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Characters of ultrafilters and tightness of products of fans

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Abstract

As applications of productivity of coreflective classes of topological spaces, the following results will be proved: (1) Characters of points of $\beta\mathbb{N} \setminus \mathbb{N}$ are not smaller than any submeasurable cardinal less or equal to 2^ω. (2) If κ is a submeasurable cardinal and S is a sequential fan with κ many spines then the tightness of the κ-power of S is equal to κ. In fact, a little more general results are proved.

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It was proved in [9] that the *productivity numbers of coreflective classes of topological spaces are, in addition to* 2, ω *and* ∞, *precisely the submeasurable cardinals*. We shall now briefly describe the concepts used in this result.

A real-valued function μ defined on all subsets of κ is a submeasure if

(1) $\mu(\emptyset) = 0$;
(2) $\mu(A) \leqslant \mu(B)$ whenever $A \subset B \subset \kappa$;
(3) $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for every $A, B \subset \kappa$.

An uncountable cardinal κ is called submeasurable if there is a nonzero κ-continuous submeasure μ defined on all subsets of κ and vanishing on singletons (if, moreover, $\mu(\kappa) = 1$, μ will be called *normalized*). Equivalently, an uncountable cardinal κ is submeasurable, if there is a noncontinuous κ-continuous real-valued function defined on 2^κ. The κ-continuity means that the map preserves convergence of nets of length less than κ. For submeasures this means that $\mu(A_\alpha) \to 0$ whenever $\{A_\alpha\}_\lambda$ is a decreasing family of subsets of κ indexed by $\lambda < \kappa$ and having empty intersection (or, equivalently, the submeasure is sequentially continuous and κ-additive on null sets).

We recall that the first submeasurable cardinal is the least sequential cardinal. Every real-measurable cardinal is submeasurable. A submeasurable cardinal is either not larger than 2^ω or is 2-measurable. See [1,8] for more details.

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A coreflective class of topological spaces is a class that contains a nonvoid space and is closed under quotients and disjoint sums. The productivity of a coreflective class $\mathcal{C}$ of topological spaces is the smallest cardinal κ such that $X^\kappa \notin \mathcal{C}$ for some $X \in \mathcal{C}$; if there is no such κ (i.e., $\mathcal{C}$ is productive) then its productivity number is denoted by ∞ (a symbol larger than any cardinal number). A class is called κ-productive if κ is not bigger than its productivity.

1. κ-productive classes

In this section, κ will stand for a submeasurable cardinal and μ for a normalized submeasure on κ that is κ-continuous and vanishes at singletons. Although some of the next concepts and results depend, in addition to κ, also on μ, we believe there can be no misunderstanding when we omit μ from notations.

Generalizing an example from [4] to submeasurable cardinals, it was shown in [9] that the following class $\mathcal{C}_{\kappa,\mu}$ is κ-productive and coreflective:

$$\mathcal{C}_\kappa = \Big\{X;\ \text{for every family } \mathcal{G} = \{G_\alpha\colon \alpha \in \kappa\} \text{ of open sets in } X \text{ and for every } r \in]0,1] \text{ the set } \mathcal{G}_r = \big\{x \in X\colon \mu\{\alpha\colon x \notin G_\alpha\} < r\big\} \text{ is open}\Big\}.$$

The set $\{\alpha\colon x \notin G_\alpha\}$ will be denoted by A_x.

It may be difficult to imagine what $\mathcal{C}_\kappa$ looks like. The next characterization may help.

Proposition 1. *$\mathcal{C}_\kappa$ coincides with the class*

$$\Big\{X;\ \text{if } x \in \overline{P} \subset X \text{ and } \{U_\alpha\}_\kappa \text{ is a family of neighborhoods of } x,\ \text{then for every } s > 0 \text{ there exists } S \subset \kappa \text{ such that } \mu(\kappa \setminus S) < s \text{ and } P \cap \bigcap_S U_\alpha \neq \emptyset\Big\}.$$

