Generalized Inverse of Linear Transformations: A Geometric Approach

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ABSTRACT

A generalized inverse of a linear transformation $A:\mathscr{V}\to\mathscr{W}$, where $\mathscr{V}$ and $\mathscr{W}$ are arbitrary finite dimensional vector spaces, is defined using only geometrical concepts of linear transformations. The inverse is uniquely defined in terms of specified subspaces $\mathscr{L}\subset\mathscr{W}$, $\mathscr{M}\subset\mathscr{V}$ and a linear transformation N satisfying some conditions. Such an inverse is called the $\mathscr{L}\mathscr{M}N$-inverse. A Moore-Penrose type inverse is obtained by choosing $N=0$. Some optimization problems are considered by choosing $\mathscr{V}$ and $\mathscr{W}$ as inner product spaces. Our results extend without any major modification of proofs to bounded linear operators with closed range on Hilbert spaces.

1. INTRODUCTION

Let $\mathscr{V}$ and $\mathscr{W}$ be finite dimensional vector spaces, and $A:\mathscr{V}\to\mathscr{W}$ a linear transformation. We denote by $\mathscr{A}\subset\mathscr{W}$ the range space of A, by $\mathscr{L}$ a direct complement of $\mathscr{A}$ (i.e., $\mathscr{A}\oplus\mathscr{L}=\mathscr{W}$), by $\mathscr{K}$ the kernel (or the null space) of A, and by $\mathscr{M}$ a direct complement of $\mathscr{K}$ (i.e., $\mathscr{M}\oplus\mathscr{K}=\mathscr{V}$). The range space of any general transformation T will be indicated by $R(T)$. The projection operator on $\mathscr{A}$ along $\mathscr{L}$ is denoted by $P_{\mathscr{A}\cdot\mathscr{L}}$, and that on $\mathscr{M}$ along $\mathscr{K}$ by $P_{\mathscr{M}\cdot\mathscr{K}}$. These projection operators are well defined (see [8, pp.

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106–113], [9], and [10]). The following properties hold from the definitions:

$$P_{\mathscr{A}\cdot\mathscr{L}} + P_{\mathscr{L}\cdot\mathscr{A}} = I \text{ (identity operator)}, \tag{1.1}$$

$$P_{\mathscr{M}\cdot\mathscr{K}} + P_{\mathscr{K}\cdot\mathscr{M}} = I, \tag{1.2}$$

$$AP_{\mathscr{M}\cdot\mathscr{K}} = A \quad \text{and} \quad AP_{\mathscr{K}\cdot\mathscr{M}} = 0. \tag{1.3}$$

If $A:\mathscr{V}\to\mathscr{W}$ is not bijective, there is no unique inverse transformation $A^{-1}:\mathscr{W}\to\mathscr{V}$. In such a case, an inverse can be defined only in some special sense and for specific purposes. Early attempts at defining such inverses in the case of a matrix transformation are due to Moore [3], Bjerhammar [1], Penrose [5], and Rao [6]. Bjerhammar and Rao were concerned with the applications in least squares theory. Later, Rao [7] showed that in applications such as solving consistent linear equations $Ax = y$, an inverse transformation $G:\mathscr{W}\to\mathscr{V}$ should be such that Gy is a preimage of y for all $y\in R(A)$. This implies that $AGA = A$, or $AG|\mathscr{A} = I$, where $AG|\mathscr{A}$ is the operator AG restricted to $\mathscr{A}$. Such a G, which may not be unique, was called a g-inverse of A in [7], and represented by A^-. Rao [7] also showed that given any A^-, all the preimages of $y\in R(A)$ are provided by the set $\{A^-y+(I-A^-A)z,\ z$ arbitrary$\}$.

While Moore and Penrose used orthogonal projection operators in defining the g-inverse, Langenhop [2] used general projection operators and obtained a class of g-inverses with the reflexive type (outer inverse) as a unique member. Nashed and Votruba [4] provided a general framework for studying different types of g-inverses constructed for specific purposes. Reference may also be made to the treatise by Rao, Radhakrishna, and Mitra [8], which contains a detailed discussion of g-inverses and their applications.

In this paper, we provide a general definition of a g-inverse using only the geometrical concept of a linear transformation, which seems to provide a unified treatment of the theory of g-inverses of linear transformations and also characterize different types of g-inverses in terms of specified subspaces $\mathscr{M}$ and $\mathscr{L}$ in $\mathscr{V}$ and $\mathscr{W}$ and a linear transformation $N:\mathscr{W}\to\mathscr{V}$.

