On the Integrability of Two-Dimensional Flows

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This paper deals with the notion of integrability of flows or vector fields on two-dimensional manifolds. We consider the following two key points about first integrals: (1) They must be defined on the whole domain of definition of the flow or vector field, or defined on the complement of some special orbits of the system; (2) How are they computed? We prove that every local flow \( \phi \) on a two-dimensional manifold \( M \) always has a continuous first integral on each component of \( M \setminus \Sigma \) where \( \Sigma \) is the set of all separatrices of \( \phi \). We consider the inverse integrating factor and we show that it is better to work with it instead of working directly with a first integral or an integrating factor for studying the integrability of a given two-dimensional flow or vector field. Finally, we prove the existence and uniqueness of an analytic inverse integrating factor in a neighborhood of a strong focus, of a non-resonant hyperbolic node, and of a Siegel hyperbolic saddle. © 1999 Academic Press

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1. INTRODUCTION

By definition a planar differential system is

\[ \frac{dx}{dt} = x = P(x, y), \quad \frac{dy}{dt} = y = Q(x, y), \quad (1) \]

where \( P \) and \( Q \) are \( C^r \) maps with \( r \geq 1 \) from an open subset \( U \) of \( \mathbb{R}^2 \) to \( \mathbb{R} \). We say that \( U \) is the domain of definition of the differential system (1), and that

\[ X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} \quad (2) \]

is the \( C^r \) vector field defined on \( U \) associated to differential system (1).

A \( C^k \) function \( H: U \to \mathbb{R} \) with \( k \geq 0 \) is a strong first integral of the differential system (1) defined in \( U \) if \( H \) is constant on each solution of this system, and \( H \) is non-constant on any open subset of \( U \). Here \( k \geq 0 \) means that \( k = 0, 1, 2, \ldots, \infty \). More precisely, \( k = 0 \) means that \( H \) is continuous, \( k = 1, 2, \ldots, \infty \) means that \( H \) is \( C^k \), and \( k = \infty \) means that \( H \) is analytic. If \( k \geq 1 \) then the previous definition of integrability implies that the derivative of \( H \) following the direction of the vector field \( X \) is zero, i.e. if \( XH = 0 \) on \( U \).

This definition of strong first integral is the usual definition of first integral which appears in the major part of books on differential equations (see for instance [1, 26]). With this definition the linear differential system

\[ x' = x, \quad y' = y, \quad (3) \]

defined on \( \mathbb{R}^2 \) has no strong first integrals. This is due to the fact that every strong first integral of system (3) must be a continuous function on \( \mathbb{R}^2 \) that must take a constant value on each straight line through the origin, because these straight lines are formed by orbits of the system. Hence it must be constant on the whole \( \mathbb{R}^2 \), consequently there are no strong first integrals for system (3). By using this argument it follows that if system (1) has a strong first integral, then it cannot have nodes, foci, center-foci, singular points having some parabolic or elliptic sectors, limit cycles and separatix cycles that be the \( \alpha \)-or \( \omega \)-limit set of some orbit of the system (see [26] for definitions). Since we do not like that differential systems so easy as system (3) have no first integrals, we will introduce the notion of weak first integral.

Let \( \Sigma \) be a set of orbits of system (1) such that \( U \setminus \Sigma \) is open. We say that a \( C^k \) function \( H: U \setminus \Sigma \to \mathbb{R} \) with \( k \geq 0 \) is a weak first integral of the differential system (1) defined in \( U \) if \( H \) is constant on each solution of system (1) contained in \( U \setminus \Sigma \), and \( H \) is non-constant on any open subset of \( U \setminus \Sigma \).
If \( k \geq 1 \) this definition implies that the derivative of \( H \) following the direction of the vector field \( X \) is zero on \( U \setminus \Sigma \).

We remark that the unique difference between the notions of strong and weak first integral is that a weak first integral does not need to be defined in the whole domain of definition \( U \) of the differential system (1). This difference has been noted by many authors. Thus, the first integrals computed by Darboux [9] in 1878 for polynomial differential systems possessing sufficient algebraic solutions are in general weak first integrals, see for more details Section 2. In that section we study the integrability of linear differential systems

\[
\frac{dx}{dt} = x' = ax + by, \quad \frac{dy}{dt} = y' = cx + dy,
\]

with \( ad - bc \neq 0 \) (non-degenerate). Thus with the notion of strong first integral only the centers and the saddles are integrable, but with the notion of weak first integral all linear systems (4) are integrable. In particular we show that system (3) has the weak first integral \( H: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \) where \( H(x, y) = xy/(x^2 + y^2) \).

We claim that the notion of weak first integral is the natural notion of integrability for two-dimensional differential systems instead of the more usual notion of strong first integral. Theorem 1 will confirm this claim. In order to present it we need some preliminary definitions and results.

For differential system (1) the following three properties are well known; see for more details [26].

(a) (Existence and uniqueness of maximal solutions for a differential system) For all \( p \in U \) there exists an open interval \( I_p \) of the real line where it is defined the unique maximal solution \( \varphi_p : I_p \to U \) of (1) such that \( \varphi_p(0) = p \).

(b) (Group property) If \( q = \varphi_{p_a}(t) \) and \( t \in I_{p_a} \), then \( I_q = I_{p_a} - t = \{ r - t : r \in I_{p_a} \} \) and \( \varphi_q(s) = \varphi_{p_a}(t + s) \) for all \( s \in I_q \).

(c) (Differentiability with respect to initial conditions) The set \( D = \{(t, p) : p \in U, t \in I_p \} \) is open in \( \mathbb{R}^3 \) and the map \( \varphi : D \to U \) defined by \( \varphi(t, p) = \varphi_p(t) \) is \( C^r \).

