ABSTRACT

A generalized inverse of a linear transformation \( A : \mathcal{V} \to \mathcal{W} \), where \( \mathcal{V} \) and \( \mathcal{W} \) are arbitrary finite dimensional vector spaces, is defined using only geometrical concepts of linear transformations. The inverse is uniquely defined in terms of specified subspaces \( \mathcal{V} \subset \mathcal{W} \), \( \mathcal{M} \subset \mathcal{V} \) and a linear transformation \( N \) satisfying some conditions. Such an inverse is called the \( \mathcal{V} \mathcal{M} \mathcal{N} \)-inverse. A Moore-Penrose type inverse is obtained by choosing \( N = 0 \). Some optimization problems are considered by choosing \( \mathcal{V} \) and \( \mathcal{W} \) as inner product spaces. Our results extend without any major modification of proofs to bounded linear operators with closed range on Hilbert spaces.

1. INTRODUCTION

Let \( \mathcal{V} \) and \( \mathcal{W} \) be finite dimensional vector spaces, and \( A : \mathcal{V} \to \mathcal{W} \) a linear transformation. We denote by \( \mathcal{A} \subset \mathcal{W} \) the range space of \( A \), by \( \mathcal{L} \) a direct complement of \( \mathcal{A} \) (i.e., \( \mathcal{A} \oplus \mathcal{L} = \mathcal{W} \)), by \( \mathcal{K} \) the kernel (or the null space) of \( A \), and by \( \mathcal{M} \) a direct complement of \( \mathcal{K} \) (i.e., \( \mathcal{M} \oplus \mathcal{K} = \mathcal{V} \)). The range space of any general transformation \( T \) will be indicated by \( R(T) \). The projection operator on \( \mathcal{A} \) along \( \mathcal{L} \) is denoted by \( P_{\mathcal{A}, \mathcal{L}} \), and that on \( \mathcal{M} \) along \( \mathcal{K} \) by \( P_{\mathcal{M}, \mathcal{K}} \). These projection operators are well defined (see [8, pp.
The following properties hold from the definitions:

\[ P_{\mathcal{M}} + P_{\mathcal{N}} = I \]  
(1.1)

\[ P_{\mathcal{N}} + P_{\mathcal{M}} = I \]  
(1.2)

\[ AP_{\mathcal{M}} = A \quad \text{and} \quad AP_{\mathcal{N}} = 0. \]  
(1.3)

If \( A: \mathcal{Y} \to \mathcal{W} \) is not bijective, there is no unique inverse transformation \( A^{-1}: \mathcal{W} \to \mathcal{Y} \). In such a case, an inverse can be defined only in some special sense and for specific purposes. Early attempts at defining such inverses in the case of a matrix transformation are due to Moore [3], Bjerhammar [1], Penrose [5], and Rao [6]. Bjerhammar and Rao were concerned with the applications in least squares theory. Later, Rao [7] showed that in applications such as solving consistent linear equations \( Ax = y \), an inverse transformation \( G: \mathcal{W} \to \mathcal{Y} \) should be such that \( Gy \) is a preimage of \( y \) for all \( y \in R(A) \). This implies that \( AGA = A \), or \( AG|_{\mathcal{K}} = I \), where \( AG|_{\mathcal{K}} \) is the operator \( AG \) restricted to \( \mathcal{K} \). Such a \( G \), which may not be unique, was called a \( g \)-inverse of \( A \) in [7], and represented by \( A^{-} \). Rao [7] also showed that given any \( A^{-} \), all the preimages of \( y \in R(A) \) are provided by the set \( \{ A^{-} y + (I - A^{-} A)z, \ z \ \text{arbitrary} \} \).

While Moore and Penrose used orthogonal projection operators in defining the \( g \)-inverse, Langenhop [2] used general projection operators and obtained a class of \( g \)-inverses with the reflexive type (outer inverse) as a unique member. Nashed and Votruba [4] provided a general framework for studying different types of \( g \)-inverses constructed for specific purposes. Reference may also be made to the treatise by Rao, Radhakrishna, and Mitra [8], which contains a detailed discussion of \( g \)-inverses and their applications.

