Bounded $K$-Spherical Functions on Heisenberg Groups*

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Communicated by the Editors

Received February 28, 1990; revised May 20, 1991

Let $H_n$ be the $(2n+1)$-dimensional Heisenberg group, and let $K$ be a compact subgroup of Aut($H_n$), the group of automorphisms of $H_n$. The pair $(K, H_n)$ is called a Gelfand pair if $L^1_K(H_n)$, the subalgebra of elements of $L^1(H_n)$ that are invariant under the action of $K$, is commutative. In this case, the continuous homomorphisms on $L^1_K(H_n)$ are given by integrating against certain $K$-invariant functions on $H_n$. These functions are the $K$-spherical functions associated to the Gelfand pair $(K, H_n)$. In this paper we show how to compute the bounded $K$-spherical functions on $H_n$.

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INTRODUCTION

Suppose that $G$ is a Lie group and $K$ is a compact Lie subgroup of Aut($G$). We say that $(K, G)$ is a Gelfand pair if the convolution algebra $L^1_K(G)$ of $K$-invariant $L^1$-functions on $G$ is commutative. Equivalently, $K \subset K \ltimes G$ is a Gelfand pair in the usual sense that $L^1(K \ltimes G//K)$, the space of $K$-bi-invariant $L^1$-functions on $K \ltimes G$, is $^*$-commutative.

* The authors were supported in part by the National Science Foundation. The third author was also supported in part by the Australian Research Council.
In [BJR] the authors considered Gelfand pairs \((K,G)\) with \(G\) a connected, solvable Lie group. The classification of such Gelfand pairs was reduced to the classification of Gelfand pairs \((K,N)\) with \(N\) a connected, nilpotent Lie group. In order for \((K,N)\) to be a Gelfand pair, it was shown that \(N\) could be at most 2-step. Gelfand pairs of the form \((K,H_n)\), where \(H_n\) is the \((2n+1)\)-dimensional Heisenberg group, play a key role in the more general theory.

Let \((K,N)\) be a Gelfand pair with \(N\) nilpotent. We denote the set of differential operators on \(N\) that are invariant under the action of \(K\) and the left action of \(N\) by \(\mathcal{U}_K(n)\). A smooth \(K\)-invariant function \(\phi: N \to \mathbb{C}\) is \(K\)-spherical if \(\phi(e) = 1\) and \(\phi\) is a joint eigenfunction for all \(D \in \mathcal{U}_K(n)\). Reference [BJR] provides a parametrization for the bounded \(K\)-spherical functions on \(N\), yielding a description of the Gelfand space for \(L^1_k(N)\). We review this below in Section 2. The main focus here is to specialize the theory to Heisenberg groups, obtaining methods to construct, in principle, all bounded \(K\)-spherical functions on \(H_n\).

Section 1 begins with some basic facts about Heisenberg groups and their representation theory. We identify \(H_n\) with \(V \times \mathbb{R}\) where \(V \cong \mathbb{C}^n\). A maximal compact connected group of automorphisms of \(H_n\) is given by the unitary group \(U(n)\) acting via \(k \cdot (z, t) = (kz, t)\). Conjugating by an automorphism of \(H_n\) if necessary, we can always assume that a given compact connected group \(K \subset \text{Aut}(H_n)\) is contained in \(U(n)\). It is well known that \((U(n), H_n)\) is a Gelfand pair (see, for example, [Ko]) and as discussed in Section 1, there are many proper closed subgroups \(K \subset U(n)\) for which \((K,H_n)\) is a Gelfand pair.

There is a representation theoretic criterion, due to Carcano [Ca], for \((K,N)\) to be a Gelfand pair. In Section 1, we show that this implies that \((K,H_n)\) is a Gelfand pair if and only if the action of \(K\) on the holomorphic polynomials \(\mathcal{P}(V)\) is multiplicity free. Let \(\mathcal{P}(V) = \sum P_\alpha\) denote the decomposition of \(\mathcal{P}(V)\) into \(K\)-irreducibles.

The general theory in [BJR] describes the bounded \(K\)-spherical functions for a Gelfand pair \((K,N)\) in terms of representation theory. In Section 2 we show that for Gelfand pairs \((K,H_n)\), there are two distinct classes of bounded \(K\)-spherical functions.

1. The first type is parametrized by pairs \((\lambda, P_\alpha)\) where \(\lambda\) is a non-zero real number. These arise from the infinite dimensional representations of \(H_n\) and we denote them by \(\phi_{\lambda, \alpha}(z, t)\). We will see that \(\phi_{\lambda, \alpha}(z, t) = \phi_{1, \alpha}(\lambda^{1/2}z, \lambda t)\) for \(\lambda > 0\) and \(\phi_{\lambda, \alpha} = \phi_{|\lambda|, \alpha}\) for \(\lambda < 0\). Subsequently, we will concentrate on \(\phi_{\alpha} := \phi_{1, \alpha}\).

2. The second type arise from the 1-dimensional representations of \(H_n\) and are parametrized by \(V/K\), the space of \(K\)-orbits in \(V \cong \mathbb{C}^n\). For \(\omega \in V\) we write \(\eta_\omega\) for the associated \(K\)-spherical function. One has \(\eta_\omega = \eta_{\omega'}\).
if $K \cdot \omega = K \cdot \omega'$. $\eta_{\omega}(z, t)$ is independent of $t$ and given by the Fourier transform of the unit mass on $K \cdot \omega$.

The type 2 $K$-spherical functions reflect the abelian component of analysis on $H_n$ and are non-generic (they have zero Plancherel measure). The main concern here is the study of the type 1 $K$-spherical functions.

The prescription for the $\phi_\omega$'s derived from [BJR] involves integrating matrix coefficients over $K$. This is difficult to carry out (at best) except in very simple cases. The main results in this paper (contained in Section 4) provide alternative approaches. We will show that $\phi_\omega(z, t) = e^{i\theta} q_\omega(z) e^{-\frac{1}{4}|z|^2}$ where $q_\omega(z)$ is a polynomial function in $(z, \bar{z})$. One also has the (real) $K$-invariant polynomial $p_\omega(z) = \langle 1/\dim(P) \rangle \sum v_i(z) v_i(\bar{z})$ where $\{v_i\}$ is an orthonormal basis for $P_z$. This turns out to be (up to sign) the homogeneous component of highest degree in $q_\omega$. In fact, the $q_\omega$'s are completely determined by the $p_\omega$'s. We provide two computational procedures:

Orthogonalization. The $q_\omega$'s are obtained (up to normalization) by applying the Gram–Schmidt algorithm using the measure $e^{-\frac{1}{2}|z|^2} dz d\bar{z}$ to the $p_\omega$'s. (One can order the $p_\omega$'s in any way that ensures $\alpha < \beta \Rightarrow \deg(p_\alpha) < \deg(p_\beta)$.)

Rodrigues' Formula. There is a constant coefficient differential operator $D_{p_\omega}$ with the property

$$D_{p_\omega}(e^{-\frac{1}{2}|z|^2}) = q_\omega(z) e^{-\frac{1}{2}|z|^2}.$$ $D_{p_\omega}$ is essentially the abelian symmetrization of $p_\omega$.

These results show that the spherical functions $\phi_\omega$ can be written down explicitly once one has the $K$-invariant polynomials $p_\omega$ in hand.

Recent results of Howe and Umeda show that $\mathcal{P}(V_R)^K$, the $K$-invariant polynomials on the underlying real vector space $V_R$ of $V$, is a polynomial ring. In fact, $\mathcal{P}(V_R)^K = \mathbb{C}[\gamma_1, \ldots, \gamma_d]$ where $\{\gamma_1, \ldots, \gamma_d\}$ is an essentially canonical subset of $\{p_\omega\}$ [HU]. In Section 3, we use $\gamma_1, \ldots, \gamma_d$ together with the symmetrization map for $H_n$ to produce two sets of $K$-invariant differential operators $L^1, \ldots, L_d$ and $L^0, \ldots, L^0_d$ on $V$. The functions $\{\psi_{\omega}(z) := q_\omega(z) e^{-\frac{1}{4}|z|^2}\}$ and $\{\eta_{\omega}(z)\}$ are simultaneous eigenfunctions for $\{L_j\}$ and $\{L^0_j\}$, respectively.

