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On the spectrum of certain subschemes of $\mathbb{P}^N$

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Abstract

We study subschemes of $\mathbb{P}^N$ over an algebraically closed field k which correspond to equidimensional Cohen–Macaulay k-algebras A such that A modulo the subalgebra generated by the elements of degree one has finite length. We show that the h-vector of these algebras satisfies a number of conditions, as suggested by the theory of the spectrum for torsion free sheaves on $\mathbb{P}^3$.

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1. Introduction

Let A be a graded finitely generated algebra over a field k. If A happens to be Cohen–Macaulay, its Hilbert function is determined by a finite sequence of positive integers sometimes called the h-vector of A [3, 12]. A is called homogeneous if it is generated over k by its elements of degree one; the set of sequences of integers which occur as h-vectors of some homogeneous Cohen–Macaulay k-algebra is described by a famous theorem of Macaulay.

In geometry, though, one often deals with Cohen–Macaulay algebras which are not quite homogeneous; for example, if C is a pure dimensional locally Cohen–Macaulay curve in the projective space $\mathbb{P}^N$, the ring

$$A_C = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(C, \mathcal{O}_C(n))$$

is Cohen–Macaulay, but not homogeneous in general.

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In this paper we study those subschemes X of $\mathbb{P}^N$ for which the ring

$$A_X = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}^0(X, \mathcal{O}_X(n))$$

is equidimensional and Cohen–Macaulay, together with the Hilbert function of A_X. We call such subschemes quasi arithmetically Cohen–Macaulay (in short quasi ACM). The corresponding algebraic notion is that of an equidimensional Cohen–Macaulay k-algebra A such that A modulo its subalgebra generated by the elements of degree one has finite length. We define the spectrum of a quasi ACM scheme X to be the h-vector of A_X.

The *index of speciality* $e(X)$ of a quasi ACM subscheme $X \subseteq \mathbb{P}^N$ is the largest integer n such that $\mathrm{H}^0(X, \omega_X(-n)) \neq 0$, where ω_X is the Grothendieck dualizing sheaf of X.

The Rao module M_X of X is defined as $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}^1(\mathbb{P}^N, \mathcal{I}_X(n))$. For X quasi ACM, M_X is a graded module of finite length. We denote by $\mu_X(n)$ the number of minimal generators in degree n of the Rao module over the coordinate ring R, that is:

$$\mu_X(n) = \dim(M_X \otimes_R R/\mathfrak{m}_R)_n.$$

The main result of this paper is the following theorem.

Theorem 1.1 (cf. Theorem 3.2 below). *Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$ and index of speciality e. Then $e + t + 1 \geq 0$ and the spectrum of X has the following properties:*

1. *$h_X(n) \geq 1 + \mu_X(n)$ for $0 \leq n \leq e(X) + t + 1$.*
2. *Suppose there is an integer l satisfying $1 \leq l \leq e + t + 1$ such that $h_X(l) = 1 + \mu_X(l)$. Then*

$$h_X(n) = 1 \quad \text{and} \quad \mu_X(n) = 0 \quad \text{for } l \leq n \leq e + t + 1.$$

   Furthermore, if $l \leq e + t$, there is a linear subspace $M \subseteq \mathbb{P}^N$ of dimension $t + 1$ such that X contains a hypersurface in M of degree $e + t + 2$.
3. *$h_X(n) = \mu_X(n) = 0$ for $n \geq e + t + 2$.*

Theorem 1.1 is modeled after the corresponding result for the spectrum of torsion-free sheaves on $\mathbb{P}^3$, which has a rather long history: see [2, 4, 5, 8–10]; for curves in $\mathbb{P}^3$, Theorem 1.1 follows from the results of [8] if one deletes $\mu_X(n)$ from the statement, but even for curves in $\mathbb{P}^N$ it seems new. For curves in $\mathbb{P}^3$, these properties of the spectrum have interesting applications [7, 11].

2. Quasi ACM subschemes of $\mathbb{P}^N$

In this section we introduce the class of quasi arithmetically Cohen–Macaulay subschemes of $\mathbb{P}^N$.

We use the following notation: we write $\mathbb{P}^N$ for the N-dimensional projective space over an algebraically closed field k. R denotes the homogeneous coordinate ring of $\mathbb{P}^N$, so that $R \cong k[x_0, \ldots, x_N]$. Given a linear subspace $L \subseteq \mathbb{P}^N$ of codimension $t+1$, we define S_L to be the polynomial subring of R generated by the linear forms vanishing on L. Thus, if $y_0 = \cdots = y_t = 0$ are equations for L, we have $S_L \cong k[y_0, \ldots, y_t]$.

If $\mathscr{F}$ is a quasi-coherent sheaf on a closed subscheme $X \subseteq \mathbb{P}^N$, we define

$$\mathrm{H}^i_*(X, \mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{H}^i(X, \mathscr{F}(n))$$

and

$$h^i(X, \mathscr{F}) = \dim_k \mathrm{H}^i(X, \mathscr{F}).$$

Given a closed subscheme $X \subseteq \mathbb{P}^N$, we will be looking at the ring $\mathrm{H}^0_*(\mathbb{P}^N, \mathcal{O}_X)$. This ring is finitely generated over R if and only if $\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0$ for $n \ll 0$, and this is the case if X has no associated points of dimension zero:

Lemma 2.1. *Let X be a closed subscheme of $\mathbb{P}^N$. If X has no associated point of dimension zero, then*

$$\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \quad \textit{for } n \ll 0.$$

Proof. Assume no associated point of X has dimension zero, and let

$$r = \dim \mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X).$$

We claim that $\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0$ for $n \leq -r$. If ξ is an associated point of X, by assumption $\dim \xi \geq 1$, hence the set of hyperplanes containing ξ has codimension at least two in $(\mathbb{P}^N)^\vee$, the projective space dual to $\mathbb{P}^N$. So if D is a general line in $(\mathbb{P}^N)^\vee$, no associated point of X is contained in a hyperplane belonging to D. It follows that, given a general two-dimensional space of linear forms V in R, the multiplication map:

$$x : \mathcal{O}_X(-1) \to \mathcal{O}_X$$

is injective for every $x \in V$.