In this paper, we provide a general definition of a \( g \)-inverse using only the geometrical concept of a linear transformation, which seems to provide a unified treatment of the theory of \( g \)-inverses of linear transformations and also characterize different types of \( g \)-inverses in terms of specified subspaces \( \mathcal{M} \) and \( \mathcal{N} \) in \( \mathcal{Y} \) and \( \mathcal{W} \) and a linear transformation \( N: \mathcal{W} \to \mathcal{Y} \).

2. THE \( LMN \)-INVERSE

Let \( G \) be such that \( AG|_{\mathcal{K}} = I \) on \( \mathcal{K} \). Then the following hold:

(i) If \( \mathcal{M} = R(GA) \), then \( \mathcal{M} \) is a direct complement of \( \mathcal{K} \subset \mathcal{Y} \), the kernel of \( A \), and

\[ A|_{\mathcal{M}}: \mathcal{M} \to \mathcal{K} \text{ is bijective}, \]  
(2.1)
in which case there exists a unique inverse of $A|\mathcal{M}$ which maps $\mathcal{A}$ onto $\mathcal{M}$, and which is the same as $G|\mathcal{A}$.

(ii) If $\mathcal{L} = R(I - AG)$, then $\mathcal{L} \subset \mathcal{W}$ is a direct complement of $\mathcal{A}$ and

$$G|\mathcal{L} : \mathcal{L} \to \mathcal{N} \subset \mathcal{X},$$

(2.2)

where $\mathcal{N} = R(G - GAG)$.

(iii) If $N = G - GAG$, then $\mathcal{N} = R(N)$ and

$$AN = 0, \quad NA = 0, \quad N|\mathcal{L} = G|\mathcal{L}. \quad (2.3)$$

Thus, given a $G \in \{ A^\dagger \}$, the class of all solutions of $AGA = A$, there exist an $\mathcal{L}, \mathcal{M}, N$ associated with it, with the properties (2.1)-(2.3). In the terminology of Nashed and Votruba [4], $N$ represents the deficiency in $G$ from being an outer (reflexive) inverse. Does there exist a $G \in \{ A^\dagger \}$ for any given set of $\mathcal{L}, \mathcal{M}, N$ as described in (i)-(iii)? The answer is contained in the following definition and theorems.

Let $\mathcal{M}$ be any complement of $\mathcal{X}$ in $\mathcal{Y}$, $\mathcal{L}$ be any complement of $\mathcal{A}$ in $\mathcal{W}$, and $N : \mathcal{W} \to \mathcal{V}$ be any linear transformation such that $AN = 0, NA = 0$.

**Definition.** Let $\mathcal{L}, \mathcal{M}, N$ be as specified above. Then a linear transformation $G : \mathcal{W} \to \mathcal{Y}$ is said to be a $\mathcal{M}N$-inverse of $A$ iff

$$G|\mathcal{A} = T_{\#}, \quad G|\mathcal{L} = N|\mathcal{L}, \quad (2.4)$$

where $T_{\#} : \mathcal{A} \to \mathcal{M}$ is the unique inverse of $A|\mathcal{M} : \mathcal{M} \to \mathcal{A}$.

We denote an $\mathcal{M}N$-inverse by $G_{\mathcal{L},\mathcal{M},N}$ and prove the following theorems.

**Theorem 2.1.** $G_{\mathcal{L},\mathcal{M},N}$ defined by (2.4) exists, and the mapping $\{ \mathcal{L}, \mathcal{M}, N \} \to \{ A^\dagger \}$ is bijective.