The map \( \varphi : D \to U \) is a local flow of class \( C^r \) with \( r \geq 1 \) on \( U \) associated to system (1), which verifies

(i) \( \varphi(0, p) = p \) for all \( p \in U \);
(ii) \( \varphi(t, \varphi(s, p)) = \varphi(t + s, p) \) for all \( p \in U \), and all \( s \) and \( t \) such that \( s, t + s \in I_p \);
(iii) \( \varphi_p(-t) = \varphi_p^{-1}(t) \) for all \( p \in U \) such that \( t, -t \in I_p \).
Let \( \varphi \) be a local flow on the two-dimensional manifold \( M \), and let \( \Sigma \) be a subset of \( M \) formed by orbits of \( \varphi \) such that \( M \setminus \Sigma \) is open. We say that a \( C^k \) function \( H: M \setminus \Sigma \to \mathbb{R} \) with \( k \geq 0 \) is a weak first integral of \( \varphi \) if \( H \circ \varphi_p \) is constant for each \( p \in M \setminus \Sigma \) and \( H \) is not constant on any open subset of \( M \setminus \Sigma \). Of course, when the local flow is the local flow associated to a \( C^r \) differential system (1) with \( r \geq 1 \), the above definition of weak first integral for system (1) and the definition of weak first integral for its associated local flow coincide.

In this paper we consider \( C^r \) local flows with \( r \geq 0 \) on an arbitrary two-dimensional manifold \( M \) (separable metric, but not necessarily compact nor orientable and possibly with boundary). Of course, when \( r = 0 \) the flow is only continuous. Two such flows, \( (M, \varphi) \) and \( (M', \varphi') \), are \( C^k \)-equivalent with \( k \geq 0 \) if there is a \( C^k \) diffeomorphism of \( M \) onto \( M' \) which takes orbits of \( \varphi \) onto orbits of \( \varphi' \) preserving sense (but not necessarily the parametrization). Of course, a \( C^0 \) diffeomorphism is a homeomorphism.

Let \( \varphi \) be a \( C^r \) local flow with \( r \geq 0 \) on the two-dimensional manifold \( M \). We call \( (M, \varphi) \) \( C^k \)-parallel if it is \( C^k \)-equivalent to one of the following flows:

1. \( \mathbb{R}^2 \) with the flow defined by \( x' = 1, \ y' = 0; \)
2. \( \mathbb{R}^2 \setminus \{0\} \) with the flow defined (in polar coordinates) by \( r' = 0, \ \theta' = 1; \)
3. \( \mathbb{R}^3 \setminus \{0\} \) with the flow defined by \( r' = r, \ \theta' = 0; \)
4. \( S^1 \times S^1 \) with rational flow (e.g., the flow induced by (1) above under the usual covering map; note in particular that all rational flows on the torus are equivalent). We call these flows as strip, annular, spiral, and toral, respectively.

Let \( p \in M \). We denote by \( \gamma(p) \) the orbit of the flow \( \varphi \) on \( M \) through \( p \), more precisely \( \gamma(p) = \{ \varphi_p(t) : t \in I_p \} \). The positive semiorbit of \( p \in M \) is \( \gamma^+(p) = \{ \varphi_p(t) : t \in I_p, \ t \geq 0 \} \). In a similar way it is defined the negative semiorbit \( \gamma^-(p) \) of \( p \in M \).

We define the \( \infty \)-limit and the \( -\infty \)-limit of \( p \) as \( (\gamma^-(p)) \) and let

\[
\alpha(p) = \text{cl}(\gamma^-(p)) \setminus \gamma^-(p), \quad \omega(p) = \text{cl}(\gamma^+(p)) \setminus \gamma^+(p),
\]

respectively. Here, as usual, \( \text{cl} \) denotes the closure.

Let \( \gamma(p) \) be an orbit of the flow \( \varphi \) defined on \( M \). A parallel neighborhood of the orbit \( \gamma(p) \) is an open neighborhood \( N \) of \( \gamma(p) \) such that \( (N, \varphi) \) is \( C^k \)-equivalent to a parallel flow for some \( k \geq 0 \).
We say that \( \gamma(p) \) is a separatrix of \( \varphi \) if \( \gamma(p) \) is not contained in a parallel neighborhood \( N \) satisfying the following two assumptions:

1. For any \( q \in N \), \( \alpha(q) = \alpha(p) \) and \( \omega(q) = \omega(p) \), and
2. \( \text{cl}(N) \setminus N \) consists of \( \alpha(p) \), \( \omega(p) \) and exactly two orbits \( \gamma(a) \), \( \gamma(b) \) of \( \varphi \), with \( \alpha(a) = \alpha(p) = \alpha(b) \) and \( \omega(a) = \omega(p) = \omega(b) \).

We denote by \( \Sigma \) the union of all separatrices of \( \varphi \). Then \( \Sigma \) is a closed invariant subset of \( M \). A component of the complement of \( \Sigma \) in \( M \), with the restricted flow, is called a canonical region of \( \varphi \).

One of our main results is the following one.

**Theorem 1.** Let \( \varphi \) be a local flow on a two-dimensional manifold \( M \), and let \( \Sigma \) be the union of all separatrices of \( \varphi \). Then the local flow \( \varphi \) has a continuous first integral on every canonical region of \( \varphi \).

The proof of Theorem 1 uses ideas of Markus [18] and Neumann [22] (see also [23]).

Theorem 1 shows that on every canonical region of a two-dimensional flow \( \varphi \) there is a continuous first integral but:

**Open Problem 1.** What is the maximal order of differentiability of the first integrals on a canonical region of a given two-dimensional flow \( \varphi \) in function of the order of differentiability of the flow?

The paper is organized as follows. In Section 2, as an example that our definition of first integral is the natural one, we present a summary of Darboux theory for computing first integrals of planar polynomial differential systems possessing sufficient algebraic solutions, and in particular we apply this theory for computing the first integrals of planar linear systems. In Section 3 we prove Theorem 1.

In Section 4 we consider the classical problem in integrability theory about the existence or nonexistence of first integrals defined in an open neighborhood of an isolated singular point. We also present two open problems.

In Section 5 we work with the inverse integrating factor and we justify that, in general, in order to understand the integrability of a two-dimensional differential system is better to use an inverse integrating factor instead of using directly a first integral, or an integrating factor.