Now let $E \subseteq \mathbb{P}^N$ be the line defined by V. Every homogeneous polynomial in $S_E = Sym(V)$ is a product of linear forms in V. Hence for every $\alpha \in \mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n))$ the map $S_E \to A_X$ which sends one to α is injective. Now assume by way of contradiction that there is a nonzero element $\alpha \in \mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n))$ for some $n \leq -r$. Then

$$\dim \mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X) \geq \dim(S_E)_{-n} = 1 - n \geq 1 + r$$

which contradicts the definition of r. Hence $\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0$ for $n \leq -r$ and we are done. □

We follow the terminology of [3] for the commutative algebra we use. In particular, if $R = k[x_0, \ldots, x_N]$ as above, and $\mathfrak{m}_R$ is the homogeneous maximal ideal of R, a nonzero finitely generated *graded* R-module M is Cohen–Macaulay if and only if

$$\operatorname{depth}_{\mathfrak{m}_R} M = \dim M,$$

that is, if and only if $\mathfrak{m}_R$ contains an M-regular sequence of length equal to $\dim M$ (see [3, Proposition 1.5.15, Exercise 2.1.27]).

Recall that a subscheme $X \subseteq \mathbb{P}^N$ is called arithmetically Cohen–Macaulay (in short ACM) if its coordinate ring R_X is Cohen–Macaulay.

Definition 2.2. Let X be a closed subscheme of $\mathbb{P}^N$ of positive dimension. We let A_X denote the ring $\mathsf{H}^0_*(\mathbb{P}^N, \mathcal{O}_X)$. We say that X is quasi arithmetically Cohen–Macaulay (in short quasi ACM) if the ring A_X is a finitely generated Cohen–Macaulay R-module.

Remark 2.3. An ACM subscheme $X \subseteq \mathbb{P}^N$ of positive dimension is quasi ACM.

To see this we consider the exact sequence:

$$0 \to \mathsf{H}^0_{\mathfrak{m}_R}(R_X) \to R_X \to A_X \to \mathsf{H}^1_{\mathfrak{m}_R}(R_X) \to 0$$

where $\mathsf{H}^j_{\mathfrak{m}_R}$ denotes the jth local cohomology group with respect to the irrelevant maximal ideal $\mathfrak{m}_R$. Since X is ACM of positive dimension, the ring R_X is Cohen–Macaulay of dimension at least 2. Therefore

$$\mathsf{H}^0_{\mathfrak{m}_R}(R_X) = \mathsf{H}^1_{\mathfrak{m}_R}(R_X) = 0.$$

We conclude that A_X is isomorphic to R_X, and hence that X is quasi ACM.

More generally, we see that a quasi ACM subscheme of positive dimension is ACM if and only if R_X has depth at least 2 over $\mathfrak{m}_R$, that is, if and only if

$$\mathsf{H}^1_{\mathfrak{m}_R}(R_X) \cong \mathsf{H}^1_*(\mathbb{P}^N, \mathcal{J}_X) = 0.$$

Given a linear subspace $L \subseteq \mathbb{P}^N$, S_L denotes as above the polynomial subring of R generated by the linear equations of L, and we can consider A_X as an S_L-module.

Proposition 2.4. *Let X be a closed subscheme of $\mathbb{P}^N$ of dimension $t \geq 1$. The following conditions are equivalent:*

1. *X is equidimensional and A_X is a Noetherian Cohen–Macaulay ring.*
2. *X is quasi ACM.*
3. *We have*

$$\mathsf{H}^i_*(X, \mathcal{O}_X) = 0 \quad \text{for } 1 \leq i \leq t-1,$$
$$\mathsf{H}^0(X, \mathcal{O}_X(n)) = 0 \quad \text{for } n \ll 0.$$

4. *A_X is a free finitely generated graded S_L-module for some (resp. every) linear subspace $L \subseteq \mathbb{P}^N$ of codimension $t+1$ which does not meet X.*

Proof. Since $\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_{\mathbb{P}^N}$-modules, any one of the conditions in the statement implies that A_X is finitely generated over R. Hence A_X is Cohen–Macaulay as an R-module if and only if it is Cohen–Macaulay as an A_X-module and for all homogeneous maximal ideals $\mathfrak{m}$ of A_X we have $\dim (A_X)_{\mathfrak{m}} = \dim_R A_X$. The latter condition means that X is equidimensional, therefore 1 and 2 are equivalent.

Next assume that condition 2 holds. Then A_X is finitely generated over R, hence

$$\mathrm{H}^0(X, \mathcal{O}_X(n)) = 0 \quad \text{for } n \ll 0$$

and

$$\dim A_X = t + 1.$$

For $i \geq 1$ we have isomorphisms

$$\mathrm{H}^i_*(X, \mathcal{O}_X) \cong \mathrm{H}^{i+1}_{\mathfrak{m}_R}(A_X).$$

Since A_X is Cohen–Macaulay of dimension $t+1$, $\mathrm{H}^j_{\mathfrak{m}_R}(A_X) = 0$ for $j \leq t$. Hence

$$\mathrm{H}^i_*(X, \mathcal{O}_X) = 0 \quad \text{for } 1 \leq i \leq t-1.$$

Thus condition 2 implies condition 3.