Proof. Suppose first that X belongs to the class defined in Proposition 1. Let $\{G_\alpha\}_\kappa$ be a family of open sets in X and $r \in]0,1]$. If $\mathcal{G}_r$ is not open, there is some $x \in \mathcal{G}_r \cap \overline{X \setminus \mathcal{G}_r}$. It follows that $\mu(A_x) < r$ and one can choose $s > 0$ such that $\mu(A_x) + s < r$. Define $U_\alpha = G_\alpha$ if $\alpha \notin A_x$ and $U_\alpha = X$ if $\alpha \in A_x$. By the assumption on X there is a set $S \subset \kappa$ with $\mu(\kappa \setminus S) < s$ and there is some $y \in \bigcap_S U_\alpha \setminus \mathcal{G}_r$. For this point y, the inclusion

$$\{\alpha;\ y \notin G_\alpha\} \subset \{\alpha;\ x \notin G_\alpha\} \cup (\kappa \setminus S)$$

implies $\mu(A_y) < r$, which contradicts $y \notin \mathcal{G}_r$.

Conversely, assume that $X \in \mathcal{C}_\kappa$. Take $x \in \overline{P} \subset X$, a family $\{U_\alpha\}_\kappa$ of open neighborhoods of x and $s > 0$. The set $\mathcal{U}_s$ is open and contains x. Thus there is some $y \in P \cap \mathcal{U}_s$. It is easy to show that $y \in P \cap \bigcap_{\kappa \setminus A_y} U_\alpha$ and, since $\mu(A_y) < s$, our space X belongs to the requested class from our proposition. □

One can see from the proof that for T_1-spaces X and $x \in \overline{P} \setminus P$ it is possible to require the intersections $P \cap \bigcap_S U_\alpha$ to be infinite sets. Indeed, in the last paragraph of the proof, the intersections $P \cap \mathcal{U}_r$ (for any $r > 0$) must be infinite if X is a T_1-space and $x \in \overline{P} \setminus P$. So one can find different $y_n \in P \cap \mathcal{U}_{s/2^n}$ and then $\{y_n\}_\omega \subset P \cap \bigcap_{\kappa \setminus \bigcup A_{y_n}} U_\alpha$; clearly, $\mu(\bigcup A_{y_n}) < s$.

Another observation is that $\mathcal{C}_\kappa$ are hereditary classes.

The characterizing property of $\mathcal{C}_\kappa$ from the last proposition can be reformulated as follows:

For every family $\{G_\alpha\}_\kappa$ of open subsets of X and any $r > 0$ it holds

$$\bigcap_\kappa G_\alpha \subset \operatorname{Int} \bigcup \Big\{\bigcap_{\kappa \setminus S} G_\alpha;\ S \subset \kappa,\ \mu(S) < r\Big\}.$$

It is easy to see that if $X \in \mathcal{C}_\kappa$ and $\{U_\alpha\}_\kappa$ is a monotone family of neighborhoods of some $x \in X$, then $\bigcap_\kappa U_\alpha$ is a neighborhood of x.

If κ is 2-measurable, then $\mathcal{C}_\kappa$ can be described as

$$\Big\{X;\ \text{if } x \in \overline{P} \subset X \text{ and } \{U_\alpha\}_\kappa \text{ is a family of neighborhoods of } x,\ \text{then there exists}$$
$$S \subset \kappa \text{ such that } \mu(\kappa \setminus S) = 0 \text{ and } P \cap \bigcap_S U_\alpha \neq \emptyset\Big\}.$$

In accordance with [4] we denote by $\lambda \oplus 1$, for an infinite cardinal λ, the space $\lambda + 1$ of ordinals less or equal to λ where all the ordinals less than λ are isolated and the point λ has the usual order neighborhoods. Then $\lambda \oplus 1 \in \mathcal{C}_\kappa$ iff $\mathrm{cf}(\lambda) \neq \kappa$. We believe it is clear how to define $D \oplus 1$ for directed ordered sets D.

2. Characters of ultrafilters

Recall that a space X is said to be generated by a class $\mathcal{P}$ of nets if it belongs to the coreflective hull of the class $\{P \oplus 1;\ P \in \mathcal{P}\}$, i.e., closures in X are formed by adding (iteratedly) limits of nets from $\mathcal{P}$.