*Proof.* Consider the decomposition $y = y_1 + y_2$ ($y \in \mathcal{W}$, $y_1 \in \mathcal{A}$, $y_2 \in \mathcal{L}$), and define

$$Gy = T_{\#}y_1 + Ny_2. \quad (2.5)$$

Then $G$ is linear and satisfies (2.4), so that $G_{\mathcal{L},\mathcal{M},N}$ exists. Let $G_1$ and $G_2$ be two solutions of (2.4) for given $\mathcal{L}, \mathcal{M}, N$. Then $(G_1 - G_2)y = 0 \forall y \in \mathcal{W} \Rightarrow G_1 = G_2$, so that $G_{\mathcal{L},\mathcal{M},N}$ is unique.
Suppose that $G_{\mathcal{L}, \mathcal{M}} N_1 = G_{\mathcal{L}, \mathcal{M}} N_2 = G$. Then $\mathcal{R}(G|\mathcal{A}) = \mathcal{M}_1 = \mathcal{M}_2$, and $AG|\mathcal{A} = I$ and $AG|(\mathcal{L}_1 \cup \mathcal{L}_2) = 0 \Rightarrow \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ (say). Finally, $G|\mathcal{L} = N_1|\mathcal{L} = N_2|\mathcal{L}$, so that $(N_1 - N_2)|\mathcal{A} = 0$. But $(N_1 - N_2)|\mathcal{L} = 0$, so that $N_1 = N_2$. The theorem is proved.

**NOTE.** If instead of $\mathcal{L}, \mathcal{M}, N$, we specify the three subspaces $\mathcal{L}, \mathcal{M}, \mathcal{N}$ where $\mathcal{N} = \mathcal{R}(N)$ as in (2.2), the $G$ so determined is not unique to the extent that there may be different choices of $N$ such that $\mathcal{N} = \mathcal{R}(N)$. Thus an $\mathcal{L}\mathcal{M}\mathcal{N}$-inverse could be defined, and a general solution could be obtained by varying $N$ such that $\mathcal{R}(N) = \mathcal{N}$.

**THEOREM 2.2.**

$$G_{\mathcal{L}, \mathcal{M}} = G_{\mathcal{L}, \mathcal{M}} AG_{\mathcal{L}, \mathcal{M}}$$

(2.6)

i.e., $G_{\mathcal{L}, \mathcal{M}}$ is a reflexive inverse of $A$, and

$$G_{\mathcal{L}, \mathcal{M}} = G_{\mathcal{L}, \mathcal{M}} + N.$$  

(2.7)

**Proof.** If $N = 0$ and $y_1$ is the component of $y \in \mathcal{W}$ in $\mathcal{A}$, then

$$G_{\mathcal{L}, \mathcal{M}} y = T_{\mathcal{M}} y_1 = G_{\mathcal{L}, \mathcal{M}} y_1$$

(2.8)

and

$$G_{\mathcal{L}, \mathcal{M}} AG_{\mathcal{L}, \mathcal{M}} y = G_{\mathcal{L}, \mathcal{M}} A T_{\mathcal{M}} y_1 = G_{\mathcal{L}, \mathcal{M}} y_1.$$ 

(2.9)

(2.8) and (2.9) $\Rightarrow$ (2.6).

It is easily verified that

$$(G_{\mathcal{L}, \mathcal{M}} + N)|\mathcal{A} = G_{\mathcal{L}, \mathcal{M}}|\mathcal{A} \quad \text{and} \quad (G_{\mathcal{L}, \mathcal{M}} + N)|\mathcal{L} = N|\mathcal{L},$$

which proves (2.7).

Note that $G_{\mathcal{L}, \mathcal{M}}$ is reflexive (or outer inverse), i.e., $G_{\mathcal{L}, \mathcal{M}} AG_{\mathcal{L}, \mathcal{M}} = G_{\mathcal{L}, \mathcal{M}}$ only if $N = 0$.