Finally, in Section 6 we also prove the existence and uniqueness of an analytic inverse integrating factor for a strong focus (see Proposition 8 and Theorem 14), for a non-resonant hyperbolic node (see Proposition 10 and Theorem 14), and for a Siegel hyperbolic saddle (see Proposition 13 and Theorem 14).
2. DARBOUX INTEGRABILITY

By definition a polynomial (differential) system is a system of the form (1) where \(P\) and \(Q\) are complex polynomials (i.e., \(P, Q \in \mathbb{C}[x, y]\)), and \(t\) is a real variable. We say that \(m = \max\{\deg P, \deg Q\}\) is the degree of the polynomial system. Here we only consider polynomial systems (1) such that \(P\) and \(Q\) are relatively prime. In other words, we only consider polynomial systems (1) having finitely many singular points.

The Darboux theory contributes to show the link between the integrability of polynomial systems and the existence of algebraic solutions. Indeed, already in 1878, Darboux [9] showed how the first integrals of polynomial systems possessing sufficient algebraic solutions are constructed (see Theorem 2). First we need some preliminary notions and definitions.

An invariant algebraic curve of system (1) is an algebraic curve \(f(x, y) = 0\) with \(f \neq \mathbb{C}[x, y]\), such that for some polynomial \(K \in \mathbb{C}[x, y]\) we have \(Xf = Kf\), where \(X\) is defined in (2). So \(f\) satisfies the equation

\[
\frac{\partial f}{\partial x}(x, y) P(x, y) + \frac{\partial f}{\partial y}(x, y) Q(x, y) = K(x, y) f(x, y).
\]

The polynomial \(K\) is called the cofactor of the invariant algebraic curve \(f = 0\).

We say that the curve \(f = 0\) with \(f \in \mathbb{C}[x, y]\) is an algebraic solution of system (1) if and only if it is an invariant algebraic curve and \(f\) is an irreducible polynomial over \(\mathbb{C}[x, y]\).

Let \(U\) be an open subset of \(\mathbb{C}^2\) and let \(R: U \to \mathbb{C}\) be a \(C^k\) function with \(k \geq 1\) which is not identically zero on \(U\). The function \(R\) is an integrating factor of system (1) on \(U\) if we have

\[
\frac{\partial (RP)}{\partial x} = -\frac{\partial (RQ)}{\partial y}.
\]

This is equivalent to the fact that \(R\) satisfies the linear partial differential equation

\[
P \frac{\partial R}{\partial x} + Q \frac{\partial R}{\partial y} = - \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) R,
\]

in \(U\).

The first integral \(H\) associated to the integrating factor \(R\) can be computed through the integral

\[
H(x, y) = \left[ R(x, y) P(x, y) dy \right] + f(x),
\]

satisfying \(\partial H/\partial x = -RQ\).
If \( S(x, y) = \sum_{i+j=0}^{m-1} a_{ij} x^i y^j \) is a polynomial of degree \( m-1 \) with \( m(m+1)/2 \) coefficients in \( \mathbb{C} \), then we write \( S \in \mathbb{C}_{m-1}[x, y] \). We identify the linear vector space \( \mathbb{C}_{m-1}[x, y] \) with \( \mathbb{C}^{m(m+1)/2} \) through the isomorphism

\[
S \mapsto (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-1,1}, \ldots, a_{0,m-1}).
\]

We say that \( p \) points \((x_k, y_k), k = 1, \ldots, p\), are independent with respect to \( \mathbb{C}_{m-1}[x, y] \) if the intersection of the \( p \) hyperplans

\[
\sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, \quad k = 1, \ldots, p,
\]

in \( \mathbb{C}^{m(m+1)/2} \) is a linear subspace of dimension \([m(m+1)/2] - p\).

We remark that the maximum number of isolated singular points of system (1) is \( m^2 \) (by Bezout Theorem), and that the maximum number of independent isolated singular points of system (1) is \( m(m+1)/2 \), and that \( m(m+1)/2 < m^2 \) for \( m \geq 2 \).

A singular point \((x_0, y_0)\) of system (1) is called weak if the divergence, \( \text{div}(P, Q) \), of system (1) at \((x_0, y_0)\) is zero.

For a proof of the following theorem see [9, 16, 6].

**Theorem 2. Darboux Theory.** Suppose that a polynomial system (1) of degree \( m \) admits \( q \) invariant algebraic curves \( f_i = 0 \) with cofactors \( K_i \), and \( p \) independent singular points \((x_k, y_k)\) such that \( f_i(x_k, y_k) \neq 0 \) for \( k = 1, \ldots, p \) and \( i = 1, \ldots, q \).

(a) If there exist \( \lambda_i \in \mathbb{C} \) not all zero such that \( \sum_{i=1}^{q} \lambda_i K_i = 0 \), then \( f_1^{\lambda_1} \cdots f_q^{\lambda_q} \) is a first integral.

(b) If \( q \geq [m(m+1)/2] + 1 - p \), then there exist \( \lambda_i \in \mathbb{C} \) not all zero such that \( \sum_{i=1}^{q} \lambda_i K_i = 0 \).

(c) If \( q \geq [m(m+1)/2] + 2 \), then there exist integers \( \lambda_i \) not all zero such that \( \sum_{i=1}^{q} \lambda_i K_i = 0 \). In particular the first integral given by (a) is rational, and consequently all solutions of the system are algebraic.

(d) If there exist \( \lambda_i \in \mathbb{C} \) not all zero such that \( \sum_{i=1}^{q} \lambda_i K_i = -\text{div}(P, Q) \), then \( R = f_1^{\lambda_1} \cdots f_q^{\lambda_q} \) is an integrating factor.

(e) If \( q \geq [m(m+1)/2] - p > 0 \) and the \( p \) points are weak, then there exist \( \lambda_i \in \mathbb{C} \) not all zero such that \( \sum_{i=1}^{q} \lambda_i K_i = -\text{div}(P, Q) \).

If \( P \) and \( Q \) are real polynomial and the differential system (1) has a complex invariant algebraic curve \( f = 0 \), then it also has its conjugate \( f = 0 \).