2. THE $\mathscr{L}\mathscr{M}N$-INVERSE

Let G be such that $AG|\mathscr{A} = I$ on $\mathscr{A}$. Then the following hold:

(i) If $\mathscr{M} = R(GA)$, then $\mathscr{M}$ is a direct complement of $\mathscr{K}\subset\mathscr{V}$, the kernel of A, and

$$A|\mathscr{M}:\mathscr{M}\to\mathscr{A} \text{ is bijective}, \tag{2.1}$$

in which case there exists a unique inverse of $A|\mathscr{M}$ which maps $\mathscr{A}$ onto $\mathscr{M}$, and which is the same as $G|\mathscr{A}$.

(ii) If $\mathscr{L} = R(I - AG)$, then $\mathscr{L} \subset \mathscr{W}$ is a direct complement of $\mathscr{A}$ and

$$G|\mathscr{L}: \mathscr{L} \to \mathscr{N} \subset \mathscr{X}, \tag{2.2}$$

where $\mathscr{N} = R(G - GAG)$.

(iii) If $N = G - GAG$, then $\mathscr{N} = R(N)$ and

$$AN = 0, \qquad NA = 0, \qquad N|\mathscr{L} = G|\mathscr{L}. \tag{2.3}$$

Thus, given a $G \in \{A^-\}$, the class of all solutions of $AGA = A$, there exist an $\mathscr{L}, \mathscr{M}, N$ associated with it, with the properties (2.1)–(2.3). In the terminology of Nashed and Votruba [4], N represents the deficiency in G from being an outer (reflexive) inverse. Does there exist a $G \in \{A^-\}$ for *any* given set of $\mathscr{L}, \mathscr{M}, N$ as described in (i)–(iii)? The answer is contained in the following definition and theorems.

Let $\mathscr{M}$ be any complement of $\mathscr{X}$ in $\mathscr{V}$, $\mathscr{L}$ be any complement of $\mathscr{A}$ in $\mathscr{W}$, and $N: \mathscr{W} \to \mathscr{V}$ be any linear transformation such that $AN = 0$, $NA = 0$.

DEFINITION. Let $\mathscr{L}, \mathscr{M}, N$ be as specified above. Then a linear transformation $G: \mathscr{W} \to \mathscr{V}$ is said to be an $\mathscr{L}\mathscr{M}N$-inverse of A iff

$$G|\mathscr{A} = T_{\mathscr{M}}, \qquad G|\mathscr{L} = N|\mathscr{L}, \tag{2.4}$$

where $T_{\mathscr{M}}: \mathscr{A} \to \mathscr{M}$ is the unique inverse of $A|\mathscr{M}: \mathscr{M} \to \mathscr{A}$.

We denote an $\mathscr{L}\mathscr{M}N$-inverse by $G_{\mathscr{L}\mathscr{M}N}$ and prove the following theorems.

THEOREM 2.1. *$G_{\mathscr{L}\mathscr{M}N}$ defined by* (2.4) *exists, and the mapping $\{\mathscr{L}, \mathscr{M}, N\} \to \{A^-\}$ is bijective.*

Proof. Consider the decomposition $y = y_1 + y_2$ ($y \in \mathscr{W}$, $y_1 \in \mathscr{A}$, $y_2 \in \mathscr{L}$), and define

$$Gy = T_{\mathscr{M}}y_1 + Ny_2. \tag{2.5}$$

Then G is linear and satisfies (2.4), so that $G_{\mathscr{L}\mathscr{M}N}$ exists. Let G_1 and G_2 be two solutions of (2.4) for given $\mathscr{L}, \mathscr{M}, N$. Then $(G_1 - G_2)y = 0\ \forall y \in \mathscr{W} \Rightarrow G_1 = G_2$, so that $G_{\mathscr{L}\mathscr{M}N}$ is unique.