**THEOREM 2.3.** The following statements are equivalent for given $\mathcal{L}, \mathcal{M}$, and $N$, where $P_{\mathcal{M}}$ and $P_{\mathcal{L}}$ are projection operators as defined in
(1.1)–(1.3):

(i) \( G \) is the \( \mathcal{M} \mathcal{N} \)-inverse, i.e., satisfies (2.4).
(ii) \( GA = P_{\mathcal{M} \mathcal{X}}, \ \text{GP}_{\mathcal{L} \mathcal{M}} = N \).
(iii) \( GA = P_{\mathcal{M} \mathcal{X}}, \ \text{AG} = P_{\mathcal{L} \mathcal{M}}, \ \text{P}_{\mathcal{X} \mathcal{M}} G = N \).
(iv) \( GA = P_{\mathcal{M} \mathcal{X}}, \ \text{AG} = P_{\mathcal{L} \mathcal{M}}, \ \text{G} \ - \text{GAG} = N \).
(v) \( AGA = A, \ \text{R}(G|_{\mathcal{A}}) = \mathcal{M}, \ \text{GP}_{\mathcal{L} \mathcal{M}} = N \).

**Proof.** First, we show that (ii) \( \Leftrightarrow \) (iii). That (iii) \( \Rightarrow \) (ii) easily follows, since

\[
P_{\mathcal{X} \mathcal{M}} G = (I - P_{\mathcal{M} \mathcal{X}}) G = (I - GA) G = G (I - AG) = G (I - P_{\mathcal{L} \mathcal{M}}) = \text{GP}_{\mathcal{L} \mathcal{M}}.
\]

To show that (ii) \( \Rightarrow \) (iii), observe that \( AGP_{\mathcal{L} \mathcal{M}} = 0 \), and from (1.3)

\[
A = AP_{\mathcal{M} \mathcal{X}} = AGA,
\]

which imply that \( AG = P_{\mathcal{L} \mathcal{M}} \). Also, we have

\[
\text{GP}_{\mathcal{L} \mathcal{M}} = G (I - P_{\mathcal{L} \mathcal{M}}) = G (I - AG) = (I - GA) G = (I - P_{\mathcal{M} \mathcal{X}}) G = P_{\mathcal{X} \mathcal{M}} G,
\]

which establishes the desired result.

That (i) \( \Rightarrow \) (ii) follows from

\[
GAx = x \quad \text{if} \quad x \in \mathcal{M},
\]

using the condition \( G|_{\mathcal{A}} = T_{\mathcal{M}} \), and

\[
GAx = 0 \quad \text{if} \quad x \in \mathcal{X},
\]

thus establishing \( GA = P_{\mathcal{M} \mathcal{X}} \), and \( G|_{\mathcal{L}} = N|_{\mathcal{L}} \Rightarrow \text{GP}_{\mathcal{L} \mathcal{M}} = N \).

To prove that (iii) \( \Rightarrow \) (iv), observe that \( P_{\mathcal{X} \mathcal{M}} = I - GA \) and \( P_{\mathcal{X} \mathcal{M}} G = G \ - GAG \).

It is easy to establish that (iv) \( \Rightarrow \) (v) and (v) \( \Rightarrow \) (i), which establishes Theorem 2.3.

**Note 1.** It is seen that when \( N = 0 \), statement (iv) of Theorem 2.3 reduces to the definition of an inverse given by Nashed and Votruba [4], so that their inverse if \( G_{|_{\mathcal{L} \mathcal{M}} 0} \).
Note 2. Let \( \mathcal{Y} \) and \( \mathcal{W} \) be Euclidean spaces of \( m \) and \( n \) dimensions respectively, in which case \( A \) can be represented by an \( m \times n \) matrix and \( G \) by an \( n \times m \) matrix.