In Section 5, we show how the analysis can be reduced to "radial directions." We can regard the functions $p_\omega(z)$ and $\psi_\omega(z)$ as living on $\Gamma \subset (\mathbb{R}^+)^d$, the space of values assumed by the fundamental invariants $\gamma = (\gamma_1, \ldots, \gamma_d)$. Let $\tilde{p}_\omega$ and $\tilde{\psi}_\omega$ be the functions on $\Gamma$ characterized by $\tilde{p}_\omega \circ \gamma = p_\omega$ and $\tilde{\psi}_\omega \circ \gamma = \psi_\omega$. The orthogonalization procedure and Rodrigues' formula can be formulated on $\Gamma$ to obtain algorithms for computing the $\tilde{\psi}_\omega$'s from the
\[ \hat{p}_\alpha \text{'s}. \] The \( \overline{\hat{p}}_\alpha \text{'s} \) solve eigenvalue problems for differential operators \( \overline{L}_1, ..., \overline{L}_d \) obtained from \( L_1, ..., L_d \).

There is an alternative radial reduction when the representation \( K \mid V \) is polar. This means that there is a cross section \( \mathcal{A} \) meeting every orbit orthogonally [Da]. We show that for Gelfand pairs \( (K, H_n) \) with \( K \mid V \) polar, the orthogonalization procedure and the Rodrigues' formula can be reformulated on \( \mathcal{A} \). These results provide algorithms for determining the restrictions \( \psi_\alpha \mid \mathcal{A} \) given \( \{ p_\alpha \mid \mathcal{A} \} \).

In Section 6 we apply the general theory to the motivating example \( (U(n), H_n) \). Most of the results here have been obtained independently by other authors using different methods and viewpoints. (See [Fa, Ko, HR, St, Str].) The type 1 spherical functions \( \phi_\alpha(z, t) = e^{\alpha t} \psi_\alpha(z) \) can be expressed in terms of even Hermite functions on a cross section \( \mathcal{A} \simeq \mathbb{R} \) and as Laguerre functions on the value space \( \Gamma = \mathbb{R}^+ \) for the fundamental invariant \( \gamma_1(z) = |z|^2/2n \). The orthogonalization procedure and Rodrigues' formula reduce to classical theorems concerning these special functions. Our main results generalize these well known theorems. We also show how an easy abstract identity (Proposition 2.4) can be used to derive non-obvious formulas involving Laguerre polynomials (Theorem 6.16).

The results in this paper reduce the study of \( K \)-spherical functions on \( H_n \) to problems in classical invariant theory. The type 1 \( K \)-spherical functions are computable given the multiplicity free decomposition \( \mathcal{P}(V) = \sum_n P_n \). The relevant differential operators and a radial reduction procedure require the fundamental invariants in \( \mathcal{P}(V_n) \). Determination of the type 2 \( K \)-spherical functions involves the dual problem of describing the \( K \)-orbits in \( V \). The connected groups \( K \) which act irreducibly on \( V \) and are multiplicity free on \( \mathcal{P}(V) \) have been classified by V. Kac [Ka]. This yields a table (Table 1.8 below) of all possible “irreducible” Gelfand pairs \( (K, H_n) \). The decompositions of \( \mathcal{P}(V) \) and the fundamental invariants for all of these groups can be found in [HU]. The authors hope to study the resulting spherical functions in a subsequent paper.

1. GELFAND PAIRS \( (K, H_n) \)

We begin with a representation theoretic criterion for \( (K, H_n) \) to be a Gelfand pair. We identify \( H_n \) with \( \mathbb{C}^n \times \mathbb{R} \) with multiplication given by

\[
(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} \omega(z, z')), \tag{1.1}
\]

where \( \omega(z, z') := -\text{Im} \langle z, z' \rangle = -\text{Im}(z \cdot z') \) for \( z, z' \in \mathbb{C}^n \), and \( t, t' \in \mathbb{R} \). The natural action of the group of \( n \times n \) unitary matrices on \( \mathbb{C}^n \) (which we
denote by \( k \cdot z \) for \( k \in U(n) \) and \( z \in \mathbb{C}^n \) gives rise to a compact subgroup of \( \text{Aut}(H_n) \), the group of automorphisms of \( H_n \), via \( k \cdot (z, t) = (k \cdot z, t) \). This subgroup, again denoted \( U(n) \), is a maximal connected, compact subgroup of \( \text{Aut}(H_n) \), and thus any connected, compact subgroup of \( \text{Aut}(H_n) \) is the conjugate of a subgroup \( K \) of \( U(n) \). Since conjugates of \( K \) form Gelfand pairs with \( H_n \) if, and only if, \( K \) does, and produce the same spherical functions, we will always assume that we are dealing with a compact subgroup of \( U(n) \).

It will occasionally be more congenial to have a “coordinate free” model for \( H_n \). In such cases, we assume that \( V \) is an \( n \)-dimensional vector space over \( \mathbb{C} \) equipped with a Hermitian inner product \( \langle \cdot, \cdot \rangle \). \( H_n \) is then identified with \( V \times \mathbb{R} \), and the multiplication is given by \( (v, t)(v', t') = (v + v', t + t' + (1/2) \omega(v, v')) \) where \( \omega(v, v') := -\text{Im}(\langle v, v' \rangle) \). We will write \( H_V \) for the Heisenberg group given by \( (V, \langle \cdot, \cdot \rangle) \). \( U(V) \cong U(n) \), the subgroup of \( GL(V) \) preserving \( \langle \cdot, \cdot \rangle \), is again naturally identified with a closed subgroup of \( \text{Aut}(H_V) \).

The irreducible unitary representations of \( H_n \) which are non-trivial on the center, \( \mathbb{R} \), are determined up to equivalence by their central character. The Fock model, for real \( \lambda > 0 \), is defined on the space \( \mathcal{F}_\lambda \) of holomorphic functions on \( \mathbb{C}^n \) which are square integrable with respect to the measure \( d\tilde{w} = (\lambda/2\pi)^n e^{-(1/2)\lambda |w|^2} dwd\tilde{w} \) [Ba]. The space \( \mathcal{P}(\mathbb{C}^n) \) of holomorphic polynomials is dense in \( \mathcal{F}_\lambda \), and contains an orthonormal basis given by \( \{u_{x, \lambda} : x \in \mathbb{Z}^+ \} \), where

\[
\tag{1.2}
\frac{\lambda^{2/3}w^x}{(2|x|!1/2)} = \frac{w^{x_1}_{1!1/2}}{((2/\lambda)^{2!}|x_1|!1/2)} \cdots \frac{w^{x_n}_{n!1/2}}{((2/\lambda)^{2!}|x_n|!1/2)}.
\]

\( \mathcal{P}_m(\mathbb{C}^n) \) \((= \mathcal{P}_m)\) will denote the subspace of homogeneous polynomials of degree \( m \), and is thus spanned by \( \{u_{x, \lambda} : |x| = m \} \). We will write \( u_x \) for \( u_{x, \lambda} \). For \( p \in \mathcal{P}_m \), let \( p(D) \) be the constant coefficient differential operator \( p(\partial/\partial w_1, ..., \partial/\partial w_n) \). One has an alternative description of the Hilbert space structure.