Now assume 3 holds and let L be a linear subspace of codimension $t+1$ which does not meet X. Consider the projection from L:

$$\phi : X \to \mathbb{P}^t.$$

Since X does not meet L, ϕ is well-defined and has finite fibers. By Stein's factorization ϕ is a finite morphism. Hence, if we set $\mathcal{G} = \phi_* \mathcal{O}_X$, $\mathcal{G}$ is a coherent sheaf on $\mathbb{P}^t$ and there are isomorphisms

$$\mathrm{H}^i(\mathbb{P}^t, \mathcal{G}(n)) \cong \mathrm{H}^i(X, \mathcal{O}_X(n)).$$

In particular, there is an isomorphism of S-modules

$$A_X \cong \mathrm{H}^0_*(\mathbb{P}^t, \mathcal{G}).$$

Now using local cohomology with respect to $\mathfrak{m}_S$ and reversing the argument above we see that A_X is a finitely generated S-module of depth $t+1$. Since S has dimension $t+1$, the graded version of the Auslander–Buchsbaum Theorem tells us that A_X is projective over S, and using Nakayama's Lemma we see that A_X is in fact a free finitely generated S-module. Thus 3 implies 4.

Next assume that 4 holds for some L not meeting X. Then the map $S_L \to A_X$ is finite and flat, hence A_X is Cohen–Macaulay and equidimensional, so 1 holds. □

Remark 2.5. It follows that the property of being quasi ACM depends on X and on the very ample sheaf $\mathcal{O}_X(1)$, but not on the linear system which defines the embedding in $\mathbb{P}^N$.

Corollary 2.6. *Let X be a quasi ACM closed subscheme of $\mathbb{P}^N$. Then X is locally Cohen–Macaulay and equidimensional.*

Corollary 2.7. *Let X be a closed subscheme of $\mathbb{P}^N$ of dimension one. The following conditions are equivalent*:

1. *X is quasi ACM.*
2. *X is locally Cohen–Macaulay and equidimensional.*
3. *Every associated point of X has dimension one.*

Proof. If X is quasi ACM, by Corollary 2.6 X is locally Cohen–Macaulay and equidimensional, and this implies that every associated point of X has dimension one.

Suppose now that every associated point of X has dimension one. By Proposition 2.4 to show that X is quasi ACM it is enough to prove that

$$\mathrm{H}^0(\mathbb{P}^N, \mathcal{O}_X(n)) = 0 \quad \text{for } n \ll 0,$$

and this follows from Lemma 2.1. □

Example 2.8. We can construct subschemes X of arbitrary dimension which are quasi ACM , but not ACM, by projecting ACM subschemes to a lower dimensional projective space. This way we obtain subschemes which are not linearly normal; however the class of quasi ACM subscheme is stable under biliaison (see [6] for the theory of biliason), so we can bilink them up until we obtain linearly normal quasi ACM subschemes which are not ACM.

We are now going to define the spectrum of a quasi ACM subscheme X as the h-vector of the Cohen–Macaulay ring A_X. We recall first what this means.

Given a numerical function $f:\mathbb{Z}\to\mathbb{Z}$, we define its first difference function ∂f by the formula:

$$\partial f(n) = f(n) - f(n-1)$$

and by induction we let

$$\partial^r f = \partial\partial^{r-1} f \quad \text{for } r \geq 2.$$

Proposition 2.9. *Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension t. Denote by*

$$a_n = \dim_k(A_X)_n$$

the Hilbert function of A_X.

There exists a unique sequence of nonnegative integers $\{h_X(n)\}$ with the following properties:

1. *For some (resp. every) linear subspace $L \subseteq \mathbb{P}^N$ of codimension $t+1$ which does not meet X, we have an isomorphism of S_L-modules*

$$A_X \cong \bigoplus_{n\in\mathbb{Z}} S_L(-n)^{h_X(n)}.$$

2. *We can express the Hilbert function in terms of* $\{h_X(n)\}$:

$$a_n = \sum_{k \le n} \binom{n-k+t}{t} h_X(k).$$

3. *Conversely, the Hilbert function determines the sequence* $\{h_X(n)\}$ *via the formula*

$$h_X(n) = \partial^{t+1} a_n.$$

Furthermore, only finitely many among the integers $h_X(n)$ *are nonzero.*

Proof. Uniqueness is obvious because of 3. We now show existence. Fix a linear subspace $L \subseteq \mathbb{P}^N$ of codimension $t+1$ which does not meet X. By Proposition 2.4 A_X is a free finitely generated graded S_L-module, hence we have

$$A_X \cong \bigoplus_{n \in \mathbb{Z}} S_L(-n)^{h_X(n)}$$

for some sequence of nonnegative integers $\{h_X(n)\}$ satisfying $h_X(n) = 0$ except for finitely many values of n. Since

$$\dim(S_L)_n = \begin{cases} \binom{n+t}{t} & \text{for } n \ge 0 \\ 0 & \text{for } n < 0 \end{cases}$$

we find that

$$a_n = \sum_{k \le n} \binom{n-k+t}{t} h_X(k).$$

Differentiating this we obtain

$$h_X(n) = \partial^{t+1} a_n$$

and we are done. □

The sequence $\{h_X(n)\}$ is essentially what we will call the spectrum of X. According to the terminology of [13] we should call it the *h-vector* of the Cohen–Macaulay ring A_X. For convenience of notation, we prefer to think the spectrum as a set of integers with multiplicities, in which the integer n occurs $h_X(n)$ times. We will use the notation $\{n^{b(n)}\}$ for the set of integers with multiplicities B in which n occurs $b(n)$ times. If B is finite, we will also write

$$B = \{n_1^{b(n_1)}, \ldots, n_r^{b(n_r)}\}$$

where $n_1, \ldots, n_r$ are the integers for which $b(n) > 0$. Finally we suppress the "exponent" $b(n)$ if $b(n) = 1$. For example, we write $\{0, 1^2, 2^3\}$ for the set $\{0, 1, 1, 2, 2, 2\}$.