Note 3. Let \( \mathcal{G} = R(G) \). When \( N = 0 \), the conditions of (iv) of Theorem 2.3,

\[
GA = P_{\mathcal{M} \cdot \mathcal{X}}, \quad AG = P_{\mathcal{A} \cdot \mathcal{Y}}, \quad G = GAG,
\]  

are equivalent to

\[
GA = P_{\mathcal{G} \cdot \mathcal{X}}, \quad AG = P_{\mathcal{A} \cdot \mathcal{Y}}, \quad \mathcal{M} = \mathcal{G}.
\]  

If we consider orthogonal projection operators, then (2.11) reduces to

\[
GA = P_{\mathcal{G}}, \quad AG = P_{\mathcal{A}},
\]  

since \( \mathcal{M} \) and \( \mathcal{L} \) are uniquely determined by \( \mathcal{X} \) and \( \mathcal{A} \), which is the definition given by Moore and Penrose.

In the next sections we consider classes of inverses obtained by not specifying one or more of \( \mathcal{L}, \mathcal{M}, N \).

3. THE \( \mathcal{L}, \mathcal{M} \)-INVERSE

If in the definition (2.4), we do not specify \( N \) but only require \( G|\mathcal{L}: \mathcal{L} \rightarrow \mathcal{X} \), then we can write the conditions in the form

\[
G|\mathcal{A} = T_{\mathcal{M}} \quad \text{and} \quad AG|\mathcal{L} = 0.
\]  

We represent a solution of (3.1) by \( G_{\mathcal{L}, \mathcal{M}} \), which may not be unique, and call it an \( \mathcal{L}, \mathcal{M} \)-inverse. We have the following theorem.

Theorem 3.1. The following statements are equivalent for given \( \mathcal{L} \) and \( \mathcal{M} \), any direct complements of \( \mathcal{A} \) and \( \mathcal{X} \) respectively:

(i) \( G \) is an \( \mathcal{L}, \mathcal{M} \)-inverse.

(ii) \( GA = P_{\mathcal{A} \cdot \mathcal{X}}, \ AG = P_{\mathcal{A} \cdot \mathcal{Y}} \).

(iii) \( AGA = A, \ R(G|\mathcal{A}) = \mathcal{M}, \ AGP_{\mathcal{L} \cdot \mathcal{M}} = 0 \).

The results are proved in the same way as in Theorem 2.3.
Note 1. The definition given in (ii) of Theorem 3.1 was proposed by Langenhop [2], who also provided a general solution for \( G \) as the sum of two parts, one of which is the \( \mathcal{L}\mathcal{M} \)-inverse. However, an alternative construction is provided by Theorem 3.2, which is a restatement of Theorem 2.4 of Langenhop [2].

**Theorem 3.2.** Let \( \mathcal{A}^{-} \) be any g-inverse of \( \mathcal{A} \), i.e., \( \mathcal{A}\mathcal{A}^{-} \mathcal{A} = \mathcal{A} \). Then

\[
G_{\mathcal{L}\mathcal{M}0} = \mathcal{P}_{\mathcal{A}\mathcal{X}} \mathcal{A}^{-} \mathcal{P}_{\mathcal{A}\mathcal{X}'}, \tag{3.2}
\]

and

\[
G_{\mathcal{L}\mathcal{M}} = G_{\mathcal{L}\mathcal{M}0} + \mathcal{P}_{\mathcal{X}\mathcal{M}} \mathcal{Z} \mathcal{P}_{\mathcal{L}\mathcal{A}} \tag{3.3}
\]

is a general solution for an \( \mathcal{L}\mathcal{M} \)-inverse, where \( Z: \mathcal{W} \to \mathcal{V} \) is arbitrary.

**Proof.** To prove (3.2), we verify the conditions (ii) of Theorem 2.3, putting \( N = 0 \). The second condition \( G_{\mathcal{L}\mathcal{M}0} \mathcal{P}_{\mathcal{A}\mathcal{X}'} = 0 \) is trivially true. To prove the first condition observe that

\[
\mathcal{A}(\mathcal{P}_{\mathcal{A}\mathcal{X}'} \mathcal{A}^{-} \mathcal{A} - I)x = 0 \quad \Rightarrow \quad (\mathcal{P}_{\mathcal{A}\mathcal{X}'} \mathcal{A}^{-} \mathcal{A} - I)x \in \mathcal{K}.
\]