**Lemma 1.3.** For \( p, q \in \mathcal{P}_m \), the inner product in \( \mathcal{F} := \mathcal{F}_1 \) is given by

\[
\langle p, q \rangle = 2^m p(D)q.
\]

**Proof.** For monomials \( w^\alpha, w^\beta \), with \( |\alpha| = |\beta| = m \), the inner product on \( \mathcal{F} \) is

\[
\langle w^\alpha, w^\beta \rangle = \begin{cases} 0, & \text{if } \alpha \neq \beta; \\ 2^m |\alpha|!, & \text{if } \alpha = \beta. \end{cases}
\]
Remark. In general, for polynomials $p, q \in \mathcal{P}(\mathbb{C}^n)$, we have $\langle p, q \rangle = \langle p(2D)q \rangle(0)$.

The representation $\pi_\lambda$ of $H_n$ of $\mathcal{F}_\lambda$ is given by

$$\pi_\lambda(z, t) u(w) = e^{i\lambda t} (1/2)^{\lambda} \langle w, z \rangle^{\lambda/2} u(w + z). \quad (1.4)$$

For $\lambda < 0$, $\mathcal{F}_\lambda$ consists of antiholomorphic functions which are square integrable with respect to $d\bar{w}_{|\lambda|}$, and the representation is given by

$$\pi_\lambda(z, t) u(\bar{w}) = e^{i\lambda t + (1/2)\lambda \langle w, z \rangle + (1/4)\lambda |z|^2} \bar{u}(w + z). \quad (1.5)$$

For $k \in U(n)$, the representation $\pi_\lambda^k(z, t) := \pi_\lambda(k \cdot z, t)$ has the same central character as $\pi_\lambda$, and hence is equivalent to $\pi_\lambda$. For $\lambda > 0$ the operator that intertwines these two representations comes from the standard action of $U(n)$ on $\mathbb{C}^n$. More precisely,

$$[\pi_\lambda(k \cdot z, t) u](k \cdot w) = [\pi_\lambda(z, t)(k^{-1} \cdot u)](w), \quad (1.6)$$

where $k \cdot u(w) = u(k^{-1} \cdot w)$. One has a similar formula for $\lambda < 0$, except that the action of $U(n)$ on antiholomorphic functions is given by $k \cdot u(\bar{w}) = u(k \cdot \bar{w})$.

The following theorem is a special case of a general result proved by G. Carcano [Ca].

**Theorem 1.7.** $(K, H_n)$ is a Gelfand pair if, and only if, the action of $K$ on $\mathcal{F}_\lambda$ decomposes into irreducible components of multiplicity one for all $\lambda \neq 0$.

Since $\mathcal{P}(\mathbb{C}^n)$ is dense in $\mathcal{F}_\lambda$, $\lambda > 0$, and $\mathcal{P}_m$ is invariant under $U(n)$, for $K \subset U(n)$, the $K$-irreducible components of $\mathcal{F}_\lambda$ have multiplicity one if, and only if, the action of $K$ on $\mathcal{P}(\mathbb{C}^n)$ is multiplicity free.

A similar statement holds for the decomposition of $\mathcal{F}_\lambda$, $\lambda < 0$, and the action of $K$ on the antiholomorphic polynomials. Thus one concludes that $(K, H_n)$ is a Gelfand pair if, and only if, $\mathcal{F}$ ($= \mathcal{F}_1$) decomposes into $K$-irreducible components of multiplicity one, which in turn is equivalent to requiring that the action of $K$ on $\mathcal{P}(\mathbb{C}^n)$ be multiplicity free.

Let $K_C \subset GL(n, \mathbb{C})$ be the complexification of $K$. Then the irreducible components of $\mathcal{P}(\mathbb{C}^n)$ with respect to $K$ and $K_C$ are identical. The connected groups $K_C$ which act irreducibly and without multiplicity have been classified by V. Kac [Ka]. These groups and their representations are given in the following table.
1.8. Multiplicity Free Representations.

<table>
<thead>
<tr>
<th>Group</th>
<th>Acting On</th>
<th>Subject To</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SL}(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 2$</td>
</tr>
<tr>
<td>$\text{GL}(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 1$</td>
</tr>
<tr>
<td>$\text{Sp}(k, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 2k$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times \text{Sp}(k, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 2k$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times \text{SO}(n, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>$\text{Gl}(k, \mathbb{C})$</td>
<td>$\mathbb{S}^1(\mathbb{C}^k) \cong \mathbb{C}^n$</td>
<td>$n = k(k + 1)/2, k \geq 2$</td>
</tr>
<tr>
<td>$\text{Sl}(k, \mathbb{C})$</td>
<td>$\mathbb{A}^2(\mathbb{C}^k) \cong \mathbb{C}^n$</td>
<td>$n = \left(\frac{k}{2}\right)$ and $k$ is odd</td>
</tr>
<tr>
<td>$\text{Gl}(k, \mathbb{C})$</td>
<td>$\mathbb{A}^2(\mathbb{C}^k) \cong \mathbb{C}^n$</td>
<td>$n = \left(\frac{k}{2}\right)$</td>
</tr>
<tr>
<td>$\text{Sl}(k, \mathbb{C}) \times \text{Sl}(l, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^l \cong \mathbb{C}^n$</td>
<td>$n = kl, k \neq l$</td>
</tr>
<tr>
<td>$\text{Gl}(k, \mathbb{C}) \times \text{Sl}(l, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^l \cong \mathbb{C}^n$</td>
<td>$n = kl$</td>
</tr>
<tr>
<td>$\text{Gl}(2, \mathbb{C}) \times \text{Sp}(k, \mathbb{C})$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \cong \mathbb{C}^n$</td>
<td>$n = 4k$</td>
</tr>
<tr>
<td>$\text{Sl}(3, \mathbb{C}) \times \text{Sp}(k, \mathbb{C})$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \cong \mathbb{C}^n$</td>
<td>$n = 6k$</td>
</tr>
<tr>
<td>$\text{Gl}(3, \mathbb{C}) \times \text{Sp}(k, \mathbb{C})$</td>
<td>$\mathbb{C}^3 \otimes \mathbb{C}^{2k} \cong \mathbb{C}^n$</td>
<td>$n = 6k$</td>
</tr>
<tr>
<td>$\text{Gl}(4, \mathbb{C}) \times \text{Sp}(4, \mathbb{C})$</td>
<td>$\mathbb{C}^4 \otimes \mathbb{C}^4 \cong \mathbb{C}^n$</td>
<td>$n = 32$</td>
</tr>
<tr>
<td>$\text{Sl}(k, \mathbb{C}) \times \text{Sp}(4, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^4 \cong \mathbb{C}^n$</td>
<td>$n = 8k, k &gt; 4$</td>
</tr>
<tr>
<td>$\text{Gl}(k, \mathbb{C}) \times \text{Sp}(4, \mathbb{C})$</td>
<td>$\mathbb{C}^k \otimes \mathbb{C}^4 \cong \mathbb{C}^n$</td>
<td>$n = 8k, k &gt; 4$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times \text{Spin}(7, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 8$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times \text{Spin}(9, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$\text{Spin}(10, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times \text{Spin}(10, \mathbb{C})$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 16$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times G_2$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 7$</td>
</tr>
<tr>
<td>$\mathbb{C}^* \times E_6$</td>
<td>$\mathbb{C}^n$</td>
<td>$n = 27$</td>
</tr>
</tbody>
</table>

2. $K$-Spherical Functions

There are several equivalent definitions for the $K$-spherical functions associated to a given Gelfand pair $(K, N)$. By one definition, a $K$-spherical function associated to such a pair is a smooth $K$-invariant complex-valued function $\phi$ on $N$ such that $\phi(e) = 1$, and $\phi$ is a joint eigenfunction for all left $N$-and $K$-invariant differential operators on $N$. Equivalently, they can be defined as the homomorphisms of the commutative convolution algebra of compactly supported, $K$-invariant distributions, or the commutative convolution algebra $L^1_k(N)$. (See [He].) In the latter case, the homomorphisms corresponding to bounded $K$-spherical functions are continuous in the $L^1$-norm. In [BJR] we give a parametrization for the bounded $K$-spherical functions. Given $\pi \in \hat{N}$, we define

$$K_{\pi} = \{ k \in K : \pi^k \text{ is unitarily equivalent to } \pi \}, \quad (2.1)$$

where $\pi^k := \pi \circ k$.

