

where $p(n/5)$ and $p(n/7)$ are zero if $n/5$ and $n/7$, respectively, are not integers. Therefore we conclude that $\overline{c\phi_5}(n) = c\phi_5(n) - p(n/5) = 5p(5n-1)$ and $\overline{c\phi_7}(n) = c\phi_7(n) - p(n/7) = 7p(7n-2)$ for all positive integers $n$.

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