But \( (\mathcal{P}_{\mathcal{A}\mathcal{X}'} \mathcal{A}^{-} \mathcal{A} - I)x \in \mathcal{M} \) if \( x \in \mathcal{M} \). Hence

\[
G_{\mathcal{L}\mathcal{M}0} \mathcal{A}x = \mathcal{P}_{\mathcal{A}\mathcal{X}'} \mathcal{A}^{-} \mathcal{A}x = x \quad \text{if} \, \, x \in \mathcal{M}. \tag{3.4}
\]

Since \( G_{\mathcal{L}\mathcal{M}0} \mathcal{A}x = 0 \) if \( x \in \mathcal{X}' \), it follows that \( G_{\mathcal{L}\mathcal{M}0} \mathcal{A} = \mathcal{P}_{\mathcal{A}\mathcal{X}} \), which is the first condition in (ii) of Theorem 2.3. The result (3.2) is proved.

Since \( G_{\mathcal{L}\mathcal{M}0} \) is a particular \( \mathcal{L}\mathcal{M} \)-inverse, we need only add a term which reduces to the null operator by both pre- and postmultiplications by \( \mathcal{A} \). Obviously a general expression for such a term is the second part of (3.3). Thus (3.3) is proved.

4. OTHER CLASSES OF INVERSES

**\( \mathcal{M} \)-Inverse**

An \( \mathcal{M} \)-inverse of \( \mathcal{A} \) is \( \mathcal{G} \) satisfying the condition

\[
\mathcal{G} \mathcal{A} = \mathcal{P}_{\mathcal{A}\mathcal{X}} \tag{4.1}
\]
with the equivalent conditions
\[ AGA = A \quad \text{and} \quad R(GA) = \mathcal{M}. \]  
(4.2)

A general solution of (4.1) is
\[ G = P_{\mathcal{M}^\perp} A^\perp + ZP_{\mathcal{M}}, \]  
(4.3)

where \( AA^\perp A = A \), and \( Z \) is arbitrary. We represent an \( \mathcal{M} \)-inverse by \( A^- \) (to be consistent with the notation developed in [8]).

If \( \mathcal{Y} \) is a vector space endowed with an inner product, then we may choose \( \mathcal{M} \) to be the orthogonal complement of \( \mathcal{X} \). In such a case, if \( Ax = y \) is a consistent equation, then
\[ \min_{Ax = y} \| x \| = \| A^- y \|, \]  
(4.4)

so that \( A^- y \) is the minimum norm solution of \( Ax = y \).

\( \mathcal{L} \)-Inverse

An \( \mathcal{L} \)-inverse of \( A \), denoted by \( A^\perp \), is \( G \) satisfying the equation
\[ AG = P_{\mathcal{L}^\perp}, \]  
(4.5)

with the equivalent conditions
\[ AGA = A \quad \text{and} \quad AGP_{\mathcal{L}^\perp} = 0. \]  
(4.6)

A general solution of (4.5) is
\[ G = A^\perp P_{\mathcal{L}^\perp} + P_{\mathcal{L}^\perp} Z. \]  
(4.7)

An \( \mathcal{L} \)-0-inverse is \( G \) satisfying the equivalent conditions
\[ AG = P_{\mathcal{L}^\perp}, \quad G = GAG \iff AG = P_{\mathcal{L}^\perp}, \quad GA = P_{\mathcal{L}^\perp}. \]  
(4.8)

If \( \mathcal{W} \) is an inner product vector space, we may choose \( \mathcal{L} \) to be an orthogonal complement of \( \mathcal{D} \). In such a case
\[ \min_{x} \| y - Ax \| = \| y - AA^\perp y \|, \]  
(4.9)

so that \( A^\perp y \) is a general least squares solution.