Suppose that $G_{\mathcal{L}_1\mathcal{M}_1N_1} = G_{\mathcal{L}_2\mathcal{M}_2N_2} = G$. Then $R(G|A) = \mathcal{M}_1 = \mathcal{M}_2$, and $AG|\mathcal{A} = I$ and $AG|(\mathcal{L}_1 \cup \mathcal{L}_2) = 0 \Rightarrow \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ (say). Finally, $G|\mathcal{L} = N_1|\mathcal{L} = N_2|\mathcal{L}$, so that $(N_1 - N_2)|\mathcal{L} = 0$. But $(N_1 - N_2)|\mathcal{A} = 0$, so that $N_1 = N_2$. The theorem is proved. ∎

NOTE. If instead of $\mathcal{L}, \mathcal{M}, N$, we specify the three subspaces $\mathcal{L}, \mathcal{M}, \mathcal{N}$ where $\mathcal{N} = R(N)$ as in (2.2), the G so determined is not unique to the extent that there may be different choices of N such that $\mathcal{N} = R(N)$. Thus an $\mathcal{L}\mathcal{M}\mathcal{N}$-inverse could be defined, and a general solution could be obtained by varying N such that $R(N) = \mathcal{N}$.

THEOREM 2.2.

$$G_{\mathcal{L}\mathcal{M}0} = G_{\mathcal{L}\mathcal{M}0}AG_{\mathcal{L}\mathcal{M}0} \tag{2.6}$$

i.e., $G_{\mathcal{L}\mathcal{M}0}$ *is a reflexive inverse of* A, *and*

$$G_{\mathcal{L}\mathcal{M}N} = G_{\mathcal{L}\mathcal{M}0} + N. \tag{2.7}$$

Proof. If $N = 0$ and y_1 is the component of $y \in \mathcal{W}$ in $\mathcal{A}$, then

$$G_{\mathcal{L}\mathcal{M}0}y = T_{\mathcal{M}}y_1 = G_{\mathcal{L}\mathcal{M}0}y_1 \tag{2.8}$$

and

$$G_{\mathcal{L}\mathcal{M}0}AG_{\mathcal{L}\mathcal{M}0}y = G_{\mathcal{L}\mathcal{M}0}AT_{\mathcal{M}}y_1 = G_{\mathcal{L}\mathcal{M}0}y_1. \tag{2.9}$$

(2.8) and (2.9) $\Rightarrow$ (2.6).

It is easily verified that

$$(G_{\mathcal{L}\mathcal{M}0} + N)|\mathcal{A} = G_{\mathcal{L}\mathcal{M}0}\mathcal{A} \quad \text{and} \quad (G_{\mathcal{L}\mathcal{M}0} + N)|\mathcal{L} = N|\mathcal{L},$$

which proves (2.7). ∎

Note that $G_{\mathcal{L}\mathcal{M}0}$ is reflexive (or outer inverse), i.e., $G_{\mathcal{L}\mathcal{M}N}AG_{\mathcal{L}\mathcal{M}N} = G_{\mathcal{L}\mathcal{M}N}$ only if $N = 0$.

THEOREM 2.3. *The following statements are equivalent for given* $\mathcal{L}$, $\mathcal{M}$, *and* N, *where* $P_{\mathcal{M}\cdot\mathcal{K}}$ *and* $P_{\mathcal{L}\cdot\mathcal{A}}$ *are projection operators as defined in*

(1.1)–(1.3):

(i) *G is the $\mathscr{L}\mathscr{M}N$-inverse, i.e., satisfies* (2.4).
(ii) $GA = P_{\mathscr{M}\cdot\mathscr{K}}$, $GP_{\mathscr{L}\cdot\mathscr{A}} = N$.
(iii) $GA = P_{\mathscr{M}\cdot\mathscr{K}}$, $AG = P_{\mathscr{A}\cdot\mathscr{L}}$, $P_{\mathscr{K}\cdot\mathscr{M}}G = N$.
(iv) $GA = P_{\mathscr{M}\cdot\mathscr{K}}$, $AG = P_{\mathscr{A}\cdot\mathscr{L}}$, $G - GAG = N$.
(v) $AGA = A$, $R(G|\mathscr{A}) = \mathscr{M}$, $GP_{\mathscr{L}\cdot\mathscr{A}} = N$.