**Theorem 2.2.** Given a Gelfand pair $(K, N)$, and an irreducible representation $\pi$ of $N$, let $H_\pi = \bigoplus_x P_x$ be the decomposition of the representation
space into $K_\pi$-irreducible subspaces. For each subspace $P_\pi$, and each representation $\pi$, we have the bounded $K$-spherical function

$$\phi_{\pi,v}(x) = \int_K \langle \pi(k \cdot x)v, v \rangle \, dk,$$

for any unit vector $v \in P_\pi$, and $x \in N$. All bounded $K$-spherical functions are obtained in this manner, and $\phi_{\pi,v} = \phi_{\pi,v'}$ if, and only if, $\pi' = \pi^k$ for some $k \in K$ and $v, v'$ belong to the same $P_\pi$.

**Notation.** We set $\phi_{\pi,v} = \phi_{\pi,v}$ for any unit vector $v \in P_\pi$.

The proof of (2.2) given in [BJR] yields an alternative formula for $\phi_{\pi,v}$ in certain cases.

**COROLLARY 2.3.** Suppose that $K_\pi = K$ and that $\{v_1, \ldots, v_l\}$ is any orthonormal basis for $P_\pi$. Then

$$\phi_{\pi,v}(x) = \sum_{j=1}^l \langle \pi(x) v_j, v_j \rangle.$$

Recall that $\pi^k \simeq \pi$ for an infinite dimensional representation $\pi$ of $H_n$ and any $k \in U(n)$. (See Eq. (1.6).) Thus, $K_\pi = K$ for any compact subgroup $K$ of $U(n)$ and (2.3) can be applied to compute the associated $K$-spherical functions $\phi_{\pi,v} := \phi_{\pi,v}$.

**Remark.** Although the group $K$ is used explicitly in (2.2) to define the $K$-spherical functions, one sees from (2.3) that when $K_\pi = K$, the $K$-spherical functions (as well as the $K$-invariant polynomials $P_\pi$ given in (3.9)) depend only upon the decomposition $\mathcal{P}(V) = \bigoplus P_\pi$, not upon the group $K$ itself. For example, the $K$-spherical functions on $H_n$ given by $\pi_\lambda$ coincide if $K_C$ is any of the first four groups in Table 1.8. These are computed in Section 5. In addition, the $C^*$-factors do not affect the decomposition of $\mathcal{P}(V)$. Thus, when two groups in Table 1.8 differ by the addition or removal of $C^*$, they yield the same spherical functions.

**PROPOSITION 2.4.** Suppose that $K'$ is a compact subgroup of $K$, and that $(K', N)$ is a Gelfand pair. Let $P_\pi = \bigoplus_{i=1}^{n_\pi} P_{\pi,i}$ be the decomposition of $P_\pi$ into $K'$-irreducible subspaces. Let $\phi_{\pi,v,i}$ denote the $K'$-spherical function corresponding to a unit vector in $P_{\pi,i}$. Then

$$\phi_{\pi,x}(x) = \int_{K/K'} \phi_{\pi,v,i}(k \cdot x) \, dk. \quad (2.5)$$

Moreover, if $K_\pi = K$ then

$$\phi_{\pi,x} = \frac{1}{\dim(P_\pi)} \sum_{i=1}^{n_\pi} \dim(P_{\pi,i}) \phi_{\pi,v,i}. \quad (2.6)$$
Proof. If $K_n = K$ then $K'_n = K' \cap K_n = K'$. Thus, (2.6) follows easily from (2.3) by using an orthonormal basis for $P_\kappa$ that is a union of orthonormal bases for the $P_{\kappa,j}$'s.

For the first equation, note that the right hand side of (2.5) is equal to

$$
\int_K \phi_{n,\kappa,\alpha}(k \cdot x) \, dk = \int_K \int_{K'} \langle \pi(k'k \cdot x)v, v' \rangle \, dk' \, dk
$$

$$
= \int_K \langle \pi(k \cdot x)v, v \rangle \, dk
$$

$$
= \phi_{n,\alpha}(x).
$$

We now return to the Heisenberg group, and first determine how the bounded $K$-spherical functions for different representations $\pi_\lambda$ are related. For each $\lambda > 0$ we have an automorphism of $H_n$,

$$
\delta_\lambda(z, y) = (\lambda^{1/2}z, \lambda t).
$$

(2.7)

called dilation and an isometry $d_\lambda : \mathcal{F}_1 \to \mathcal{F}_\lambda$,

$$
d_\lambda u(w) = u(\lambda^{1/2}w).
$$

(2.8)

For $\lambda > 0$, $\pi_1$ and $\pi_\lambda$ are related by the formula

$$
d_\lambda \pi_1(\delta_\lambda(z, t)) = \pi_\lambda(z, t) d_\lambda.
$$

(2.9)

We temporarily adopt the notation $\langle \cdot, \cdot \rangle_\lambda$ for the inner product on $\mathcal{F}_\lambda$ given by $d\tilde{\omega}_\lambda$. For $\lambda > 0$ and $v \in P_\lambda$ with $\langle v, v \rangle_\lambda = 1$, one has the associated $K$-spherical function,

$$
\phi_{\lambda,\alpha}(z, t) = \int_K \langle \pi_\lambda(k \cdot z, t)v, v \rangle_\lambda \, dk = e^{it} \psi_{\lambda,\alpha}(z),
$$

where

$$
\psi_{\lambda,\alpha}(z) = \int_K \langle \pi_\lambda(k \cdot z, 0)v, v \rangle_\lambda \, dk
$$

$$
\int_K \langle d_\lambda \pi_1(\delta_\lambda(k \cdot z, 0)) d_\lambda^{-1}v, v \rangle_\lambda \, dk \quad \text{(by formula (2.9))}
$$

$$
= \int_K \langle \pi_1(\delta_\lambda(k \cdot z, 0)) d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 \, dk
$$

$$
= \int_K \langle \pi_1(k \cdot \delta_\lambda(z, 0)) d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 \, dk
$$

$$
= \psi_{1,\alpha}(\lambda^{1/2}z)
$$

(since $\langle d_\lambda^{-1}v, d_\lambda^{-1}v \rangle_1 = 1$).
That is, for $\lambda > 0$

$$\phi_{\lambda, z}(z, t) = \phi_{1, z}(\lambda^{1/2}z, \lambda t) = e^{i\lambda t} \psi_{1, z}(\lambda^{1/2}z). \quad (2.10)$$

For $\lambda < 0$, one has

$$\phi_{\lambda, z} = \overline{\phi_{|\lambda|, z}}(\lambda^{1/2}z, |\lambda| t) = e^{i|\lambda| t} \overline{\psi_{1, z}(\lambda^{1/2}z)}. \quad (2.11)$$

Indeed, $\pi_\lambda = \overline{\pi_{|\lambda|}}$ and hence the matrix coefficients of $\pi_\lambda$ are obtained via complex conjugation from those for $\pi_{|\lambda|}$.

Thus the $K$-spherical functions arising from infinite dimensional representations of $H_n$ are all obtained by dilating certain $K$-spherical functions associated to $\pi (= \pi_1)$. We will focus our attention on this representation. Henceforth $\pi$, $\mathcal{F}$, $\phi_\lambda$, and $P_\lambda$ will be used instead of $\pi_1$, $\mathcal{F}_1$, $\phi_{1, \lambda}$, and $P_{1, \lambda}$.