\( \mathcal{L} \mathcal{M} \mathcal{N} \)-Inverse

Let us consider inner product spaces \( \mathcal{Y} \) and \( \mathcal{W} \), and the problem of minimizing \( \| x \| \) subject to \( x \) being a least squares solution of \( Ax = y \). If
\( y_1 = P_{\mathcal{M}^\perp}y \), then the problem reduces to minimizing \( ||x|| \) subject to the consistent equation \( Ax = y_1 \). Then the optimum \( x \) is obtained by using an \( \mathcal{M} \)-inverse. The solution is \( x = \Lambda^{-}_m y_1 = \Lambda^{-}_m P_{\mathcal{M}^\perp}y \). It is seen that if \( G = \Lambda^{-}_m P_{\mathcal{M}^\perp} \), then

\[
GA = P_{\mathcal{M}^\perp} \quad \text{and} \quad GP_{\mathcal{M}^\perp} = 0,
\]

so that \( G \) is the \( \mathcal{LMN} \)-inverse of \( A \) with \( N = 0 \). This inverse (when \( N = 0 \), \( \mathcal{L} = A^\perp \), \( \mathcal{M} = \mathcal{N}^\perp \)) may be denoted by \( A^+ \); it is the Moore-Penrose inverse.

All the above results can be extended without any major modification of the proofs to bounded linear operators with closed range of Hilbert spaces.

5. EXPRESSIONS FOR \( g \)-INVERSES OF MATRICES

We derive explicit expressions for \( g \)-inverses of matrices, for which we consider the linear transformation \( A \) as an \( m \times n \) matrix and take \( \mathcal{N} = E^n \) and \( \mathcal{M} = E^m \). We prove the following lemma, where \( A' \) represents the transpose of \( A \); \( K(T) \), the kernel of a matrix transformation \( T \); and \( R(T) \), the range space of \( T \).

**Lemma 5.1.** Let a matrix \( C \) be such that

\[
R(A') \cap R(C') = 0, \quad R(A') \cap K(C) = 0, \quad R(A') \cap R(C') = E^n. \quad (5.1)
\]

Then \( K(A) \cap K(C) = 0 \) and

\[
K(A) \oplus K(C) = E^n. \quad (5.2)
\]

**Proof.** Let \( x \in K(A) \cap K(C) \). Then \( Ax = 0 \), \( Cx = 0 \) \( \Rightarrow x = 0 \) in view of (5.1), i.e., \( K(A) \cap K(C) = 0 \). Further note that

\[
\dim K(A) + \dim K(C) = [n - \operatorname{rank}(A)] + [n - \operatorname{rank}(C)] = n,
\]

which establishes (5.2). \( \square \)

The following theorem is a consequence of Lemma 5.1.

**Theorem 4.1.** Let \( C \) be such that (5.1) holds, and \( F \) be a matrix such that \( R(F) \) is the direct complement of \( K(A) \). Then

\[
P_{R(F) \cdot K(A)} = P_{K(C) \cdot K(A)} \Leftrightarrow CP_{R(F) \cdot K(A)} = 0. \quad (5.3)
\]
Further

\[(P_{R(A')} - R(C')) = p_{K(C) - K(A)}. \quad (5.4)\]

**Proof.** (5.3) is easy to establish. To prove (5.4), we may observe that
\[R(1 - P') = K(A) \quad \text{and} \quad R(P') = K(C),\]
implies that \(PA' = A'\) and \(PC' = C'\), where \(P = p_{R(A')} - R(C')\).

**Lemma 5.2.** Let \(B\) be a matrix such that \(R(B)\) is a direct complement of \(R(A)\), and define \(S_B = I - BB'\) and \(Q_B = I - B(B'B)^{-1} B'\). Then

\[p_{R(A')} - R(B) = A(S_B A)^{-} S_B, \quad (5.5)\]

\[= A(A'Q_B A) A'Q_B, \quad (5.6)\]

\[= AA'(AA' + BB')^{-1}. \quad (5.7)\]

A proof of Lemma 5.2 is given in [9]. Using Theorem 5.1 and Lemma 5.2, it is easy to establish the following lemma.