Definition 2.10. Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme and let h_X be the sequence of integers associated to X as in Proposition 2.9. The set of integers with multiplicities $\{n^{h_X(n)}\}$ is called the spectrum of X and is denoted by the symbol sp_X.

Example 2.11. If X is connected and reduced or if X is arithmetically Cohen–Macaulay, then 0 occurs exactly once in sp_X and every other integer in the spectrum is positive. Indeed, under these assumptions $\dim(A_X)_n$ equals zero for $n<0$ and one for $n=0$. Similarly, if X is reduced, every integer in sp_X is nonnegative.

Example 2.12. On the other hand, the spectrum of a nonreduced scheme may contain negative integers. The simplest example is given by double structures on a line in $\mathbb{P}^3$. We write $R=k[x,y,z,w]$ for the coordinate ring of $\mathbb{P}^3$. Let C be a double structure on the line $z=w=0$ with arithmetic genus $-r\leq 0$. The homogeneous ideal of C is generated by $z^2, zw, w^2, zf+wg$ where $f,g\in k[x,y]$ are homogeneous polynomials of degree r, forming a regular sequence if $r>0$ (and not both zero if $r=0$). One can check that A_C is a free graded $k[x,y]$-module with generators 1 and $z/g=-w/f$. Hence $sp_C=\{0,1-r\}$.

Example 2.13. If X has positive dimension and X is the disjoint union of Y and Z, then the spectrum of X is the union of the spectra of Y and Z, that is,

$$h_X(n)=h_Y(n)+h_Z(n).$$

This follows from the fact that A_X is the direct sum of A_Y and A_Z. In particular, the spectrum of X is the union of the spectra of the connected components of X.

Proposition 2.14. *Let X be a quasi ACM subscheme of $\mathbb{P}^N$ of dimension t with spectrum $sp_X=\{n^{h_X(n)}\}$. Then the Hilbert polynomial $P_X(n)$ of X is given by the formula*:

$$P_X(n)=\sum_{k\in\mathbb{Z}}\binom{n-k+t}{t}h_X(k).$$

In particular

$$\deg X=\sum_{n\in\mathbb{Z}}h_X(n),$$

while for the arithmetic genus $g(X)$ we have

$$(-1)^t g(X)=\sum_{k\in\mathbb{Z}}\binom{t-k}{t}h_X(k)-1.$$

Proof. For $n\gg 0$ we have by Proposition 2.9

$$\dim(A_X)_n=\sum_{k\in\mathbb{Z}}\binom{n-k+t}{t}h_X(k).$$

Since the right-hand side is a polynomial in n and $\dim(A_X)_n = P_X(n)$ for $n \gg 0$, we see that the right-hand side is the Hilbert polynomial of X. Its leading term is

$$\sum_{k\in\mathbb{Z}} h_X(k)\frac{n^t}{t!}.$$

Therefore $\deg X = \sum h_X(n)$. By definition of the arithmetic genus, we have

$$(-1)^t g(X) = P_X(0) - 1 = \sum_{k\in\mathbb{Z}} \binom{t-k}{t} h_X(k) - 1.$$

Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$. The canonical module of X is by definition the graded A_X module

$$\Omega_X = Ext_R^{N-t}(A_X, R(-N-1)).$$

Note that Ω_X is isomorphic to $\mathrm{H}^0_*(X, \omega_X)$ where

$$\omega_X \cong \mathscr{E}xt^{N-t}_{\mathbb{P}^N}(\mathscr{O}_X, \mathscr{O}_{\mathbb{P}^N}(-N-1))$$

is the Grothendieck dualizing sheaf of X.

Proposition 2.15. *Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$, and let L be a linear subspace of codimension $t+1$ which does not meet X. Then there is an isomorphism of A_X-modules*

$$\Omega_X \cong \mathrm{Hom}_{S_L}(A_X, S_L(-t-1)).$$

In particular

$$h^0(X, \omega_X(n)) = \sum_{k\leq n} \binom{n-k+t}{t} h_X(t+1-k).$$

Proof. Let $S = S_L$. The A_X-module structure on $\mathrm{Hom}_S(A_X, S(-t-1))$ is defined by

$$(a\phi)(b) = \phi(ab)$$

for $a \in A_X$, $\phi \in \mathrm{Hom}_S(A_X, S(-t-1))$ and $b \in A_X$. The statement follows from Proposition 3.6.12 in [3]: A_X is a direct sum of finitely many *local R-algebras A_i's, which correspond to the connected components of X, all of dimension $t+1$ and finitely generated as S-modules. It is then enough to prove the statement for the A_i's:

$$Ext_R^{N-t}(A_i, R(-N-1)) \cong \mathrm{Hom}_S(A_i, S(-t-1)).$$

To see this, apply Proposition 3.6.12 in [3] to the finite maps of *local rings $S \to A_i$ and $R \to A_i$. □

The following proposition reduces the study of the spectrum of an arbitrary quasi ACM subscheme to the one-dimensional case.