Then, if we are under the assumptions of Theorem 2(a) the first integral \( f_1^{\lambda_1} \cdots f_q^{\lambda_q} \) has a factor of the form \( f_1^{\alpha_1} \cdots f_q^{\alpha_q} \), which is just the real function

\[
[\text{Re} \, f]^2 + [\text{Im} \, f]^2 \text{Re} \, \lambda \exp \left(-2 \text{Im} \, \lambda \arctan \left( \frac{\text{Im} \, f}{\text{Re} \, f} \right) \right).
\]
Moreover, when \( P \) and \( Q \) are real polynomials it is easy to show that we can choose the \( \lambda_i \)'s of Darboux Theorem in such a way that the function \( f_1^* \cdots f_n^* \) be real.

The following result shows that all non-degenerate linear differential systems in \( \mathbb{R}^2 \) have a weak first integral and that the centers and the saddles have a strong first integral. By commodity we only consider the non-degenerate linear differential systems.

**Proposition 3.** We consider the real linear differential system (4) with \( ad - bc \neq 0 \). Then after a linear change of variables and a rescaling of the independent variable \( t \) one of the following statements holds.

(a) The matrix of the system can be written as

\[
\begin{pmatrix}
\alpha & -1 \\
1 & \alpha
\end{pmatrix}
\]

and \( r = C \exp(-\lambda \theta) \) is a first integral of the system on \( \mathbb{R}^2 \setminus \{0\} \) given in polar coordinates \( x = r \cos \theta, y = r \sin \theta \). Of course, if \( \alpha \neq 0 \) the origin is a focus, and the first integral \( r = C \exp(-\lambda \theta) \) is weak; but if \( \alpha = 0 \) the origin is a center, and the first integral \( r = C \) is defined in the whole \( \mathbb{R}^2 \), and consequently it is strong.

(b) The matrix of the system can be written as

\[
\begin{pmatrix}
1 & 0 \\
0 & \lambda
\end{pmatrix}
\]

with \( \lambda > 0 \), and \( y |x|^{1/4} (y^2 + x^2) = C \) is a weak first integral of the system defined on \( \mathbb{R}^2 \setminus \{0\} \). The origin is a diagonalizable node.

(c) The matrix of the system can be written as

\[
\begin{pmatrix}
1 & 0 \\
0 & -\lambda
\end{pmatrix}
\]

with \( \lambda > 0 \), and \( |x|^4 y = C \) is a strong first integral. The origin is a saddle.

(d) The matrix of the system can be written as

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

and a weak first integral of the system on \( \mathbb{R}^2 \setminus \{y = 0\} \) is \( y \exp(-x/y) = C \). The origin is a non-diagonalizable node.
Proof. We assume that the matrix of the system is written as in statement (a). Then system (4) has the algebraic curve \( f(x, y) = x^2 + y^2 = 0 \) as a solution. Its cofactor is \( K(x, y) = 2x \), and consequently it coincides with the divergence of the system. Therefore, by Theorem 2 (d) it follows that \( R(x, y) = (x^2 + y^2)^{-1} \) is an integrating factor of the system. Hence, we can compute the first integral of system (4) using (6). Thus we obtain the first integral \( \frac{y}{x} + \frac{|x|^2}{|y|^2} = \frac{y^2}{|x|^2} + \frac{|x|^2}{y^2} \), and consequently, in polar coordinates the first integral \( r = C \exp(-\pi \theta) \) is defined on \( \mathbb{R}^2 \setminus \{x = 0\} \).

Suppose now that the matrix of the system is written as in statement (b). Then system (4) has the algebraic curves \( f_1(x, y) = x = 0 \) and \( f_2(x, y) = y = 0 \) as solutions. Their cofactors are \( K_1(x, y) = 1 \), \( K_2(x, y) = -2 \). Therefore, by Theorem 2(a) it follows that \( H : \mathbb{R}^2 \setminus \{x = 0\} \to \mathbb{R} \) defined by \( H(x, y) = y/|x|^2 \) is a first integral. Since the inverse of a first integral is a first integral (of course, where it is well-defined) and the sum of two first integrals is again a first integral, we obtain that

\[
\frac{y}{x} + \frac{|x|^2}{y} = \frac{y^2 + |x|^2}{|y|^2},
\]

is a first integral on \( \mathbb{R}^2 \setminus \{x = 0\} \). Consequently, \( |x|^2/\left| y^2 + x^2 \right| \) is a first integral defined on \( \mathbb{R}^2 \setminus \{0\} \).

Assume that the matrix of the system is written as in statement (c). Then system (4) has the algebraic curves \( f_1(x, y) = x = 0 \) and \( f_2(x, y) = y = 0 \) as solutions. Their cofactors are \( K_1(x, y) = 1 \) and \( K_2(x, y) = -\lambda \), respectively. Therefore, by Theorem 2(a) it follows that \( |x|^2 y = C \) is a first integral defined on \( \mathbb{R}^2 \).

Finally we suppose that the matrix of the system is written as in statement (d). Then system (4) has the algebraic curve \( f(x, y) = y = 0 \) as a solution. Its cofactor is \( K(x, y) = 1 \), and the divergence of the system is equal to 2. Therefore, by Theorem 2(d) it follows that \( R(x, y) = y^{-2} \) is an integrating factor of the system. Hence, we can compute the first integral of system (4) using (6). Thus we obtain the first integral \( y \exp(-x/y) = C \) defined on \( \mathbb{R}^2 \setminus \{y = 0\} \).

In the next proposition we present an example of a cubic polynomial differential system which has a weak first integral and a limit cycle. Different examples of this kind were given by Dolov [10], Kooij and Christopher [17], and Christopher [7].

Proposition 4. The differential system

\[
x' = -y - x(x^2 + y^2 - 1), \quad y' = x - y(x^2 + y^2 - 1),
\]

(7)
has the algebraic solution \( x^2 + y^2 - 1 = 0 \) as a limit cycle. Its separatrices are
\( \Sigma = \{ x^2 + y^2 - 1 = 0 \} \cup \{ 0 \} \), and in polar coordinates \( H(r, \theta) = (r^2 - 1) \exp(2\theta)/r^2 = C \) is a first integral defined on \( \mathbb{R}^2 \setminus \Sigma \). We remark that \( H \) is not continuous on \( \Sigma \).