Proposition 2. *If the topology of X is generated by nets of length less than κ then $X \in \mathcal{C}_\kappa$.*

Proof. It suffices to show that every space $P \oplus 1$ belongs to $\mathcal{C}_\kappa$, where P is a net of length less than κ. Denote by ∞ the unique accumulation point of $P \oplus 1$. Take a family $\{G_\alpha\}_\kappa$ of open sets in $P \oplus 1$, a number $r > 0$ and assume that $\infty \in \mathcal{G}_r$. Thus $\mu\{\alpha;\ \infty \notin G_\alpha\} < r$ and we can find an $s > 0$ such that $\mu\{\alpha;\ \infty \notin G_\alpha\} + s < r$. We must prove that there is a $p \in P$ such that the whole interval $[p, \infty]$ is a part of $\mathcal{G}_r$.

If $\infty \in G_\alpha$ then $[p_\alpha, \infty] \subset G_\alpha$ for some $p_\alpha \in P$. Take a family $\{H_\alpha\}$ of open sets defined as follows:

$$H_\alpha = \begin{cases} P \oplus 1, & \text{if } \infty \notin G_\alpha; \\ [p_\alpha, \infty], & \text{otherwise.} \end{cases}$$

Denote now $A_p = \{\alpha;\ p \notin H_\alpha\}$. If $p \leqslant q$ then $A_p \supset A_q$, and the net $\{A_p;\ p \in P\}$ converges to $\emptyset$ in the usual convergence of subsets of κ (i.e., in 2^κ). Consequently, by the κ-continuity of μ, the net $\{\mu(A_p);\ p \in P\}$ converges to 0. So, there is some $p \in P$ with $\mu(A_p) < s$. It is easy to see that $[p, \infty] \subset H_\alpha$ for every $\alpha \in \kappa \setminus A_p$.

Now, the set $\{\alpha;\ \infty \notin G_\alpha\} \cup A_p$ has the μ-value less than r and for each $\alpha \in \kappa$ not belonging to that set we have $[p, \infty] \subset G_\alpha$, which means $[p, \infty] \subset \mathcal{G}_r$. □

Corollary 3. *If characters of all points of X are less than κ then $X \in \mathcal{C}_\kappa$.*

For a free ultrafilter u on $\mathbb{N}$ denote by $\mathbb{N}_u$ the subspace $\mathbb{N} \cup \{u\}$ of the Čech–Stone compactification $\beta(\mathbb{N})$.

For the proof of the main result of this section we need a combinatorial result (stated in Proposition 5) based on Theorem 3.4 from [10]: *Every uniformly exhaustive submeasure is equivalent to a measure.* The quoted paper deals with submeasures on Boolean algebras; equivalence of submeasures μ, ν means that $\mu(A_n) \to 0$ iff $\nu(A_n) \to 0$; a submeasure μ is uniformly exhaustive if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that every disjoint collection of n members from the algebra contains a member of submeasure at most ε.

We do not need that result in its full generality and state here a special case convenient for our purpose:

Proposition 4. *If κ is submeasurable and not real-measurable then it bears a normalized κ-continuous submeasure vanishing on singletons having an additional property κ has a decomposition to n sets of submeasure* 1 *for every natural n,*

Proof. Since κ is not real-measurable, a corresponding submeasure μ on κ cannot be equivalent to a measure in the above sense. By the quoted Theorem 3.4 from [10], μ is not uniformly exhaustive and, thus, there exists some $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ one can find a disjoint collection of n subsets of κ each having its submeasure at least ε (enlarging a member of the collection, we may assume that the collection is a cover of κ). The submeasure $\min\{1, \mu/\varepsilon\}$ satisfies the required properties. □

The next proposition is known for measures.

Proposition 5. *If κ is a submeasurable cardinal less or equal to 2^ω, then there exists a normalized κ-continuous submeasure μ on κ vanishing at singletons such that for every $s \in (0,1)$ one can find a sequence $\{A_n\}$ of subsets of κ having the properties*

(1) $\mu(A_n) \geqslant 1/2$, $\mu(\kappa \setminus A_n) \geqslant 1/2$ *for all* n;
(2) *for every infinite* $K \subset \mathbb{N}$ *and choice* $B_n = A_n$ *or* $B_n = \kappa \setminus A_n$ *for* $n \in K$ *one has* $\mu(\bigcup_{n\in K} B_n \setminus \bigcap_{n\in K} B_n) \geqslant s$.