Proof. First, we show that (ii) $\Leftrightarrow$ (iii). That (iii) $\Rightarrow$ (ii) easily follows, since

$$P_{\mathscr{K}\cdot\mathscr{M}}G = (I - P_{\mathscr{M}\cdot\mathscr{K}})G = (I - GA)G = G(I - AG)$$
$$= G(I - P_{\mathscr{A}\cdot\mathscr{L}}) = GP_{\mathscr{L}\cdot\mathscr{A}}.$$

To show that (ii) $\Rightarrow$ (iii), observe that $AGP_{\mathscr{L}\cdot\mathscr{A}} = 0$, and from (1.3)

$$A = AP_{\mathscr{M}\cdot\mathscr{K}} = AGA,$$

which imply that $AG = P_{\mathscr{A}\cdot\mathscr{L}}$. Also, we have

$$GP_{\mathscr{L}\cdot\mathscr{A}} = G(I - P_{\mathscr{A}\cdot\mathscr{L}}) = G(I - AG)$$
$$= (I - GA)G = (I - P_{\mathscr{M}\cdot\mathscr{K}})G = P_{\mathscr{K}\cdot\mathscr{M}}G,$$

which establishes the desired result.

That (i) $\Rightarrow$ (ii) follows from

$$GAx = x \qquad \text{if} \quad x \in \mathscr{M},$$

using the condition $G|\mathscr{A} = T_{\mathscr{M}}$, and

$$GAx = 0 \qquad \text{if} \quad x \in \mathscr{K},$$

thus establishing $GA = P_{\mathscr{M}\cdot\mathscr{K}}$, and $G|\mathscr{L} = N|\mathscr{L} \Rightarrow GP_{\mathscr{L}\cdot\mathscr{A}} = N$.

To prove that (iii) $\Rightarrow$ (iv), observe that $P_{\mathscr{K}\cdot\mathscr{M}} = I - GA$ and $P_{\mathscr{K}\cdot\mathscr{M}}G = G - GAG$.

It is easy to establish that (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i), which establishes Theorem 2.3.

NOTE 1. It is seen that when $N = 0$, statement (iv) of Theorem 2.3 reduces to the definition of an inverse given by Nashed and Votruba [4], so that their inverse if $G_{\mathscr{L}\mathscr{M}0}$.

NOTE 2. Let $\mathscr{V}$ and $\mathscr{W}$ be Euclidean spaces of m and n dimensions respectively, in which case A can be represented by an $m \times n$ matrix and G by an $n \times m$ matrix.

NOTE 3. Let $\mathscr{G} = R(G)$. When $N = 0$, the conditions of (iv) of Theorem 2.3,

$$GA = P_{\mathscr{M}\cdot\mathscr{K}}, \qquad AG = P_{\mathscr{A}\cdot\mathscr{L}}, \qquad G = GAG, \tag{2.10}$$

are equivalent to

$$GA = P_{\mathscr{G}\cdot\mathscr{K}}, \qquad AG = P_{\mathscr{A}\cdot\mathscr{L}}, \qquad \mathscr{M} = \mathscr{G}. \tag{2.11}$$

If we consider orthogonal projection operators, then (2.11) reduces to

$$GA = P_{\mathscr{G}}, \qquad AG = P_{\mathscr{A}}, \tag{2.12}$$

since $\mathscr{M}$ and $\mathscr{L}$ are uniquely determined by $\mathscr{K}$ and $\mathscr{A}$, which is the definition given by Moore and Penrose.

In the next sections we consider classes of inverses obtained by not specifying one or more of $\mathscr{L}$, $\mathscr{M}$, N.

3. THE $\mathscr{L}\mathscr{M}$-INVERSE

If in the definition (2.4), we do not specify N but only require $G|\mathscr{L}: \mathscr{L} \to \mathscr{K}$, then we can write the conditions in the form

$$G|\mathscr{A} = T_{\mathscr{M}} \quad \text{and} \quad AG|\mathscr{L} = 0. \tag{3.1}$$

We represent a solution of (3.1) by $G_{\mathscr{L}\mathscr{M}}$, which may not be unique, and call it an $\mathscr{L}\mathscr{M}$-inverse. We have the following theorem.

THEOREM 3.1. *The following statements are equivalent for given $\mathscr{L}$ and $\mathscr{M}$, any direct complements of $\mathscr{A}$ and $\mathscr{K}$ respectively*:

(i) *G is an $\mathscr{L}\mathscr{M}$-inverse.*
(ii) $GA = P_{\mathscr{M}\cdot\mathscr{K}}$, $AG = P_{\mathscr{A}\cdot\mathscr{L}}$.
(iii) $AGA = A$, $R(G|\mathscr{A}) = \mathscr{M}$, $AGP_{\mathscr{L}\cdot\mathscr{A}} = 0$.