Suppose now that $K_i$ is a compact subgroup of $U(n_i)$ for $i = 1, \ldots, d$, and that $K^0 = \prod_{i=1}^d K_i$. Let $n = \sum n_i$. Then $K^0 \subset U(n)$, and from [BJR] one has that $(K^0, H_n)$ is a Gelfand pair if, and only if, $(K_i, H_{n_i})$ is a Gelfand pair for $i = 1, \ldots, d$. Let $H_n = V \times \mathbb{R}$ and let $V = \bigoplus_{i=1}^d V_i$ be such that $K^0|_{V_i} = K_i$. Then $\mathcal{P}(V) = \bigotimes_{i=1}^d \mathcal{P}(V_i)$. Let $\mathcal{P}(V_i) = \bigoplus_{j=1}^{n_i} P_{i,j}$ be the decomposition of $\mathcal{P}(V_i)$ into $K_i$-irreducible subspaces. Then $\mathcal{P}(V) = \bigoplus_{j_1, \ldots, j_d}^{P_{n_1} \cdots P_{n_d}}$, where $P_{n_1} \cdots P_{n_d} = P_{1, j_1} \otimes \cdots \otimes P_{d, j_d}$, is the decomposition of $\mathcal{P}(V)$ into $K^0$-irreducible subspaces. Denote by $\phi_{j_1, \ldots, j_d}$ the $K^0$-spherical function associated to $\pi$ and $P_{n_1} \cdots P_{n_d}$, and by $\phi_{i, j}$ the $K_i$-spherical function on $H_{n_i}$ associated to $\pi$ and $P_{i, j}$, for $i = 1, \ldots, d$.

**Proposition 2.12.** Let $u_i \in V_i$, for $i = 1, \ldots, d$. Then

$$\phi_{j_1, \ldots, j_d}(u_1 + \cdots + u_d, t) = e^{it} \prod_{i=1}^d \psi_{i, j}(u_i).$$

**Proof.** By an obvious induction argument we may reduce to the case $d = 2$. Now note that since $K^0$ acts trivially on the center of $H_n$, $\phi_{j_1, j_2}(z, t) = e^{it} \psi_{j_1, j_2}(z)$. Let $\{v_p^i: 1 \leq p \leq m_i\}$ be a basis for $P_{i, j}$, for $i = 1, 2$. By Corollary 2.2,

$$\phi_{j_1, j_2}(u_1 + u_2, t) = e^{it} \psi_{j_1, j_2}(u_1 + u_2)$$

$$= \frac{e^{it}}{m_1 m_2} \sum_{p, q} \langle \pi(u_1 + u_2, 0) v_p^1 \otimes v_q^2, v_p \otimes \overline{v_q^2} \rangle$$

$$= \frac{e^{it}}{m_1 m_2} \sum_{p, q} \langle \pi(u_1, 0) v_p^1 \otimes \pi(u_2, 0) v_q^2, v_p \otimes \overline{v_q^2} \rangle$$

$$= \frac{e^{it}}{m_1 m_2} \sum_p \langle \pi(u_1, 0) v_p^1, v_p^1 \rangle \sum_q \langle \pi(u_2, 0) v_q^2, v_q^2 \rangle.$$

The two latter sums are $m_i \psi_{j_i}^i$ for $i = 1, 2$, respectively.
In addition to the infinite dimensional representations $\pi$ of $H_n$, one also has infinitely many one dimensional representations corresponding to one-point coadjoint orbits. These are parametrized by $C^n$ where for $\omega \in C^n$, $\chi_{\omega}(z, t) = e^{i \Re \langle \omega, z \rangle}$ for $(z, t) \in H_n$. If $(K, H_n)$ is a Gelfand pair, then for each such character, one has by Theorem 2.1 a $K$-spherical function, $\eta_{\omega}$, that is the $K$-average of $\chi_{\omega}$, i.e.,

$$\eta_{\omega}(z, t) = \int_K e^{i \Re \langle \omega, k \cdot z \rangle} dk.$$

Thus $\eta_{\omega}$ depends only on the $K$-orbit through $\omega$, and Theorem 2.1 ensures that if $\omega' \notin K \cdot \omega$ then $\eta_{\omega'} \neq \eta_{\omega}$.

Let $\mu_{K \cdot \omega}$ denote the unit mass supported on the $K$-orbit through $\omega$. As a distribution this is given by

$$\langle \mu_{K \cdot \omega}, f \rangle = \int_K f(k^{-1} \cdot \omega, 0) \, dk,$$

for all $f \in C_c^\infty(H_n)$. We define the (Euclidean) Fourier transform for such a function by

$$\hat{f}(\omega, \tau) = \int_{C^n \times R} f(z, t) e^{i \Re \langle \omega, z \rangle + \tau t} \, dz \, d\bar{z} \, dt.$$

Note that

$$\int_{H_n} \eta_{\omega}(z, t) f(z, t) \, dz \, d\bar{z} \, dt = \int_{H_n} \int_K e^{i \Re \langle k^{-1} \cdot \omega, z \rangle} f(z, t) \, dk \, dz \, d\bar{z} \, dt$$

$$= \int_K \hat{f}(k^{-1} \cdot \omega, 0) \, dk$$

$$= \langle \hat{\mu}_{K \cdot \omega}, f \rangle,$$

for $f \in C_c^\infty(H_n)$. Therefore we have

**Proposition 2.13.** $\eta_{\omega} = \hat{\mu}_{K \cdot \omega}$.

Finally we observe that if $K = K_1 \times K_2$, where $K_i \subset U(n_i)$, for $i = 1, 2$, then the one point $K$-orbits are naturally parametrized by $(\omega_1, \omega_2)$, where $\omega_i \in C^{n_i}$ with $K \cdot (\omega_1, \omega_2) = \{(k_1 \cdot \omega_1, k_2 \cdot \omega_2) : k_i \in K_i\}$. One easily checks that $\eta_{(\omega_1, \omega_2)}(z_1, z_2, t) = \eta_{\omega_1}(z_1) \eta_{\omega_2}(z_2)$, which establishes

**Proposition 2.14.** $\eta_{(\omega_1, \omega_2)} = \hat{\mu}_{K_1 \cdot \omega_1} \otimes \hat{\mu}_{K_2 \cdot \omega_2}$.

The Gelfand space of a commutative Banach algebra $E$ is the subspace of $E'$, the dual of $E$, consisting of all continuous, complex valued
homomorphisms of $E$ equipped with the weak topology. For a Gelfand pair $(K, H,)$, the Gelfand space of $L^1_K(H_n)$ is the point set $(\mathbb{R}^* \times \{a\}) \cup V/K$, where $\{a\}$ is the index set for the decomposition $\mathcal{P}(V) = \bigoplus_a P_a$ into irreducible $K$-modules and $V/K$ denotes the $K$-orbits in $V$. The topology on the first subset is the product of the usual topology on $\mathbb{R}^*$ with the discrete topology on $\{a\}$, and $V/K$ has the quotient topology.

One knows from general principles that $V/K$ is in the closure of $\mathbb{R}^* \times \{a\}$. For example, at $0 = K \cdot 0 \in V/K$, one easily sees from (2.10) and (2.11) that as $\lambda \to 0$, $\phi_{\lambda, a} \to 1$ uniformly on compacta, for each $a$. Thus, each neighborhood of $0$ contains $(\lambda, a)$ for each $a$ and all sufficiently small $\lambda = \lambda(a)$. It would be interesting to know how $\lambda$ must behave as $\lambda \to 0$ in order to produce convergence to a non-zero point in $V/K$. For $K = U(n)$, this was studied in [Str].

3. $K$-INARIANT POLYNOMIALS AND DIFFERENTIAL OPERATORS

Suppose that $N$ is a nilpotent Lie group with Lie algebra $\mathfrak{n}$ and let $K$ be a compact group of automorphisms of $N$. Given a differential operator $D \in \mathcal{U}(\mathfrak{n})$, the universal enveloping algebra of $\mathfrak{n}$, and $k \in K$, we define the differential operator $D^k$ by

$$D^k(f) = D(f \circ k) \circ k^{-1} \quad \text{for} \quad f \in C^\infty(N). \quad (3.1)$$

$D$ is said to be $K$-invariant if $D^k = D$ for all $k \in K$. We write $\mathcal{U}_K(\mathfrak{n})$ for the algebra of $K$-invariant differential operators.