Proposition 2.16. *Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 2$, let H be a hyperplane meeting X properly, and let Y be the scheme theoretic intersection $X \cap H$. Then Y is quasi ACM, and X and Y have the same spectrum.*

Proof. Let h be the equation of H: since H meets X properly and X is locally Cohen–Macaulay, h is not a zero divisor for $\mathcal{O}_X$ and we have an exact sequence:

$$0 \to \mathcal{O}_X(-1) \xrightarrow{h} \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Now use 3 in Proposition 2.4 to see that Y is quasi ACM, and that $A_Y \cong A_X/hA_X$; hence X and Y have same spectrum. □

3. Properties of the spectrum

The main result in this section is Theorem 3.2 which describes some fundamental properties of the spectrum of a quasi ACM subscheme X of $\mathbb{P}^N$. This theorem is modeled after corresponding results from the theory of the spectrum for torsion-free sheaves on $\mathbb{P}^3$; in particular, the hard part of the theorem has its origin in [5, Proposition 5.1], and [8, Theorem 4.1]; the latter implies our statement, except for the bounds on the number of generators of the Rao module, in the special case of a curve X in $\mathbb{P}^3$. Note that no such theory has been developed for sheaves on $\mathbb{P}^N$.

Definition 3.1. We define the *index of speciality* $e(X)$ of a quasi ACM subscheme $X \subseteq \mathbb{P}^N$ to be the largest integer n such that $\mathrm{H}^0(X, \omega_X(-n)) \neq 0$, or equivalently, such that Ω_X is nonzero in degree $-n$.

The Rao module M_X of X is defined as $A_X/R_X \cong \mathrm{H}^1_*(\mathbb{P}^N, \mathcal{J}_X)$. For X quasi ACM, M_X is a graded module of finite length. We denote by $\mu_X(n)$ the number of minimal generators in degree n of the Rao module over the coordinate ring R, that is:

$$\mu_X(n) = \dim(M_X \otimes_R R/m_R)_n.$$

Theorem 3.2. *Let $X \subseteq \mathbb{P}^N$ be a quasi ACM subscheme of dimension $t \geq 1$ and index of speciality e. Then $e+t+1 \geq 0$ and the spectrum of X has the following properties*:

1. $h_X(n) \geq 1 + \mu_X(n)$ for $0 \leq n \leq e(X) + t + 1$.
2. *Suppose there is an integer l satisfying $1 \leq l \leq e+t+1$ such that $h_X(l) = 1 + \mu_X(l)$. Then*

   $$h_X(n) = 1 \quad \textit{and} \quad \mu_X(n) = 0 \quad \textit{for} \quad l \leq n \leq e+t+1.$$

   Furthermore, if $l \leq e+t$, there is a linear subspace $M \subseteq \mathbb{P}^N$ of dimension $t+1$ such that X contains a hypersurface in M of degree $e+t+2$.
3. $h_X(n) = \mu_X(n) = 0$ for $n \geq e+t+2$.

To prove Theorem 3.2 we need the following lemma:

Lemma 3.3. *Let X be a quasi ACM subscheme of $\mathbb{P}^N$ of dimension t, let L be a linear subspace of codimension $t+1$ which does not meet X and let $S = S_L$. For a subset U of the set of minimal generators of A_X over S define $\hat{U}$ to be the S-submodule of A_X generated by U. Assume that*
- *$\hat{U}$ is an R-submodule of A_X.*
- *$\hat{U}$ contains the image of $1 \in R$.*

Then $\hat{U} = A_X$, that is, U is a basis for A_X over S.

Proof. Suppose that U is not a basis. Then there is a minimal generator g of A_X over S independent from U. We can find a positive integer N such that $x^N g$ maps to zero in the Rao module, hence $x^N g$ belongs to the image of R/I_X and therefore to $\hat{U}$, since the latter is an R-submodule containing 1. But then there is a relation over S between U and g, a contradiction. □

Proof of Theorem 3.2. Fix a linear space $L \subseteq \mathbb{P}^N$ of codimension $t+1$ which does not meet X and let $S = S_L$. First note that $h_X(n) \geq \mu(n)$ for all n: indeed, $h_X(n)$ is the number of generators in degree n of A_X over S, which are at least as many as the generators in degree n of A_X over R. Since M_X is a quotient of A_X, we see that $h_X(n) \geq \mu(n)$ for all n. The last statement in the theorem follows from Proposition 2.15 which implies that $e(X)+t+1$ is the largest integer n such that $h_X(n) > 0$. Thus to show that $e(X)+t+1 \geq 0$, it is enough to prove that $h_X(0) \geq 1+\mu(0)$. Now the constant function one is a minimal generator of A_X over R (if it were a combination of elements of negative degrees, it would be nilpotent, which is nonsense), which maps to zero in the Rao module, hence A_X has at least $1+\mu(0)$ minimal generators over R in degree zero, that is,

$$h_X(0) \geq 1 + \mu(0).$$

Next we consider

$$B = A_X \otimes_S k$$

which is a finite length module over $T = R \otimes_S k$; if we write B_n for the nth-graded piece of B and we let $v(n)$ denote the dimension of the image of the multiplication map $B_{n-1} \otimes T_1 \to B_n$, we have for $n \geq 1$

$$h_X(n) = \dim_k B_n = v(n) + \mu(n). \tag{1}$$

To prove the first statement of the theorem, we have to show that $v(n) \geq 1$ for $1 \leq n \leq e+t+1$. Suppose by way of contradiction that $v(n) = 0$ for some n with $1 \leq n \leq e+t+1$. Then the S-submodule of A_X generated by the elements of degree less than or equal to $n-1$ is an R-submodule of A_X; since $n-1 \geq 0$, it follows from Lemma 3.3 that A_X is generated over S by elements of degree less or equal to $n-1$, hence $h_X(k) = 0$ for $k \geq n$. This is impossible because $h_X(e+t+1) > 0$.