The results are proved in the same way as in Theorem 2.3.

NOTE 1. The definition given in (ii) of Theorem 3.1 was proposed by Langenhop [2], who also provided a general solution for G as the sum of two parts, one of which is the $\mathscr{L}\mathscr{M}0$-inverse. However, an alternative construction is provided by Theorem 3.2, which is a restatement of Theorem 2.4 of Langenhop [2].

THEOREM 3.2. *Let A^- be any g-inverse of A, i.e., $AA^-A = A$. Then*

$$G_{\mathscr{L}\mathscr{M}0} = P_{\mathscr{M}\cdot\mathscr{K}} A^- P_{\mathscr{A}\cdot\mathscr{L}}, \tag{3.2}$$

and

$$G_{\mathscr{L}\mathscr{M}} = G_{\mathscr{L}\mathscr{M}0} + P_{\mathscr{K}\cdot\mathscr{M}} Z P_{\mathscr{L}\cdot\mathscr{A}} \tag{3.3}$$

is a general solution for an $\mathscr{L}\mathscr{M}$-inverse, where $Z: \mathscr{W} \to \mathscr{V}$ is arbitrary.

Proof. To prove (3.2), we verify the conditions (ii) of Theorem 2.3, putting $N = 0$. The second condition $G_{\mathscr{L}\mathscr{M}0} P_{\mathscr{L}\cdot\mathscr{A}} = 0$ is trivially true. To prove the first condition observe that

$$A(P_{\mathscr{M}\cdot\mathscr{K}} A^-A - I)x = 0 \quad \Rightarrow \quad (P_{\mathscr{M}\cdot\mathscr{K}} A^-A - I)x \in \mathscr{K}.$$

But $(P_{\mathscr{M}\cdot\mathscr{K}} A^-A - I)x \in \mathscr{M}$ if $x \in \mathscr{M}$. Hence

$$G_{\mathscr{L}\mathscr{M}0} Ax = P_{\mathscr{M}\cdot\mathscr{K}} A^-Ax = x \qquad \text{if} \quad x \in \mathscr{M}. \tag{3.4}$$

Since $G_{\mathscr{L}\mathscr{M}0} Ax = 0$ if $x \in \mathscr{K}$, it follows that $G_{\mathscr{L}\mathscr{M}0} A = P_{\mathscr{M}\cdot\mathscr{K}}$, which is the first condition in (ii) of Theorem 2.3. The result (3.2) is proved.

Since $G_{\mathscr{L}\mathscr{M}0}$ is a particular $\mathscr{L}\mathscr{M}$-inverse, we need only add a term which reduces to the null operator by both pre- and postmultiplications by A. Obviously a general expression for such a term is the second part of (3.3). Thus (3.3) is proved. ■

4. OTHER CLASSES OF INVERSES

$\mathscr{M}$-Inverse

An $\mathscr{M}$-inverse of A is G satisfying the condition

$$GA = P_{\mathscr{M}\cdot\mathscr{K}} \tag{4.1}$$

with the equivalent conditions

$$AGA = A \quad \text{and} \quad R(GA) = \mathscr{M}. \tag{4.2}$$

A general solution of (4.1) is

$$G = P_{\mathscr{M}\cdot\mathscr{K}} A^{-} + Z P_{\mathscr{L}\cdot\mathscr{A}}, \tag{4.3}$$

where $AA^{-}A = A$, and Z is arbitrary. We represent an $\mathscr{M}$-inverse by A_m^- (to be consistent with the notation developed in [8]).

If $\mathscr{V}$ is a vector space endowed with an inner product, then we may choose $\mathscr{M}$ to be the orthogonal complement of $\mathscr{K}$. In such a case, if $Ax = y$ is a consistent equation, then

$$\min_{Ax = y} \|x\| = \|A_m^- y\|, \tag{4.4}$$

so that $A_m^- y$ is the minimum norm solution of $Ax = y$.