The derived action of $K$ on $\mathfrak{n}$ is given by

$$\exp(k \cdot X) = k \cdot \exp(X) \quad (3.2)$$

and $K$ acts on $\mathfrak{n}^* = \text{Hom}_\mathbb{R}(\mathfrak{n}, \mathbb{R})$ via

$$(k \cdot \alpha)(X) = \alpha(k^{-1} \cdot X) \quad (3.3)$$

and on $\mathcal{P}(\mathfrak{n}^*)$, the $\mathbb{C}$-valued polynomial functions on the $\mathbb{R}$-vector space $\mathfrak{n}^*$ via

$$(k \cdot p)(\alpha) = p(k^{-1} \cdot \alpha). \quad (3.4)$$

We identify $\mathcal{P}(\mathfrak{n}^*)$ with the complexified symmetric algebra $S(\mathfrak{n})_\mathbb{C}$ so that the symmetric product $X_1 X_2 \cdots X_n$ of $X_1, X_2, \ldots, X_n \in \mathfrak{n}$ becomes the polynomial $\mathfrak{n}^* \to \mathbb{C}$ given by

$$p_{X_1 \cdots X_n}(\alpha) = \alpha(X_1) \cdots \alpha(X_n). \quad (3.5)$$
One has
\[ k \cdot p_{x_1 \ldots x_n}(x) = p_{(k \cdot X_1) \ldots (k \cdot X_n)} \]
\[ = p_{(k \cdot X_1) \ldots (k \cdot X_n)}(x). \] (3.6)

The symmetrization map \( \lambda : \mathcal{M}(n^\ast) \to \mathcal{U}(n) \) is defined by
\[ \lambda(p)(f)(n) = p \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n} \right) \bigg|_{t_i = 0} f \left( n \exp \left( \sum_{i=1}^{n} t_i X_i \right) \right). \] (3.7)

where \( \{X_1, \ldots, X_n\} \) is any basis for \( n \). This definition is independent of the basis chosen and \( \lambda \) is characterized by the fact that \( \lambda(X^m) = (\tilde{X})^m \) where \( \tilde{X} \) denotes the first order differential operator (left invariant vector field) given by \( X \in n \). \( \lambda \) is a linear bijection, but not an algebra map. We refer the reader to [He] for these facts about the symmetrization map.

**Lemma 3.8.** Let \( (K, N) \) be a Gelfand pair. Then \( \lambda(k \cdot p) = \lambda(p)^k \) for all \( k \in K, p \in \mathcal{M}(n^\ast) \). Hence \( \lambda \) yields a linear isomorphism between \( \mathcal{M}(n^\ast)^K \simeq S(n)^K \), the \( K \)-invariant elements in \( \mathcal{M}(n^\ast) \simeq S(n)_C \) and \( \mathcal{U}_x(n) \).

**Proof.** The result follows immediately from the fact that \( \lambda \) is natural and hence equivariant with respect to Lie algebra automorphisms.

We turn now to the Heisenberg group \( H_n = V \times \mathbb{R} \) and the action of a compact subgroup \( K \subset U(n) \) for which \( (K, H_n) \) is a Gelfand pair. As before, \( \mathcal{P}(V) = \bigoplus_z P_z \) denotes the decomposition of the holomorphic polynomials into \( K \)-irreducible components of multiplicity one. Since the constant polynomials \( \mathcal{P}_0(V) \) are the component for the trivial representation of \( K \), we must have that \( \mathcal{P}(V)^K = \mathbb{C} \). That is, the only \( K \)-invariant holomorphic polynomials \( V \to \mathbb{C} \) are constant. However, one can also consider the larger space \( \mathcal{P}(V_R) \) of \( \mathbb{C} \)-valued polynomial functions on the underlying real vector space \( V_R \) of \( V \). As a \( \mathbb{C} \)-vector space, \( \mathcal{P}(V_R) = \mathcal{P}(V) \otimes \mathbb{C} \tilde{F}(V) \) (and identifying \( V \) with \( \mathbb{C}^n \) one can write \( \mathcal{P}(V) = \mathbb{C}[z_1, \ldots, z_n], \mathcal{P}(V_R) = \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n] \)). \( K \) acts on \( \mathcal{P}(V_R) \) via \( \tilde{v}(k) = v(k) \otimes \bar{v}(k) \), where \( v(k) \) denotes the standard action on \( \mathcal{P}(V) \). We have the following description of the \( K \)-invariant elements \( \mathcal{P}(V_R)^K \) in \( \mathcal{P}(V_R) \).

**Proposition 3.9.** Let \( (K, H_n) \) be a Gelfand pair and let \( \sum_z P_z \) be the decomposition of \( \mathcal{P}(V) \) (or equivalently \( \mathcal{P} \)) into \( K \)-irreducible subspaces. Let \( \{v_1, \ldots, v_l\} \) be an orthonormal basis for \( P_z \). Then
\[ P_z = \frac{1}{l} \sum_{i=1}^{l} v_i \delta_i, \]
is a $K$-invariant element of $\mathcal{P}(V_R)$. Moreover, $\{p_\alpha\}$ is a vector space basis for $\mathcal{P}(V_R)^K$.

**Proof.** One has $\mathcal{P}(V_R) = \mathcal{P}(V) \otimes \overline{P(V)} = \sum_{\alpha, \beta} P_\alpha \otimes \overline{P}_\beta$. Decompose a given $v \in \mathcal{P}(V_R)^K$ as $v = \sum v_{\alpha, \beta} v_{\alpha, \beta}$, where $v_{\alpha, \beta} \in P_\alpha \otimes \overline{P}_\beta$. (This must be a finite sum, i.e., $v_{\alpha, \beta} = 0$ for a.e. $(\alpha, \beta)$.) Since the $P_\alpha \otimes \overline{P}_\beta$'s are $K$-invariant and pair-wise orthogonal, each $v_{\alpha, \beta}$ must also be $K$-invariant. Thus, the $K$-invariant elements in each $P_\alpha \otimes \overline{P}_\beta$ span $\mathcal{P}(V_R)^K$.

As $\mathcal{P}(V)$ is multiplicity free, an application of Schur's Lemma shows that $P_\alpha \otimes \overline{P}_\beta$ has no $K$-fixed vectors when $\alpha \neq \beta$ and that $P_\alpha \otimes \overline{P}_\alpha$ has a one-dimensional space of $K$-fixed vectors which is spanned by $p_\alpha$.

Finally, the set $\{p_\alpha\}$ is linearly independent since there are no non-trivial linear dependence relations between the subspaces $P_\alpha \otimes \overline{P}_\beta$ of $\mathcal{P}(V_R)$.

**Remark.** The polynomial $p_\alpha$ is homogeneous of degree $2 \deg(P_\alpha)$. Thus, all invariant polynomials have even degree. If $q$ is an invariant polynomial of degree $2m$ then we must have

$$q \in \text{Span}(p_\alpha : \deg(P_\alpha) \leq m).$$

This fact is used later in the proof of Proposition 4.2.

Proposition 3.9 yields a vector space basis for $\mathcal{P}(V_R)^K$. Since $K$ is compact, $\mathcal{P}(V_R)^K$ is finitely generated. Recent results of Howe and Umeda show that $\mathcal{P}(V_R)^K$ is in fact a polynomial algebra. We summarize some ideas from [HU].

If $f(z)$ and $g(z)$ are $K$-highest weight vectors in $\mathcal{P}(V)$ (with respect to some fixed choice of a Cartan subalgebra and set of positive roots for $K$) then so is $f(z)g(z)$. A highest weight vector in $\mathcal{P}(V)$ is called primitive if it can't be written as a product of highest weight vectors in this way. These will be finite in number. Let $P_{\alpha_1}, \ldots, P_{\alpha_d}$ be the irreducible components in $\mathcal{P}(V)$ that contain primitive highest weight vectors and let $y_i := p_{\alpha_i}$. We call $y_1, \ldots, y_d$ the fundamental invariants.