**Proof.** System (7) in polar coordinates becomes
\[
\frac{dr}{dt} = 2r^2(r^2 - 1), \quad \frac{d\theta}{dt} = 1.
\]
Therefore, \( r = 1 \) is the unique limit cycle of the system. Consequently, this
limit cycle and the origin which is a focus are the unique separatrices of
system (7).

Clearly, system (7) has the algebraic curves \( f_1(x, y) = x^2 + y^2 = 0 \) and
\( f_2(x, y) = x^2 + y^2 - 1 = 0 \) as solutions. Their cofactors are \( K_1(x, y) = -2(x^2 + y^2 - 1) \) and \( K_2(x, y) = -2(x^2 + y^2) \). Since the divergence of
system (7) is equal to \( K_1 + K_2 \), by Darboux Theorem (d) it follows that
\( (f_1 f_2)^{-1} \) is an integrating factor of the system. Hence, we can compute the
first integral of system (7) using (6). Thus we obtain the first integral
\( (r^2 - 1) \exp(2\theta)/r^2 = C \) defined on \( \mathbb{R}^2 \setminus \Sigma \).

We note that system (7) has only two canonical regions, namely
\( 0 < r < 1 \) and \( r > 1 \).

The limit cycles of the differential system
\[
x' = -y + ax(x^2 + y^2 - 1), \quad y' = x + by(x^2 + y^2 - 1),
\]
have been studied in [11].

3. PROOF OF THEOREM 1

A key point in the proof of Theorem 1 is the following lemma of
Neumann [22]. Since its proof is short we give it.

**Lemma 5.** Let \( \varphi \) be a local flow on the two-dimensional manifold \( M \). Then
every canonical region of \( (M, \varphi) \) is \( C^0 \)-parallel.

**Proof.** Let \( (R, \varphi^t = \varphi \mid R) \) be a canonical region. There are no
separatrices in \( R \), so the set consisting of orbits homeomorphic with \( S^1 \)
open, and similarly the set consisting of orbits homeomorphic with \( \mathbb{R} \)
on open. Hence \( R \) consists entirely of closed orbits or entirely of line orbits.

Also, two orbits of \( \varphi^t \) can be separated with disjoint parallel neighborhoods. To prove this we suppose \( \gamma(p) \) and \( \gamma(q) \) are distinct orbits (closed
or not) which cannot be separated. Then, for any parallel neighborhood $N_p$ of $p$, we have $q \in \text{cl}(N_p)$; i.e.

$$q \in \bigcap_{N_p} \text{cl}(N_p) = \gamma(p) \cup \omega(p).$$

But then $q \in \gamma(p)$ (or $q \in \omega(p)$) and this is impossible because $q \notin N_q \subset R$ and $\gamma(p) \cup \omega(p) \subset \text{cl}(N_q) \setminus N_q \not\subset R$.

It follows that the quotient space $R/\varphi'$, obtained by collapsing orbits of $(R, \varphi')$ to points, is a (Hausdorff) one-dimensional manifold. Hence the natural projection $\pi : R \to R/\varphi'$ is a locally trivial fibering of $R$ over $R$ or $S^1$. Since the flow provides a natural orientation on the fibers, there are only four possibilities, the four classes of parallel flows described above.

**Proof of Theorem 1.** From Lemma 5 it follows that every canonical region $R$ of $(M, \varphi)$ is $C^0$-parallel; i.e. there is a homeomorphism $h$ of $R$ onto $M'$ which takes orbits of $\varphi$ onto orbits of $\varphi'$ preserving the sense (but not necessarily the parametrization), and $(M', \varphi')$ is one of the following flows:

1. $M' = R^2$ with the flow defined by $x' = 1, y' = 0$;
2. $M' = R^2 \setminus \{0\}$ with the flow defined by $r' = 0, \theta' = 1$;
3. $M' = R^2 \setminus \{0\}$ with the flow defined by $r' = r, \theta' = 0$;
4. $M' = S^1 \times S^1$ with the rational flow.

Clearly $H(x, y) = y = C$ is a first integral for the flows (1) and (4), $H(r, \theta) = r = C$ a first integral for the flow (2), and $H(r, \theta) = \theta = C$ a first integral for the flow (3). Hence, $H \circ h$ is a continuous valued first integral for the flow $\varphi$ on the canonical region $R$.

### 4. LOCAL INTEGRABILITY PROBLEM

It is well known that every regular point of a $C^k$ differential system (1) has a $C^k$ parallel neighborhood, see the Box Flow Theorem in [26]. A classical problem in integrability theory is the existence or nonexistence of a $C^k$ first integral defined in an open neighborhood of an isolated singular point for $k \in \{0, 1, \ldots, \infty, \omega\}$. We call this problem the $C^k$ local integrability problem.

In Section 1 we shown that the unique isolated singular points which allow the existence of a continuous first integral defined in an open neighborhood are centers or points which are union of hyperbolic sectors. A **linear center** is a singular point of system (1) whose eigenvalues are imaginary. It is known that a center whose linear part is a linear center of a $C^\omega$ differential system (1) is $C^\omega$ locally integrable, see Poincaré [24].
But, in general, it is difficult to compute this analytic first integral, because the proof of Poincaré does not allow to calculate it. For instance, the analytic differential system
\[ x' = y[2(x^2 - y^2) - 1], \quad y' = x + x^2 + y^2 + 4xy, \]
has center at the origin whose linear part is a linear center, and consequently it has an analytic first integral in its neighborhood, but we do not know how to compute it. For this system we know the following \( C^\infty \) first integral
\[ H(x, y) = \frac{x^2 + y^2}{(1 + x)^2 + y^2} \exp\left(\frac{2x - 1}{x^2 + y^2}\right). \]
This analytic differential system was found by Chavarriga and García; see [3a].

**Open Problem 2.** It is an open question to characterize what nonlinear centers of a \( C^m \) differential systems are \( C^m \) locally integrable.