The proof of the second statement of the theorem is more complicated and we break it into several steps. Assume that $h_X(l) = 1+\mu(l)$ for some l satisfying $1 \leq l \leq e+t+1$.

Then by Eq. (1) we have

$$\dim_k \operatorname{Image}(B_{l-1} \otimes T_1 \to B_l) = 1. \tag{2}$$

We define H to be the R submodule of A_X generated by the homogeneous elements of degree $\leq l-1$, and we let

$$D = \mathsf{H} \otimes_S k.$$

Note that the inclusion $\mathsf{H} \hookrightarrow A_X$ induces a map $D \to B$ which is an isomorphism in degrees $\leq l-1$ because H and A_X are equal in degrees $\leq l-1$.

Step 1.

$$\dim_k D_l = 1 \quad \textit{and} \quad D_l \hookrightarrow B_l.$$

Proof. By (2) we have $\dim_k D_l \leq 1$. If we had $D_l = 0$, then the S-submodule of A_X generated by the homogeneous elements of degree $\leq l-1$ would be all of H and by Lemma 3.3 we would have $\mathsf{H} = A_X$. But then $h_X(l) = 0$, contradicting the first statement of the theorem. Hence $\dim_k D_l = 1$. The map $D_l \to B_l$ is injective because H and A_X coincide in degrees $\leq l-1$.

Step 2. *There is an integer $a \geq 0$ such that*

$$\dim_k D_n = \begin{cases} 1 & \textit{if } l \leq n \leq l+a, \\ 0 & \textit{if } n \geq l+a+1. \end{cases}$$

Proof. Since $\dim_k D_l = 1$, if $z \in R_1$ is the preimage of a general linear form in T_1, we have

$$D_l = z\, D_{l-1}.$$

Therefore $zD_l = z\, T_1\, D_{l-1} = T_1\, D_l = D_{l+1}$ as D is generated over T by its elements of degree $\leq l-1$. By induction we see that $z\, D_{n-1} = D_n$ for all $n \geq l$. This proves our claim.

We fix such a linear form z through the rest of the proof.

Step 3. *Assume that $a = 0$, i.e., that $D_{l+1} = 0$. Then $l = e+t+1$ and $h_X(e+t+1) = 1$ and $\mu(e+t+1) = 0$.*

Proof. By Step 1 D is a submodule of B and hence H is a free S-submodule of A_X. By Lemma 3.3 it follows that $\mathsf{H} = A$ and $D = B$. Hence $h_X(n) = \dim_k D_n$ for all n. Since $a = 0$, we see that $h_X(l) = 1$ and $h_X(l+1) = 0$. Now the first statement of the theorem gives the desired conclusion.

From now on we assume $a \geq 1$: by Step 3 this happens if $l \leq e+t$ and it is the difficult case of the theorem. We define $\mathsf{H}^{(n)}$ to be the S-submodule of H generated by elements of degree $\leq n$. Clearly, $\mathsf{H}^{(n-1)} \subseteq \mathsf{H}^{(n)}$ and $\mathsf{H}^{(n)} = \mathsf{H}$ for $n \geq l + a$. By Step 2

$$\mathsf{H}^{(n)} = \mathsf{H}^{(n-1)} + z\,\mathsf{H}^{(n-1)} \quad \text{for } n \geq l. \tag{3}$$

We define E to be the kernel of the surjective map:

$$z : \mathsf{H}^{(l-1)} \to \mathsf{H}^{(l)}/\mathsf{H}^{(l-1)}.$$

By Step 1 $\mathsf{H}^{(l)}/\mathsf{H}^{(l-1)}$ is a free S-module; on the other hand $\mathsf{H}^{(l-1)}$ is the S-direct summand of A_X generated by the homogeneous elements of degree $\leq l-1$. Hence $\mathsf{H}^{(l-1)}$ and E are free S-modules. Furthermore, by definition of E, we have

$$zE \subseteq \mathsf{H}^{(l-1)}. \tag{4}$$

Next we consider the dual module of D:

$$D^* = \mathrm{Hom}_k(D, k).$$

Recall that D^* is a graded T-module with grading given by

$$D^*_n = \mathrm{Hom}_k(D_{-n}, k).$$

By Step 2 we have

$$\dim_k D^*_{j-l-a} = 1 \quad \text{if } 0 \leq j \leq a.$$

If $\phi \in D^*_{-l-a}$ is nonzero, $z^j \phi$ is a k-basis of D^*_{j-l-a} for each j satisfying $0 \leq j \leq a$.

Step 4. *Let $U = \{u \in T_1 : u\phi = 0\}$. Then $UD_n = 0$ for all $n \geq l - 1$.*