**Theorem 3.10** (See [HU]). $\mathcal{P}(V_R)^K = C[y_1, \ldots, y_d]$.

**Remark.** When $V$ is $K$-irreducible, $V^* \subset \mathcal{P}(V)$ is one of the primitive $K$-irreducible components. We will always let $y_1$ be the corresponding invariant in $\mathcal{P}(V_R)$. One has $y_1(z) = |z|^2/2n$ (where $n = \dim_c(V)$).

The symplectic structure $\omega$ on $V_R$ sets up an isomorphism $V_R \simeq V_R^*$ and an induced algebra isomorphism

$$\Omega: \mathcal{P}(V_R) \to \mathcal{P}(V_R^*).$$

(3.11)
Define $\tilde{x}: \mathcal{D}(V_J) \to \mathcal{U}(h_n)$ by

$$\tilde{x} := \lambda \circ j \circ \Omega,$$

where $j$ is the obvious inclusion $\mathcal{D}(V_J) \subset \mathcal{D}(h_n)$ and $\lambda$ is the symmetrization map for $H_n$. Let $(z_1, z_2, \ldots, z_n)$ be coordinates on $V$ with respect to any $\langle \cdot, \cdot \rangle$-orthonormal basis. We obtain isomorphisms $V \simeq \mathbb{C}^n$ and $\mathcal{D}(V_J) \simeq \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$ and $\tilde{x}$ is given by

**Lemma 3.13.**

$$(\tilde{x}(p)f)(z, t) = p \left( \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \zeta} \right) f \left( z + \xi, t + \frac{1}{2} \omega(z, \xi) \right) \bigg|_{\xi = 0}.$$

**Proof.** Write $z_j = x_j + iy_j$, with corresponding basis $\{X_j, Y_j\}$ for the Lie algebra $h_n$ of $H_n$. Then by (3.7), for $p \in \mathcal{D}(h_n)$,

$$(\lambda(p)f)(z, t) = p \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) f \left( \sum u_j X_j + \sum v_j Y_j \right) \bigg|_{u = v = 0} = p \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) f \left( z + (u + iv), t + \frac{1}{2} \omega(z, u + iv) \right) \bigg|_{u = v = 0}.$$

Let $\zeta = u + iv$. Then for $p \in \mathbb{C}[z, \bar{z}]$,

$$(\tilde{x}(p)f)(z, t) = p \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v}, \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) f \left( z + (u + iv), t + \frac{1}{2} \omega(z, u + iv) \right) \bigg|_{u = v = 0} = p \left( 2 \frac{\partial}{\partial \xi}, 2 \frac{\partial}{\partial \zeta} \right) f \left( z + \xi, t + \frac{1}{2} \omega(z, \xi) \right) \bigg|_{\xi = 0}.$$

We have $\tilde{x}(\mathcal{D}(V_J)^K) \subset \mathcal{U}_K(h_n)$ since $\lambda, j$, and $\Omega$ are all $K$-equivariant. In fact, $\mathcal{U}_K(h_n)$ is generated by $\tilde{x}(\mathcal{D}(V_J)^K)$ and the operator $\partial/\partial t$. Let $L_1, \ldots, L_d \in \mathcal{U}_K(h_n)$ be defined by

$$L_j := \tilde{x}(\gamma_j).$$

**Lemma 3.15.** A smooth $K$-invariant function $\phi: H_n \to \mathbb{C}$ with $\phi(e) = 1$ is $K$-spherical if and only if $\phi$ is a joint eigenfunction for $L_1, \ldots, L_d$ and $\partial/\partial t$.

Indeed, $\tilde{x}$ induces an algebra map on the associated graded algebras and
one can use induction to show that eigenfunctions of $L_1, \ldots, L_d$ and $\partial/\partial t$ are eigenfunctions of any element in $\mathcal{U}_K(h_n)$.

We know (see (2.10)) that $\phi_{\lambda,\alpha}(z, t)$ has the form $\phi_{\lambda,\alpha}(z, t) = e^{i\omega t} \psi_{\lambda,\alpha}(z)$. (Here $\phi_{0, \alpha} := \eta_{\alpha}$.) Thus,

$$L_j(\phi_{\lambda,\alpha}) = e^{i\omega t} L_j^\lambda(\psi_{\alpha}),$$  

(3.16)

where $L_j^\lambda \in \mathcal{U}_K(h_n)$ is the operator obtained by replacing each copy of $\partial/\partial t$ in $L_j$ by $i\lambda$.

**Theorem 3.17.** The $K$-spherical functions $\{\phi_{\alpha}(z, t)\}$ are obtained from the bounded $K$-invariant simultaneous eigenfunctions $\{\psi_{\alpha}(z)\}$ of $L_1^\lambda, \ldots, L_d^\lambda$ with $\psi_{\alpha}(0) = 1$ via $\phi_{\alpha}(z, t) = e^{i\omega t} \psi_{\alpha}(z)$. The $K$-spherical functions $\{\eta_{\alpha}(z)\}$ are the bounded $K$-invariant eigenfunctions for $L_1^\lambda, \ldots, L_d^\lambda$ with $\eta_{\alpha}(0) = 1$.

**Remark.** For $K$ acting irreducibly on $V$, $\gamma_1 = |z|^2/2n$ and one computes

$$L_1 = \frac{1}{2n} \left( 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + i z \frac{\partial}{\partial z} \frac{\partial}{\partial t} - i \bar{z} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial t} + \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} \right)$$

(3.18)

and

$$L_1^\lambda = \frac{1}{2n} \left( 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{1}{4} |z|^2 \right)$$

(3.19)

$$L_1^0 = \frac{4}{2n} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}.$$

$L_1$ is a scalar multiple of the usual Heisenberg sub-Laplacian.

We conclude this section with the following observation on the spectrum of an element $D \in \mathcal{U}_K(h_n)$.

**Proposition 3.20.** Let $(K, H_n)$ be a Gelfand pair, and let $\mathcal{F} = \bigoplus P_a$ be the decomposition of Fock space into $K$-irreducible subspaces. Then for $D \in \mathcal{U}_K(h_n)$, $D \phi_{\alpha,z} = \lambda_D \phi_{\alpha,z}$, where the eigenvalue $\lambda_D$ is obtained from the equation $\pi(D) \nu_a = \lambda_D \nu_a$ for any $\nu_a \in P_a$.

**Proof.** Note that $D$ is a $K$-invariant distribution supported at the identity, and hence one can find a sequence $\{g_n\}$ of smooth, compactly supported, $K$-invariant functions on $H_n$ which converge weakly to $D$. Then $\langle \pi(g_n)u, v \rangle \rightarrow \langle \pi(D)u, v \rangle$ for all $u, v \in \mathcal{F}$. Since each $g_n$ is $K$-invariant, $\pi(g_n)$ commutes with the action of $K$ on $\mathcal{F}$. Since the representations of $K$ on the various $P_a$'s are inequivalent, $\pi(g_n)$ preserves each $P_a$. Thus, by
Schur's Lemma, \( \pi(g_a) \) is constant on each \( P_a \). It follows that \( \pi(D) \) also is a constant multiple of the identity on each \( P_a \), say \( \lambda_a I_{P_a} \). Then

\[
D\phi_{\pi,a}(x) = D_x \left( \int_K \langle \pi(k \cdot x)v_x, v_x \rangle \, dk \right)
= \int_K \langle \pi(k \cdot x)\pi(D)v_x, v_x \rangle \, dk
= \lambda_a \phi_{\pi,a}(x),
\]

and the result follows.

4. **K-Invariant Polynomials and K-Spherical Functions**

We now fix a Gelfand pair \((K, H_a)\) (with \( H_a = V \times \mathbb{R} \)) and decompose the space \( \mathcal{P}(V) = \bigoplus P_a \) into \( K \)-irreducible subspaces. For a given \( P_a \) with orthonormal basis \( \{v_1, ..., v_l\} \), we have

(i) The \( K \)-invariant polynomial \( p_a = \sum_{j=1}^l v_j \bar{v}_j \).