Proof. It suffices to prove the result for the choice $B_n = A_n$ in the condition (2). Indeed, the choice $B_n = \kappa \setminus A_n$ gives the same set $\bigcup_{n\in K} B_n \setminus \bigcap_{n\in K} B_n$ as the choice $B_n = A_n$; in general case, one of those possibilities appears infinitely many times (say, for $n \in L$) and $\bigcup_{n\in L} B_n \setminus \bigcap_{n\in L} B_n \subset \bigcup_{n\in K} B_n \setminus \bigcap_{n\in K} B_n$.

If μ is real-measurable, then a μ-independent system $\{A_n\}$ works. Since μ is not 2-measurable, every $A \subset \kappa$ can be decomposed into two sets with measure equal to $\mu(A)/2$. So, κ has such a decomposition A_1^1, A_2^1, the set A_1^1 has such a decomposition A_1^2, A_2^2, the set A_2^1 has such a decomposition A_3^2, A_4^2, and so on. One gets decompositions $\{A_i^k\}_{i\leqslant 2^k}$ of measures 2^{-k}. It suffices to put $A_n = \bigcup_{i=1}^{2^{n-1}} A_{2i}^n$. Then for every infinite $K \subset \mathbb{N}$ one has $\mu(\bigcup_{n\in K} B_n) = 1$, $\mu(\bigcap_{n\in K} B_n) = 0$ for any choice of B_n to be A_n or $\kappa \setminus A_n$. The result follows.

Assume now that κ is submeasurable and nonreal-measurable. According to Proposition 4 there is a normalized κ-continuous submeasure μ on κ vanishing on singletons such that for every $n \in \mathbb{N}$ one can find a decomposition $\{D_1^n, \ldots, D_n^n\}$ of κ with $\mu(D_i^n) = 1$ for all possible n, i.

Suppose that there is an infinite set $K_1 \subset \mathbb{N}$ and a sequence $i_n \leqslant n$, $n \in K_1$, such that

$$\mu\left(\bigcup_{n\in K_1} D_{i_n}^n \setminus \bigcap_{n\in K_1} D_{i_n}^n\right) < s.$$

We may assume that $i_n = 1$ for all $n \in K_1$. Denote $\bigcap_{n\in K_1} D_1^n = P_1$ (then $\mu(P_1) \geqslant 1 - s$).

Take $Q_1 = \kappa \setminus P_1$ and decompositions $\mathcal{D}_n^1$ of Q_1 defined for $n \in K_1$ as

$$\mathcal{D}_n^1 = \left\{D_2^n, \ldots, D_{n-1}^n, D_n^n \cup \left(Q_1 \setminus \bigcup_{i=2}^n D_i^n\right)\right\}.$$

Now, $\mu(D) = 1$ for any $D \in \mathcal{D}_n^1$. If again there is an infinite subset K_2 of K_1 and a choice of members D_n of $\mathcal{D}_n^1$ such that $\mu(\bigcup_{n\in K_2} D_n \setminus \bigcap_{n\in K_2} D_n) < s$ we get in a similar way the sets $P_2 \subset Q_1$ with $\mu(P_2) \geqslant 1 - s$, a set $Q_2 = Q_1 \setminus P_2$ with $\mu(Q_2) = 1$ and decompositions $\mathcal{D}_n^2$, $n \in K_2$ such that each member of any of those decompositions has μ-submeasure 1.

In case the described process is infinite, the constructed sets P_n are disjoint and $\mu(P_n) \geqslant 1 - s > 0$, which contradicts the sequential continuity of μ. So, after finitely many steps the process must finish and one gets a set Q_k of cardinality κ and decompositions $\mathcal{D}_n$, $n \in K_k$. The restriction of μ to Q_k gives a normalized submeasure on κ and any choice of A_n from $\mathcal{D}_n$, $n \in K_k$ gives a requested sequence. □

The next result was proved in [4] for real-measurable cardinals using a μ-independent system. Our previous Proposition 5 allows to prove it for submeasurable cardinals, too.