Proof. Pick $u \in U$ and $d \in D_n$: if $n \geq l + a$, then $D_{n+1} = 0$ thus $ud = 0$. On the other hand, if $l - 1 \leq n \leq l + a - 1$, then D_{n+1} has dimension one and a basis for its dual is $z^j\phi$ where $0 \leq j = l + a - n - 1 \leq a$. We have

$$z^j \phi(ud) = u\phi(z^j d) = 0$$

hence $ud = 0$. □

Step 5. *There is a subspace $W \subseteq R_1$ of dimension $N - t - 1$ with the following property:*

$$w\,\mathsf{H} \subseteq E \quad \textit{for all } w \in W.$$

Proof. Let U be as in Step 4. Then U is the kernel of the multiplication map

$$D^*_{-l-a} \otimes T_1 \to D^*_{-l-a+1}.$$

Hence U has codimension one in T_1 hence dimension $N - t - 1$. Therefore, it is enough to show that every $u \in U$ has a unique preimage $w \in R_1$ such that

$$w\,\mathsf{H} \subseteq E.$$

Fix $u \in U$ and a preimage $v \in R_1$ of u. We first note that

$$v\,\mathsf{H}^{(n)} \subseteq \mathsf{H}^{(n)} \quad \text{for } n \geq l-1$$

since by Step 4 $u D_n = 0$ for $n \geq l-1$. In particular, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \mathsf{H}^{(l-1)} & \xrightarrow{z} & \mathsf{H}^{(l)}/\mathsf{H}^{(l-1)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow v & & \downarrow v & & \\ 0 & \longrightarrow & E & \longrightarrow & \mathsf{H}^{(l-1)} & \xrightarrow{z} & \mathsf{H}^{(l)}/\mathsf{H}^{(l-1)} & \longrightarrow & 0 \end{array}$$

from which we see that $vE \subseteq E$. Hence multiplication by v defines a degree one homogeneous endomorphism of $\mathsf{H}^{(l-1)}/E$, which is a free S-module of rank one being isomorphic to $\mathsf{H}^{(l)}/\mathsf{H}^{(l-1)}$ by definition of E. If 1 is a generator of $\mathsf{H}^{(l-1)}/E$, we have

$$v1 = x1$$

for some $x \in S_1$. We let $w = v - x$: by construction, w is the unique preimage of u with the property $w\,\mathsf{H}^{(l-1)} \subseteq E$. We now need to show that $w\,\mathsf{H} \subseteq E$, or equivalently that

$$w\mathsf{H}^{(n)} \subseteq E \quad \text{for } n \geq l-1.$$

We prove this last statement by induction: assuming $w\mathsf{H}^{(l-1+j)} \subseteq E$ for some $j \geq 0$ we show that $w\,\mathsf{H}^{(l+j)} \subseteq E$ as well.

Thus assume $w\,\mathsf{H}^{(l-1+j)} \subseteq E$; by definition we have $z^j\mathsf{H}^{(l-1)} \subseteq \mathsf{H}^{(l-1+j)}$. Hence

$$wz^j\,\mathsf{H}^{(l-1)} \subseteq w\mathsf{H}^{(l-1+j)} \subseteq E.$$

Now using Eq. (3) we get

$$wz^j\,\mathsf{H}^{(l)} = wz^j\,(\mathsf{H}^{(l-1)} + z\mathsf{H}^{(l-1)}) \subseteq w(\mathsf{H}^{(l-1+j)} + z\mathsf{H}^{(l-1+j)}) \subseteq E + zE \subseteq \mathsf{H}^{(l-1)}$$

as $zE \subseteq \mathsf{H}^{(l-1)}$ by definition of E. We have therefore a commutative diagram:

$$\begin{array}{ccc} \mathsf{H}^{(l-1)}/E & \xrightarrow{wz^j} & E/E \\ z \downarrow & & \downarrow z \\ \mathsf{H}^{(l)}/\mathsf{H}^{(l-1)} & \xrightarrow{wz^j} & \mathsf{H}^{(l-1)}/E \end{array}$$

where the vertical map on the left is an isomorphism. It follows that the horizontal map on the bottom is the zero map, that is

$$wz^j\,\mathsf{H}^{(l)} \subseteq E.$$

From this we deduce

$$w\mathsf{H}^{(l+j)} = w(\mathsf{H}^{(l-1+j)} + z^j\mathsf{H}^{(l)}) \subseteq E. \qquad \square$$

Step 6. Let W be as in Step 5. For all $n \leq t+1-l$ and all $\xi \in \mathsf{H}^0(\omega_X(n))$, we have

$$w\xi = 0.$$

Proof. By definition we have $R_X \subseteq \mathsf{H} \subseteq A_X$. Since A_X/R_X has finite length, Proposition 2.15 implies that

$$\Omega_X \cong \mathrm{Hom}_S(\mathsf{H}, S(-t-1))$$

as R-modules. Pick $\xi \in (\Omega_X)_n = \mathsf{H}^0(\omega_X(n))$. We can think of ξ as being a homogeneous morphism $\mathsf{H} \to S$ of degree $n-t-1$. Now any element in a basis of the free S-submodule E of H has degree $\leq l-1$ by construction, so any nonzero homogeneous morphism $E \to S$ has degree greater than $-l$. Therefore, if $n \leq t+1-l$, the degree of ξ is at most $-l$ and the restriction of ξ to E is zero. Now any $w \in W$ maps H into E by Step 5, hence

$$w\xi(\mathsf{H}) = \xi(w\mathsf{H}) \subseteq \xi(E) = \{0\}. \qquad \square$$

Step 7. *We have*

$$h_X(n) = 1 \quad \textit{for } l \leq n \leq e(X)+t+1.$$

Furthermore there is a linear subspace $M \subseteq \mathbb{P}^N$ such that X contains a hypersurface in M of degree $e+t+2$.