Nemitskii and Stepanov in [21, p. 122] proved that the system
\[ x' = -y[2x^2 + y^2 + (x^2 + y^2)^2], \quad y' = x[2x^2 + y^2 + 2(x^2 + y^2)^2], \]
has a nonlinear center at the origin since it has the symmetry \((x, y, t) \rightarrow (x, -y, -t)\). But this system is not analytically (nor formally) integrable in a neighborhood of the origin. Thus the equation \( H = 0 \) implies that \( H = H_{2m} + H_{2m+2} + \text{H.O.T.} \), where \( H_{2m} = (x^2 + y^2)^m \) with \( m \geq 1 \), and this is in contradiction with the terms of degree 2\( m + 4 \) of \( H' = 0 \). For this system we know the \( C^\infty \) first integral \((2x^2 + y^2)\exp(-1/(x^2 + y^2))\).

Some partial results related with the Open Problem 2 can be found in [5].

**Open Problem 3.** Another open question is to characterize the analytic systems having an isolated singular point formed by hyperbolic sectors which are \( C^m \) locally integrable at this singular point.

### 5. INVERSE INTEGRATING FACTOR

Let \( U \) be the domain of definition of differential system (1), and let \( W \) be an open subset of \( U \). A function \( V: W \rightarrow \mathbb{R} \) that satisfies the linear partial differential equation
\[ P \frac{\partial V}{\partial x} + Q \frac{\partial V}{\partial y} = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V, \]

is called an inverse integrating factor.
is called an inverse integrating factor of system (1) on \( W \). We note that \( \{ V = 0 \} \) is formed by orbits of system (1). This function \( V \) is very important because

1. \( R = 1/V \) defines on \( W \backslash \{ V = 0 \} \) an integrating factor of system (1) (which allows to compute a first integral of the system on \( W \backslash \{ V = 0 \} \));

2. \( \{ V = 0 \} \) contains the limit cycles of system (1) which are in \( W \), see [11]. This fact allows to study the limit cycles which bifurcate from periodic orbits of a center (Hamiltonian or not) and compute their shape, see [12, 13]. For doing that we develop the function \( V \) in power series of the small perturbation parameter. A remarkable fact is that the first term of this expansion coincides with the first non-identically zero Melnikov function (see [14]).

For a linear differential system \( x' = ax + by, \ y' = cx + dy \), always there exists an easy inverse integrating factor \( V(x, y) = cx^2 + (d - a) xy - by^2 \) (a quadratic polynomial), but the first integrals of this system are not so easy, see Proposition 3.

For system (1), if \( P \) and \( Q \) are homogeneous polynomials of the same degree, then the polynomial function \( V = yP - xQ \) satisfies Eq. (1). This follows easily from the Euler Theorem for homogeneous functions.

For system (1), if \( P \) and \( Q \) are quadratic polynomials and the origin is a center, then always there exists a polynomial function \( V: \mathbb{R}^2 \to \mathbb{R} \) of degree 3 or 5 satisfying Eq. (1), see [3, 20].

For system (1), if \( P = -y + P_3(x, y) \) and \( Q = x + Q_3(x, y) \) with \( P_3 \) and \( Q_3 \) homogeneous polynomials of degree 3, and the origin is a center, then always there exists a polynomial function \( V: \mathbb{R}^2 \to \mathbb{R} \) of degree at most 10 satisfying Eq. (1), see [3].

We remark that in all these previous examples the inverse integrating function \( V \) is a polynomial of small degree, but that their first integrals are more complicated functions.

System (8) has no analytic first integral in a neighborhood of the origin, but it has the simple inverse integrating factor \( V = (x^2 + y^2)^2 (2x^2 + y^2) \).

In short, we think that the best way to understand the integrability of a two-dimensional differential system is through the inverse integrating factor \( V \) because

1. \( V \) also allows to compute the first integral;

2. \( \{ V = 0 \} \) contains the limit cycles which are in the domain of definition of \( V \);

3. the expression of \( V \) is simpler than the expressions of the integrating factors which cannot be defined on \( \{ V = 0 \} \), and than the expressions of the first integrals;
(4) usually the domain of definition of $V$ is larger than the domain of definition of the first integral and of the integrating factor.

6. EXISTENCE AND UNIQUENESS OF ANALYTIC INVERSE INTEGRATING FACTORS

We will show that the inverse integrating factor exists, is unique (except for a multiplicative constant factor), and is analytic in an open neighborhood of convenient singular points.

Let $\lambda_1, ..., \lambda_n$ be complex numbers. Then we say that $\lambda_1, ..., \lambda_n$ are resonant if there exist some nonnegative integers $\alpha_1, ..., \alpha_n$ satisfying

$$\sum_{j=1}^{n} \alpha_j \lambda_j - \lambda_j = 0,$$

for some $j \in \{1, ..., n\}$ with $\alpha_1 + \cdots + \alpha_n \geq 2$.

If the convex hull of $\{\lambda_1, ..., \lambda_n\}$ in the complex plane does not contain the origin of $\mathbb{C}$, then $\{\lambda_1, ..., \lambda_n\}$ is said to be in the Poincaré domain.

The next theorem will allow us to study the existence of inverse integrating factors in an open neighborhood of generic foci and nodes.

**Theorem 6 (Poincaré).** Let $A = \text{diag}(\lambda_1, ..., \lambda_n)$. If $\{\lambda_1, ..., \lambda_n\}$ is in the Poincaré domain and the resonant conditions do not hold for any nonnegative integers $\alpha_1, ..., \alpha_n$ and $j$ with $\alpha_1 + \cdots + \alpha_n \geq 2$ and $1 \leq j \leq n$, then there exists an analytic change of variables $x = y + F(y)$, $y \in \Omega$ where $F(y) = O(|y|^{2})$ as $y \to 0$ and $\Omega \subset \mathbb{C}^n$ is a neighborhood of the origin, which transform the differential system $x' = Ax + f(x)$, $f(x) = O(|x|^{2})$ as $x \to 0$, and $f(x)$ is analytic in $x$, into $x' = Ax$.

For a proof of Theorem 6 see [24, 8]. From this theorem it is easy to prove the following result.