Theorem 6. *If u is a free ultrafilter on $\mathbb{N}$ and κ is a submeasurable cardinal less or equal to 2^ω, then $\mathbb{N}_u \notin \mathcal{C}_\kappa$ for a convenient submeasure μ on κ.*

Proof. To prove $\mathbb{N}_u \notin \mathcal{C}_\kappa$, we shall use Proposition 1. We can find a submeasure μ on κ and a sequence $\{A_n\}$ of subsets of κ having the properties described in Proposition 5 for $s = 1/4$. Define $U_\alpha \in u$ to be either $\{n;\ \alpha \in A_n\}$ or $\{n;\ \alpha \notin A_n\}$ depending on which of both sets belongs to u. Take any $S \subset \kappa$ with $\mu(S) < 1/4$ and let $\bigcap\{U_\alpha;\ \alpha \notin S\}$ contains an infinite set $K \subset \mathbb{N}$. Then $S \supset \bigcup_{n\in K} A_n \setminus \bigcap_{n\in K} A_n$ and, so, $\mu(S) \geqslant 1/4$, which contradicts our choice of S. Consequently, intersections $\bigcap\{U_\alpha;\ \alpha \notin S\}$ are finite whenever $\mu(S) < 1/4$. According the remark about T_1-spaces following Proposition 1, the proof is finished. □

Corollary 7. *If u is a free ultrafilter on $\mathbb{N}$ and κ is a submeasurable cardinal less or equal to 2^ω, then $\chi(\mathbb{N}_u) \geqslant \kappa$.*

Corollary 8. *If a nontrivial net in $\mathbb{N}$ converges in $\beta\mathbb{N}$ then its lenght must be at least* $\sup\{\kappa \leqslant 2^\omega;\ \kappa$ *submeasurable*$\}$.

Proof. If κ is submeasurable and $u \in \beta\mathbb{N} \setminus \mathbb{N}$ then $\mathbb{N}_u \notin \mathcal{C}_\kappa$ and, thus, $\mathbb{N}_u$ cannot be generated by nets having length less than κ. Therefore, there exists a set $A \subset \mathbb{N}$ with $u \in \overline{A}$ (i.e., $A \in u$) such that no net in A of length less than κ converges to u. It is easy to see (by taking a convenient bijection of $\mathbb{N}$ onto A) that no net in $\mathbb{N}$ of length less than κ converges to u. □

Compare the last Corollary with the fact that $\beta\mathbb{N} \setminus \mathbb{N}$ always contains a nontrivial converging net of length ω_1 (see [2,12]).

3. Tightness of products of fans

Tightness of a point $x \in \overline{A}$ with respect to a set A in a space X is $t(x, A) = \min\{|B|;\ B \subset A,\ x \in \overline{B}\}$. Tightness of the whole space X is supremum of all $t(x, A)$, where $x \in \overline{A} \subset X$.

It follows from Theorem 6 that $\mathcal{C}_\kappa$ need not contain all spaces having tightness less than κ. Nevertheless, that may happen only when $\kappa \leqslant 2^\omega$:

Proposition 9. *If κ is 2-measurable and $\overline{P} = \bigcup\{\overline{A};\ A \in [P]^{<\kappa}\}$ for every $P \subset X$, then $X \in \mathcal{C}_\kappa$.*

Proof. Suppose X has the property from the assertion. If $x \in \overline{P}$ then $x \in \overline{A}$ for some $A \subset P$ with $|A| < \kappa$. Since κ is a strongly inaccessible cardinal, the character of x with respect to A is less than κ and, hence, there is a net of length less than κ in A converging to x. By Proposition 2 we have $X \in \mathcal{C}_\kappa$. □

If one defines $\tilde{t}(X) < \kappa$ if $t(x, A) < \kappa$ for every $A \subset X, x \in \overline{A}$, then the previous proposition says that $X \in \mathcal{C}_\kappa$ provided $\tilde{t}(X) < \kappa$ (we say in this case that *small tightness* of X is less than κ). Proposition 9 does not characterize $\mathcal{C}_\kappa$ since the class contains arbitrarily large spaces $\lambda \oplus 1$, but it has a partial conversion valid for any submeasurable cardinal:

Proposition 10. *If $X \in \mathcal{C}_\kappa$ and $A \subset X$ is a sum of an increasing family $\{A_\alpha\}_\kappa$, then $\overline{A} = \bigcup_\kappa \overline{A_\alpha}$.*

Proof. Suppose the situation from the assertion. For $x \in \overline{A} \setminus \bigcup_\kappa \overline{A_\alpha}$ it suffices to take $U_\alpha = X \setminus \overline{A_\alpha}$ as a family of neighborhoods of x that does not satisfy the characterization of Theorem 1. □

Corollary 11. *If $X \in \mathcal{C}_\kappa$ and $A \subset X$ is of cardinality κ then every $x \in \overline{A}$ belongs to the closure of some subset of A of cardinality less than κ.*

In other words, if $X \in \mathcal{C}_\kappa$ and $|X| = \kappa$, then $\tilde{t}(X) < \kappa$.