$\mathscr{L}$-Inverse

An $\mathscr{L}$-inverse of A, denoted by A_l^-, is G satisfying the equation

$$AG = P_{\mathscr{A}\cdot\mathscr{L}} \tag{4.5}$$

with the equivalent conditions

$$AGA = A \quad \text{and} \quad AGP_{\mathscr{L}\cdot\mathscr{A}} = 0. \tag{4.6}$$

A general solution of (4.5) is

$$G = A^{-} P_{\mathscr{A}\cdot\mathscr{L}} + P_{\mathscr{K}\cdot\mathscr{M}} Z. \tag{4.7}$$

An $\mathscr{L}$0-inverse is G satisfying the equivalent conditions

$$AG = P_{\mathscr{A}\cdot\mathscr{L}}, \quad G = GAG \Leftrightarrow AG = P_{\mathscr{A}\cdot\mathscr{L}}, \quad GA = P_{\mathscr{G}\cdot\mathscr{K}} \tag{4.8}$$

If $\mathscr{W}$ is an inner product vector space, we may choose $\mathscr{L}$ to be an orthogonal complement of $\mathscr{A}$. In such a case

$$\min_x \|y - Ax\| = \|y - AA_l^- y\|, \tag{4.9}$$

so that $A_l^- y$ is a general least squares solution.

$\mathscr{L}\mathscr{M}$N-Inverse

Let us consider inner product spaces $\mathscr{V}$ and $\mathscr{W}$, and the problem of minimizing $\|x\|$ subject to x being a least squares solution of $Ax = y$. If

$y_1 = P_{\mathscr{A}\cdot\mathscr{L}}y$, then the problem reduces to minimizing $\|x\|$ subject to the consistent equation $Ax = y_1$. Then the optimum x is obtained by using an $\mathscr{M}$-inverse. The solution is $x = A_m^- y_1 = A_m^- P_{\mathscr{A}\cdot\mathscr{L}}y$. It is seen that if $G = A_m^- P_{\mathscr{A}\cdot\mathscr{L}}$, then

$$GA = P_{\mathscr{M}\cdot\mathscr{K}} \quad \text{and} \quad GP_{\mathscr{L}\cdot\mathscr{A}} = 0,$$

so that G is the $\mathscr{L}\mathscr{M}N$-inverse of A with $N = 0$. This inverse (when $N = 0$, $\mathscr{L} = A^{\perp}$, $\mathscr{M} = \mathscr{K}^{\perp}$) may be denoted by A^+; it is the Moore-Penrose inverse.

All the above results can be extended without any major modification of the proofs to bounded linear operators with closed range of Hilbert spaces.

5. EXPRESSIONS FOR g-INVERSES OF MATRICES

We derive explicit expressions for g-inverses of matrices, for which we consider the linear transformation A as an $m \times n$ matrix and take $\mathscr{V} = E^n$ and $\mathscr{W} = E^m$. We prove the following lemma, where A' represents the transpose of A; $K(T)$, the kernel of a matrix transformation T; and $R(T)$, the range space of T.

LEMMA 5.1. *Let a matrix C be such that $R(A') \cap R(C') = \varnothing$, the null vector, and*

$$R(A') \oplus R(C') = E^n. \tag{5.1}$$

Then $K(A) \cap K(C) = \varnothing$ and

$$K(A) \oplus K(C) = E^n. \tag{5.2}$$

Proof. Let $x \in K(A) \cap K(C)$. Then $Ax = 0$, $Cx = 0 \Rightarrow x = 0$ in view of (5.1), i.e., $K(A) \cap K(C) = \varnothing$. Further note that

$$\dim K(A) + \dim K(C) = [n - \text{rank}(A)] + [n - \text{rank}(C)] = n,$$

which establishes (5.2). ■

The following theorem is a consequence of Lemma 5.1.

THEOREM 4.1. *Let C be such that* (5.1) *holds, and F be a matrix such that $R(F)$ is the direct complement of $K(A)$. Then*

$$P_{R(F)\cdot K(A)} = P_{K(C)\cdot K(A)} \Leftrightarrow CP_{R(F)\cdot K(A)} = 0. \tag{5.3}$$

Further

$$(P_{R(A')\cdot R(C')})' = P_{K(C)\cdot K(A)}. \tag{5.4}$$

Proof. (5.3) is easy to establish. To prove (5.4), we may observe that $R(I-P') = K(A)$ and $R(P') = K(C)$, implying that $PA' = A'$ and $PC' = C'$, where $P = P_{R(A')\cdot R(C')}$. ■

LEMMA 5.2. *Let B be a matrix such that $R(B)$ is a direct complement of $R(A)$, and define $S_B = I - BB^-$ and $Q_B = I - B(B'B)^- B'$. Then*

$$P_{R(A)\cdot R(B)} = A(S_B A)^- S_B, \tag{5.5}$$

$$= A(A'Q_B A)^- A'Q_B, \tag{5.6}$$

$$= AA'(AA' + BB')^{-1}. \tag{5.7}$$

A proof of Lemma 5.2 is given in [9]. Using Theorem 5.1 and Lemma 5.2, it is easy to establish the following lemma.