**Corollary 7.** We consider the planar real analytic system

$$x' = \alpha x - \beta y + g_1(x, y), \quad y' = \beta x + \alpha y + g_2(x, y),$$

(10)

with $\alpha \beta \neq 0$, and $g_1$ and $g_2$ are of second order in $x$ and $y$. Then there exists a real analytic change of variables $(x, y) = (X, Y) + F(X, Y)$, $(X, Y) \in \Omega$, $F(X, Y) = O(X^2, XY, Y^2)$ as $(X, Y) \to 0$, and $\Omega \subset \mathbb{R}^2$ is a neighborhood of the origin, which transforms system (10) into

$$X' = \alpha X - \beta Y, \quad Y' = \beta X + \alpha Y.$$  

(11)
The next proposition shows the existence of an inverse integrating factor in a neighborhood of a strong focus, i.e., a singular point with eigenvalues $x \pm iy$ with $x \neq 0$.

**Proposition 8.** Under the assumptions of Corollary 7 let $X = f_1(x, y)$, $Y = f_2(x, y)$ be the local real analytic change of variables that transforms system (10) into the form (11). Then an inverse integrating factor of system (10) is

$$V(x, y) = \frac{f_2^2(x, y) + f_1^2(x, y)}{J(x, y)},$$

where

$$J(x, y) = \left( \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial x} \right)(x, y).$$

**Proof.** The linear system (11) has the first integral

$$H = -\beta \ln(X^2 + Y^2) + 2\alpha \tan^{-1} \left( \frac{Y}{X} \right).$$

Therefore, from Corollary 7 we have also a first integral for system (10) given by

$$H = -\beta \ln(f_1^2(x, y) + f_2^2(x, y)) + 2\alpha \tan^{-1} \left( \frac{f_2^2(x, y)}{f_1^2(x, y)} \right).$$

From this first integral we find the inverse integrating factor of the proposition.

This last proposition is due to Giacomini and Viano; see [15]. From Theorem 6 it follows easily the next result.

**Corollary 9.** We consider the planar real analytic system

$$x' = \lambda_1 x + g_1(x, y), \quad y' = \lambda_2 y + g_2(x, y),$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1, \lambda_2 > 0$, the resonant conditions do not hold for any non-negative integers $n_1$, $n_2$ and $j$ with $n_1 + n_2 \geq 2$ and $1 \leq j \leq 2$, and $g_1$ and $g_2$ are of second order in $x$ and $y$. Then there exists a real analytic change of variables $(x, y) = (X, Y) + F(X, Y)$, $(X, Y) \in \Omega$, $F(X, Y) = O(X^2, XY, Y^2)$ as $(X, Y) \to 0$, and $\Omega \subset \mathbb{R}^2$ is a neighborhood of the origin, which transforms system (12) into

$$X' = \lambda_1 X, \quad Y' = \lambda_2 Y.$$
The next proposition shows the existence of an inverse integrating factor for a non-resonant hyperbolic node, i.e., a node satisfying the assumptions of Corollary 9.

**Proposition 10.** Under the assumptions of Corollary 9 let \( X = f_1(x, y), \ Y = f_2(x, y) \) be the local real analytic change of variables that transforms system (12) into the form (13). Then an inverse integrating factor of system (12) is

\[
V(x, y) = \frac{f_1(x, y) f_2(x, y)}{J(x, y)}.
\]

**Proof.** This proposition can be proved as Proposition 8 taking into account that the first integral of system (13) is

\[
H = \lambda_1 \ln |X| - \lambda_2 \ln |Y|.
\]

The next theorem will allow us to study the existence of inverse integrating factors in a neighborhood of generic saddles.

**Theorem 11 (Siegel).** Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of the \( n \times n \) matrix \( A \). If there exists \( C > 0 \) and \( \mu > 0 \) such that for any nonnegative integers \( \alpha_1, \ldots, \alpha_n \) with \( \alpha_1 + \cdots + \alpha_n \geq 2 \),

\[
|\alpha_1 \lambda_1 + \cdots + \alpha_n \lambda_n - \lambda_j| \geq \frac{C}{|\alpha_1 + \cdots + \alpha_n|^{\mu}},
\]

\( 1 \leq j \leq n \), then the differential system \( x' = Ax + f(x), \ f(x) = O(|x|^3) \) as \( x \to 0 \), and \( f(x) \) is analytic in \( x \) can be transformed to \( x' = Ax \) in a neighborhood of the origin of \( \mathbb{C}^n \) by an analytic transformation.

For a proof of Theorem 11 see [25, 8]. From Theorem 11 it follows easily the next result.

**Corollary 12.** We consider the planar real analytic differential system

\[
x' = \lambda_1 x + g_1(x, y), \quad y' = \lambda_2 y + g_2(x, y),
\]

(14)

with \( \lambda_1 < 0 < \lambda_2 \), if there exists \( C > 0 \) and \( \mu > 0 \) such that for any nonnegative integers \( \alpha_1, \alpha_2 \) with \( \alpha_1 + \alpha_2 \geq 2 \),

\[
|\alpha_1 \lambda_1 + \alpha_2 \lambda_2 - \lambda_j| \geq \frac{C}{|\alpha_1 + \alpha_2|^{\mu}},
\]
1 \leq j \leq 2$, and $g_1$ and $g_2$ are of second order in $x$ and $y$. Then there exists a real analytic change of variables $(x, y) = (X, Y) + F(X, Y)$, $(X, Y) \in \Omega$, $F(X, Y) = O(1, 2, XY, Y^2)$ as $(X, Y) \to 0$, and $\Omega \subset \mathbb{R}^2$ is a neighborhood of the origin, which transforms system (14) into

$$
X' = \lambda_1 X, \quad Y' = \lambda_2 Y.
$$

(15)

The next proposition shows the existence of an inverse integrating factor in a neighborhood of a Siegel hyperbolic saddle, i.e., a saddle satisfying the assumptions of Corollary 12.

**Proposition 13.** Under the assumptions of Corollary 12 let $X = f_1(x, y)$, $Y = f_2(x, y)$ be the local real analytic change of variables that transforms system (14) into the form (15). Then an inverse integrating factor of system (14) is

$$
V(x, y) = \frac{f_1(x, y) f_2(x, y)}{J(x, y)}.
$$

**Proof.** Repeat the proof of Proposition 10.

The main result of this section is the following one.