Proof. We let M be the $t+1$ dimensional linear subspace of $\mathbb{P}^N$ defined by W. By Step 6, the equations of M kill $(\Omega_X)_{-e}$, hence $M \cap X$ has dimension t. We consider the beginning of the Koszul resolution of $\mathcal{O}_M$:

$$\mathcal{O}_{\mathbb{P}^N}(-1)^{N-t-1} \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_M \to 0.$$

Let $\mathbb{P}^t = \mathrm{Proj}(S)$. Tensoring with $\mathcal{O}_X$ and applying the functor $\mathcal{H}om(-, \omega_{\mathbb{P}^t})$ we obtain an exact sequence:

$$0 \to \omega_{M \cap X} \to \omega_X \to \omega_X(1)^{N-t-1}$$

where the last map is given by a basis of W. Now we use Step 6 to conclude that

$$\mathsf{H}^0(\omega_{M\cap X}(n)) \cong \mathsf{H}^0(\omega_X(n)) \quad \text{for } n \leq t+1-l.$$

If we let F denote the largest hypersurface of M contained in $M \cap X$, then $\omega_F = \omega_{M\cap X}$ because F and $M \cap X$ coincide in dimension t. Thus we have

$$\mathsf{H}^0(\omega_F(n)) \cong \mathsf{H}^0(\omega_X(n)) \quad \text{for } n \leq t+1-l.$$

By Proposition 2.15 this implies

$$h_X(n) = h_F(n) \quad \text{for } n \geq l.$$

Now F is a hypersurface in M, so its spectrum is $\{0,1,\dots,\deg F-1\}$. We conclude that $\deg F-1=e(X)+t+1$ and

$$h_X(n)=1 \quad \text{for } l\le n\le e(X)+t+1$$

which finishes the proof of Step 7 and of the theorem, since $h_X(n)=1$ implies $\mu(n)=0$ by the first part of the theorem. □

Next we are going to study those subschemes for which the index of speciality e is as small (resp. as big) as possible.

Corollary 3.4. *Let $X\subseteq\mathbb{P}^N$ be a quasi ACM subscheme of dimension $t\ge 1$ and degree d. Then*

$$-t-1\le e(X)\le d-t-2.$$

Equality holds on the right if and only if X is a hypersurface in a linear subspace of $\mathbb{P}^N$ of dimension $t+1$. If equality holds on the left, then the support of X is the disjoint union of at most $h_X(0)$ linear spaces of dimension t.

Proof. By Theorem 3.2 we have $e+t+1\ge 0$ and $0,1,\dots,e+t+1$ are precisely the nonnegative integers which occur in the spectrum at least once. Since the spectrum consists of d elements, we have $e+t+2\le d$.

If $e=d-t-2$, then the spectrum of X must be $\{0,1,\dots,d-1\}$. Then Theorem 3.2 tells us that X is ACM. If $d=1$, X itself is a linear subspace, hence a hyperplane in a $t+1$ linear subspace. If $d\ge 2$, the spectrum gives $h^0(\mathbb{P}^N,\mathscr{I}_X(1))=N-1-t$, hence X is contained in a linear subspace of dimension 1. Conversely, if X is a hypersurface of degree d in some linear subspace, we easily compute that $sp_X=\{0,1,\dots,d-1\}$.

Finally, assume that $e(X)=-t-1$. If X is reduced, then there cannot be any negative integer in the spectrum, and the spectrum does not contain any positive integer either, since $e=-t-1$. Hence $sp_X=\{0^d\}$, and each connected component of X must have spectrum $\{0\}$. Thus X is the disjoint union of d linear subspaces of dimension t. In general, note that, if Y is any quasi ACM subscheme of X of dimension t, we have an injective map $\Omega_Y\to\Omega_X$; in particular, $e(X)\ge e(Y)$. Thus, if $e(X)=-t-1$, we must have $e(X_{red})=-t-1$ and hence the support of X is the disjoint union of $h_{X_{red}}(0)$ linear subspaces. Since $e(X)=e(X_{red})=-t-1$, we have

$$h_{X_{red}}(0)\le h_X(0),$$

and the last part of our statement follows. □

Remark 3.5. In the case of curves in $\mathbb{P}^3$ we can be more precise. Recall (see [1]) that a quasi-primitive multiplicity structure on a line in $\mathbb{P}^3$ is a curve $C\subseteq\mathbb{P}^3$ whose support is a line and which is almost everywhere locally contained in a smooth surface, that is, the embedded dimension of C is (at most) 2 at all but finitely many points. We claim that a curve C in $\mathbb{P}^3$ has $e(C)=-2$ if and only if each connected component of C is a quasi-primitive multiplicity structure on a line which does not contain a planar double structure. Suppose first that $e(C) = -2$. Since $e(C)$ equals the maximum value for $e(D)$ as D varies among the connected components of C, we see from the above corollary that each connected component D of C is a multiplicity structure on a line L with $e(D) = -2$. The second infinitesimal neighborhood L_2 of L has the same cohomology as the twisted cubic curve, in particular it has $e = -1$. Hence D cannot contain L_2, that is, D is quasi-primitive [1]. Similarly, since a conic has $e = -1$, D cannot contain a planar double structure.

To prove the converse, assume D is a quasi-primitive multiplicity structure on a line L which does not contain a planar double structure. Recall from [1] that there is a chain of inclusions $L = D_1 \subseteq D_2 \subseteq \cdots \subseteq D_d = D$, where D_k is a multiplicity structure of degree k , and we have exact sequences:

$$0 \to \mathscr{L}_k \to \mathcal{O}_{D_{k+1}} \to \mathcal{O}_{D_k} \to 0$$

where $\mathscr{L}_k \in Pic(L)$ and $\deg \mathscr{L}_k \geq k \deg \mathscr{L}_1$. Now, since D does not contain a planar double structure, we must have $g(D_2) < 0$, that is, $\deg \mathscr{L}_1 \geq 0$. Hence $\deg \mathscr{L}_k \geq 0$ for $1 \leq k \leq d-1$, and this implies $H^1(\mathscr{L}_k(-1)) = 0$. By induction on k it follows that $e(D_k) = -2$.

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