Let us recall that a general fan $S(\{\lambda_\alpha\}_\lambda)$ is a quotient of a disjoint sum $\bigcup_\lambda(\lambda_\alpha \oplus 1)$, where the accumulation points of $\lambda_\alpha \oplus 1$ are sewed together. The space $S(\{\omega\}_\lambda)$ is called the sequential fan with λ many spines. If $\lambda \leqslant \kappa$ and all $\lambda_\alpha < \kappa$ we shall call $S(\{\lambda_\alpha\}_\lambda)$ a κ-fan. By the previous investigation, every κ-fan belongs to $\mathcal{C}_\kappa$.

Many papers are devoted to situations when a product of fans increases tightness (see e.g., [3,5,6,11,13] in which one can find other references). We are in a different position, namely we show a situation when the tightness of products of fans cannot increase:

Theorem 12. *Small tightness of finite products of κ-fans is less than κ.*

Proof. Use Corollary 11. □

Tightness of finite products of κ-fans may be (and probably is) equal to κ. Nevertheless, e.g. for $S = S(\{\omega\}_\kappa)$ with the accumulation point ∞, there is no subset $A \subset (S \setminus \{\infty\})^2$ of cardinality κ such that $(\infty, \infty) \in \overline{A}$ and $(\infty, \infty) \notin \overline{B}$ for every $B \subset A, |B| < \kappa$.

Since tightness of products of κ-many spaces is determined by tightness of finite subproducts if not greater than κ, we get the following assertion.

Corollary 13. *Tightness of a product of κ-many κ-fans is κ.*

A space X is said to be λ-*collectionwise Hausdorff*, if for any closed discrete subset A of X with $|A| \leqslant \lambda$ there is a disjoint collection $\{U_a\}_A$ of neighborhoods of points of A. The concept $<\lambda$-collectionwise Hausdorff space should be clear. By [7], every $<\lambda$-collectionwise Hausdorff space having character less than λ is λ-collectionwise Hausdorff provided λ is weakly compact. Submeasurable cardinals need not be weakly compact if they are not 2-measurable and so the following modification of the quoted Fleissner's result describes other situations.

Proposition 14. *If κ is submeasurable, then every $<\kappa$-collectionwise Hausdorff space that has point characters less than κ is κ-collectionwise Hausdorff.*

Proof. It is proved in [5] that every first countable $<\kappa$-collectionwise Hausdorff space is κ-collectionwise Hausdorff iff $\tilde{t}(S(\{\omega\}_\kappa)^2) < \kappa$. The proof can be easily extended to point-characters less than κ instead of first countability and to κ-fans. Then the assertion of proposition follows from Theorem 12. $\square$

There are some results asserting that a product of two sequential fans of cardinalities at most κ has tightness κ provided some conditions are fulfilled (see, e.g., [3–6,11,13]). So, if κ is submeasurable, the conditions cannot be fulfilled. The next Proposition comprises several such conditions. In some cases, the nonvalidity of conditions may be easy to prove directly without using previous Theorem 12, nevertheless it may be worth to summarize them as consequences of properties of $\mathcal{C}_\kappa$. For definitions of the concepts and symbols used in the next proposition see the papers cited above.

Proposition 15. *Let κ be an uncountable submeasurable cardinal. Then*

(1) *Every monotone κ-family is extendible to $(\kappa+1)$-family.*
(2) $\square(\kappa)$ *does not hold.*
(3) $E(\kappa)$ *does not hold.*
(4) *Cardinal b is not submeasurable.*
(5) *There are no (κ, λ)-good sets for $\lambda \leqslant \kappa$ in the sense of Brendle and LaBerge from* [3].

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