**Theorem 14.** The analytic inverse integrating factors given in Propositions 8, 10, and 13 are unique.

**Proof.** To simplify the calculations we assume that the linear part of systems (10), (12), (14), has been transformed by a linear change of variables to its diagonal form

$$
x' = p_1 x + p_3 x^2 + p_4 xy + p_5 y^2 + \cdots, \quad y' = q_2 y + q_3 x^2 + q_4 xy + q_5 y^2 + \cdots,
$$

(16)

where $p_1$ and $q_2$ are the eigenvalues of the linear part (here the expressions of $x'$ and $y'$ are complex).

Equation (1) evaluated at the singular point located at the origin gives $(p_1 + q_2) V(0, 0) = 0$. If $p_1 + q_2 \neq 0$ then $V(0, 0) = 0$. We note that, for a non-degenerate focus or node, and a non-degenerate saddle satisfying the assumptions of Corollary 12, $p_1 + q_2$ is always nonzero. Consequently, the function $V$ always vanishes on these three types of singular points.

Taking partial derivatives of (9) at the origin we obtain

$$
p_1 \frac{\partial V}{\partial y}(0, 0) = 0, \quad q_2 \frac{\partial V}{\partial x}(0, 0) = 0,
$$

where $p_1$ and $q_2$ are of second order in $x$ and $y$. Then there exists a real analytic change of variables $(x, y) = (X, Y) + F(X, Y)$, $(X, Y) \in \Omega$, $F(X, Y) = O(1, 2, XY, Y^2)$ as $(X, Y) \to 0$, and $\Omega \subset \mathbb{R}^2$ is a neighborhood of the origin, which transforms system (14) into

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$$
V(x, y) = \frac{f_1(x, y) f_2(x, y)}{J(x, y)}.
$$

**Proof.** Repeat the proof of Proposition 10.

The main result of this section is the following one.

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$$
x' = p_1 x + p_3 x^2 + p_4 xy + p_5 y^2 + \cdots, \quad y' = q_2 y + q_3 x^2 + q_4 xy + q_5 y^2 + \cdots,
$$

(16)

where $p_1$ and $q_2$ are the eigenvalues of the linear part (here the expressions of $x'$ and $y'$ are complex).

Equation (1) evaluated at the singular point located at the origin gives $(p_1 + q_2) V(0, 0) = 0$. If $p_1 + q_2 \neq 0$ then $V(0, 0) = 0$. We note that, for a non-degenerate focus or node, and a non-degenerate saddle satisfying the assumptions of Corollary 12, $p_1 + q_2$ is always nonzero. Consequently, the function $V$ always vanishes on these three types of singular points.

Taking partial derivatives of (9) at the origin we obtain

$$
p_1 \frac{\partial V}{\partial y}(0, 0) = 0, \quad q_2 \frac{\partial V}{\partial x}(0, 0) = 0,
$$
Then
\[ \frac{\partial V}{\partial x} (0, 0) = 0, \quad \frac{\partial V}{\partial y} (0, 0) = 0. \]

Therefore, the expansion of \( V \) in powers of \( x \) and \( y \) can be written as
\[ V(x, y) = c_{20} x^2 + c_{11} xy + c_{02} y^2 + \cdots + \left( \sum_{k=0}^{n} c_{n-k, k} x^{n-k} y^k \right) + \cdots. \tag{17} \]

Replacing (17) in (1) we find \( c_{20} = c_{02} = 0 \) and \( c_{11} \) arbitrarily valued. The calculation of the coefficients of terms of degree 3 yields the following \( 4 \times 4 \) linear system of equations
\[
\begin{pmatrix}
2p_1 - q_2 & 0 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & q_2 & 0 \\
0 & 0 & 2q_2 - p_1 & 0
\end{pmatrix}
\begin{pmatrix}
c_{30} \\
c_{21} \\
c_{12} \\
c_{03}
\end{pmatrix}
= \begin{pmatrix}
- q_3 \\
p_3 \\
q_5 \\
-p_5
\end{pmatrix}.
\]

Since the origin is either a non-degenerate focus, or a non-resonant node, or a non-degenerate saddle satisfying the assumptions of Corollary 12, the above linear system has a unique solution, each coefficient \( c_{ij} \) being proportional to \( c_{11} \). This result can be generalized to all orders. At order \( n \), the \( n+1 \) coefficients \( c_{ij} \) of the homogeneous polynomial of degree \( n \) are also calculated from a diagonal linear system of equations. The determinant \( A_n \) of this system is a function only of \( p_1 \) and \( q_2 \) and is given by
\[ A_n = \prod_{k=0}^{n} \left[ (n - 1 - k) p_1 + (k - 1) q_2 \right]. \]

Again, since the origin is either a non-degenerate focus, or a non-resonant node, or a non-degenerate saddle satisfying the assumptions of Corollary 12, we obtain that \( A_n \neq 0 \). Therefore, the coefficients \( c_{n-k, k} \) with \( 0 \leq k \leq n \) are determined in a unique way and can be written as
\[ c_{n-k, k} = \frac{c_{11} f_{n,k}(p_1, q_2)}{(n - 1 - k) p_1 + (k - 1) q_2}, \]
where \( f_{n,k} \) is a polynomial in the variables \( p_1 \) and \( q_2 \).

Hence, under the assumptions of Propositions 8, 10, and 13, Eq. (9) has a unique formal series solution in powers of \( x \) and \( y \), up to a multiplicative constant factor. Therefore, the theorem follows.

The conditions we have imposed in order to ensure the existence and uniqueness of the formal series solution of (9) in the proof of Theorem 14
are sufficient but not necessary. We note that the cases that are excluded are not generic. An arbitrarily small perturbation of the system can always be performed in a nongeneric case in such a way that the new system will satisfy these conditions.

Theorem 14 when the origin is a non-degenerate focus or a node is essentially due to [15].

From Theorem 14 it follows easily the next result.

**Corollary 15.** Let $U$ be an open subset of $\mathbb{R}^2$ such that all singular points of system (1) in $U$ are focus, nodes or saddles satisfying the assumptions of Propositions 8, 10, and 13 respectively. Let $V_1$ and $V_2$ be analytic inverse integrating factors defined in $U$. Then $V_1 = CV_2$ for some $C \in \mathbb{R}$.

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**References**