**Lemma 5.3.** Let \(S_C = I - C'C\) and \(Q_C = I - C'(CC')^{-1} C\). Then

\[p_{K(C) - K(A)} = S_C (AS_C) - A, \quad (5.8)\]

\[= Q_C A'(AQ_C A') - A', \quad (5.9)\]

\[= (A'A + C'C)^{-1} A'A. \quad (5.10)\]

Using these results, we give representations of \(g\)-inverses of matrices.

**Theorem 5.2.**

(i) If we choose \(M = K(C)\), then the \(M\)-inverse of \(A\) can be written as

\[A_m = S_C (AS_C) + ZP_{R(B) - R(A)} \quad (5.11)\]

\[= Q_C A'(AQ_C A') + ZP_{R(B) - R(A)} \quad (5.12)\]

\[= (A'A + C'C)^{-1} A' + ZP_{R(B) - R(A)}, \quad (5.13)\]

where \(Z\) is an arbitrary matrix.
(ii) With \( B \) as defined in Lemma 5.2 \([i.e., \mathcal{L} = R(B)]\), the \( L \)-inverse of \( A \) can be written as

\[
A^*_L = (S_B A)^{-} S_B + P_{K(A), K(C)} Z \tag{5.14}
\]

\[
= (A'Q_B A) A'Q_B + P_{K(A), K(C)} Z \tag{5.15}
\]

\[
= A'(AA' + BB')^{-1} + P_{K(A), K(C)} Z. \tag{5.16}
\]

(iii) With \( \mathcal{L} = R(B) \) and \( \mathcal{M} = K(C) \), \( N = 0 \), the \( \mathcal{L}\mathcal{M}\)-inverse of \( A \) can be written as

\[
A^*_{\mathcal{L}\mathcal{M}} = S_C (AS_C) A (S_B A)^{-} S_B \tag{5.17}
\]

\[
= Q_C A'(AQ_C A')^{} A (A'Q_B A)^{-} A'Q_B \tag{5.18}
\]

\[
= (A'A + C'C)^{-1} A'AA'(AA' + BB')^{-1}. \tag{5.19}
\]

Corollary.

(i) If, in particular, \( R(B) = R(A)^{\perp} \) under the Euclidean inner product, then

\[
A^*_{\mathcal{M}l} = Q_C A'(AQ_C A')^{} A (A'A)^{-} A' = (A'A + C'C)^{-1} A'. \tag{5.20}
\]

(ii) If \( K(C) = K(A)^{\perp} \), then

\[
A^*_{\mathcal{M}l} = A'(AA')^{} A (A'Q_B A)^{-} A'Q_B = A'(AA' + BB')^{-1}. \tag{5.21}
\]

(iii) If \( R(B) = R(A)^{\perp} \) and \( K(C) = K(A)^{\perp} \) hold simultaneously, then

\[
A^*_{\mathcal{M}l} = A'(AA')^{} A (A'A)^{-} A', \tag{5.22}
\]

which is exactly the Moore-Penrose inverse of \( A \).
NOTE. $A^+_{ml}$ as obtained in Theorem 5.2 is the Moore-Penrose inverse of the matrix $(Q_B A Q_C)$, since $A^+_{ml}$ satisfies the following conditions:

1. $(Q_B A Q_C) A^+_{ml} (Q_B A Q_C) = Q_B A Q_C,$
2. $A^+_{ml} (Q_B A Q_C) A^+_{ml} = A^+_{ml},$
3. $(Q_B A Q_C) A^+_{ml} = Q_B A Q_C A^+_{ml},$
4. $(A^+_{ml} Q_B A Q_C)^t = A^+_{ml} Q_B A Q_C.$

Thus, $A^+_{ml}$ is uniquely determined for any choices of matrices $B$ and $C$ spanning $\mathcal{L} = R(B)$ and $\mathcal{M} = K(C)$ respectively.

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