LEMMA 5.3. *Let $S_C = I - C^- C$ and $Q_{C'} = I - C'(CC')^- C$. Then*

$$P_{K(C)\cdot K(A)} = S_C(AS_C)^- A, \tag{5.8}$$

$$= Q_{C'}A'(AQ_{C'}A')^- A', \tag{5.9}$$

$$= (A'A + C'C)^{-1} A'A. \tag{5.10}$$

Using these results, we give representations of g-inverses of matrices.

THEOREM 5.2.

(i) *If we choose $\mathcal{M} = K(C)$, then the $\mathcal{M}$-inverse of A can be written as*

$$A_m^- = S_C(AS_C)^- + ZP_{R(B)\cdot R(A)} \tag{5.11}$$

$$= Q_{C'}A'(AQ_{C'}A')^- + ZP_{R(B)\cdot R(A)} \tag{5.12}$$

$$= (A'A + C'C)^{-1}A' + ZP_{R(B)\cdot R(A)}, \tag{5.13}$$

where Z is an arbitrary matrix.

(ii) *With B as defined in Lemma* 5.2 [*i.e.*, $\mathscr{L} = R(B)$], *the* $\mathscr{L}$*-inverse of A can be written as*

$$A_l^- = (S_B A)^- S_B + P_{K(A) \cdot K(C)} Z \tag{5.14}$$

$$= (A'Q_B A)^- A'Q_B + P_{K(A) \cdot K(C)} Z \tag{5.15}$$

$$= A'(AA' + BB')^{-1} + P_{K(A) \cdot K(C)} Z. \tag{5.16}$$

(iii) *With* $\mathscr{L} = R(B)$ *and* $\mathscr{M} = K(C)$, $N = 0$, *the* $\mathscr{L}\mathscr{M}N$*-inverse of A can be written as*

$$A_{ml}^+ = S_C (AS_C)^- A(S_B A)^- S_B \tag{5.17}$$

$$= Q_{C'} A'(AQ_{C'}A')^- A(A'Q_B A)^- A'Q_B \tag{5.18}$$

$$= (A'A + C'C)^{-1} A'AA'(AA' + BB')^{-1}. \tag{5.19}$$

COROLLARY.

(i) *If, in particular,* $R(B) = R(A)^{\perp}$ *under the Euclidean inner product, then*

$$A_{ml}^+ = Q_{C'} A'(AQ_{C'}A')^- A(A'A)^- A' = (A'A + C'C)^{-1} A'. \tag{5.20}$$

(ii) *If* $K(C) = K(A)^{\perp}$, *then*

$$A_{ml}^+ = A'(AA')^- A(A'Q_B A)^- A'Q_B = A'(AA' + BB')^{-1}. \tag{5.21}$$

(iii) *If* $R(B) = R(A)^{\perp}$ *and* $K(C) = K(A)^{\perp}$ *hold simultaneously, then*

$$A_{ml}^+ = A'(AA')^- A(A'A)^- A', \tag{5.22}$$

which is exactly the Moore-Penrose inverse of A.

NOTE. A_{ml}^{+} as obtained in Theorem 5.2 is the Moore-Penrose inverse of the matrix $(Q_B A Q_{C'})$, since A_{ml}^{+} satisfies the following conditions:

(i) $(Q_B A Q_{C'}) A_{ml}^{+} (Q_B A Q_{C'}) = Q_B A Q_{C'}$,
(ii) $A_{ml}^{+} (Q_B A Q_{C'}) A_{ml}^{+} = A_{ml}^{+}$,
(iii) $(Q_B A Q_{C'} A_{ml}^{+})' = Q_B A Q_{C'} A_{ml}^{+}$,
(iv) $(A_{ml}^{+} Q_B A Q_{C'})' = A_{ml}^{+} Q_B A Q_{C'}$.

Thus, A_{ml}^{+} is uniquely determined for any choices of matrices B and C spanning $\mathscr{L} = R(B)$ and $\mathscr{M} = K(C)$ respectively.

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