(ii) The \( K \)-spherical function \( \phi_a(x) = \sum_{j=1}^l \langle \pi(x)v_j, v_j \rangle \).

**Proposition 4.2 (Orthogonalization Procedure).** \( \phi_a(z,t) = e^{it}q_a(z) \) \( e^{-(1/4)|z|^2} \), where \( q_a \) is a polynomial with homogeneous component of highest degree \( (-1)^m p_a \) where \( m = (1/2) \deg(p_a) \). More specifically, the polynomials \( \{q_a\} \) are obtained, up to scalar multiples, by applying Gram–Schmidt orthogonalization with respect to the measure \( d\tilde{z} = e^{-(1/2)|z|^2} \, dz \, d\tilde{z} \) to the polynomials \( \{p_a\} \) and normalizing so that \( q_a(0) = 1 \). The sequence \( \{p_a\} \) may be ordered in any way that ensures \( \deg(p_a) < \deg(p_\beta) \).

**Proof.** First note that since \( \mathcal{P}_m(V) \), the homogeneous polynomials of degree \( m \) on \( V \), is invariant under \( U(n) \) and \( K \subset U(n) \), \( P_a \subset \mathcal{P}_m(V) \) for some \( m \). Then

\[
\langle \pi(z, t)v, v \rangle = \int [\pi(z, t)v](w) \bar{v}(w) \, d\tilde{w}
= \int e^{it - (1/2)\langle w, z \rangle - (1/4)|z|^2} v(w + z) \bar{v}(w) \, d\tilde{w}
= e^{it}q(z) e^{-(1/4)|z|^2},
\]
where

\[ q(z) = \int e^{-(1/2) \langle w, z \rangle} v(w + z) \bar{v}(w) \, dw. \]

We use two expansions:

\[ e^{-(1/2) \langle w, z \rangle} = \sum_{r=0}^{\infty} \frac{1}{r!} \left( - \frac{1}{2} \langle w, z \rangle \right)^r, \]

and the Taylor's series

\[ v(w + z) = v(w) + \cdots + v(z). \]

Thus,

\[ q(z) = \sum_{r=0}^{\infty} \frac{1}{r!} \left( - \frac{1}{2} \right)^r \int (w \cdot \bar{z})^r (v(w) + \cdots + v(z)) \bar{v}(w) \, dw. \]

Note that distinct monomials in \( P(V) \) are orthogonal with respect to \( dw \), so that in the integrand only terms of degree \( m \) in \( w \) will give a non-zero integral against \( \bar{v}(w) \). (In particular, the integral will be zero for \( r > m \).) Thus \( q \) is a polynomial in \( z \) and \( \bar{z} \). The highest order term of \( q \) is obtained from

\[ \frac{1}{m!} \left( - \frac{1}{2} \right)^m \int (w \cdot \bar{z})^m v(z) \bar{v}(w) \, dw. \]

Since for \( r \neq m \)

\[ \int (w \cdot \bar{z})^r \bar{v}(w) \, dw = 0, \]

we have

\[ \frac{1}{m!} \left( - \frac{1}{2} \right)^m \int (w \cdot \bar{z})^m v(w) \, dw = (-1)^m \sum_{r=0}^{\infty} \frac{1}{r! 2^r} \int (w \cdot \bar{z})^r \bar{v}(w) \, dw \]

\[ = (-1)^m \int e^{(1/2)w \cdot \bar{z}} \bar{v}(w) \, dw \]

\[ = (-1)^m \bar{n}(z), \]

since \( e^{(1/2)w \cdot \bar{z}} \) is the reproducing kernel in Fock space \([Ba]\). Thus,

\[ q(z) = (-1)^m v(z) \bar{v}(z) + \text{lower order terms}. \]
Applying this result to $\phi_{z}$, we obtain

$$\phi_{z}(z, t) = \frac{1}{l} \sum_{j=1}^{l} \langle \pi(z, t) v_{z}, v_{z} \rangle \quad = e^{itq_{z}(z)} e^{-(1/4)|z|^2},$$

where

$$q_{z}(z) = \frac{(-1)^m}{l} \sum_{j=1}^{l} v_{j}(z) \overline{\tilde{v}_{j}(z)} + \text{lower order terms.}$$

To complete the determination of the polynomials $\{q_{z}\}$ we next note that $\pi$ is a square integrable representation modulo the centre of $H_{n}$, so that for $v_{z} \in \mathcal{P}_{z}$ and $v_{z'} \in \mathcal{P}_{z'}$,

$$\int \frac{(n(z, 0) v_{z}) (n(z, 0) v_{z'})}{(n(z, 0) v_{z})} \frac{dz}{d\bar{z}} = \text{Const} \left\| \langle v_{z}, v_{z'} \rangle \right\|^2 = 0,$$

if $z \neq z'$ (cf. [Ki]). Thus if $z \neq z'$,

$$\int q_{z}(z) q_{z'}(z) e^{-(1/2)|z|^2} \frac{dz}{d\bar{z}}$$

$$= \int \phi_{z}(z, 0) \overline{\phi_{z'}(z, 0)} \frac{dz}{d\bar{z}}$$

$$= \int \left( \int_{k} \frac{\langle \pi(k \cdot z, 0) v_{z}, v_{z} \rangle}{\langle \pi(k \cdot z, 0) v_{z}, v_{z} \rangle} \frac{dk}{d\bar{k}} \right) \frac{dz}{d\bar{z}}$$

$$= \int \int_{k} \int_{k'} \left( \int \frac{\langle \pi(z, 0) k \cdot v_{z}, k' \cdot v_{z} \rangle}{\langle \pi(z, 0) k \cdot v_{z}, k' \cdot v_{z} \rangle} \frac{dk}{d\bar{k}} \right) \frac{dz}{d\bar{z}}$$

$$= \text{Const} \int_{k} \int_{k'} \left| \langle k \cdot v_{z}, k' \cdot v_{z} \rangle \right|^2 \frac{dk}{d\bar{k}}$$

$$= 0.$$

To ease the notation, take for the discrete index set $\{z\}$ the numbers $\{0, 1, 2, 3, \ldots\}$ and suppose that the various $P_j$'s have been ordered so that $i < j \Rightarrow \deg(P_i) \leq \deg(P_j)$. So $P_0 = \mathcal{P}_0(V); \ P_1 + \cdots + P_{l_1} = \mathcal{P}_1(V); \ P_{l_1+1} + \cdots + P_{l_2} = \mathcal{P}_2(V)$; etc.

Up to normalization constants, there is a unique sequence of non-zero, pair-wise orthogonal polynomials $r_0, r_1, r_2, \ldots$ with the property that

$$\text{Span}(r_0, \ldots, r_d) = \text{Span}(p_0, \ldots, p_d) \quad \text{for all } d. \quad (4.3)$$
This sequence can be obtained from \( \{ p_0, p_1, \ldots \} \) by applying Gram-Schmidt (using the measure \( d\overline{z} \)).

In view of the orthogonality established above, we need only show that Condition (4.3) holds with \( r_0, \ldots, r_d \) replaced by \( q_0, \ldots, q_d \). This is clear for \( d = 0 \) since \( q_0(z) \) and \( p_0(z) \) are both constants. Assume inductively that (4.3) holds. Then

\[
\text{Span}(q_0, \ldots, q_{d+1}) = \text{Span}(p_0, \ldots, p_d, q_{d+1})
\]

\[
= \text{Span}(p_0, \ldots, p_d, p_{d+1} + l_{d+1}),
\]

where \( l_{d+1}(z) \) denotes the lower order terms in \((-1)^{\deg(p_{d+1})}q_{d+1}\). As \( q_{d+1} \) and \( p_{d+1} \) both belong to \( \mathcal{P}(V_R)^k \), so does their difference \( l_{d+1} \). Thus, by (3.9) \( l_{d+1} \in \text{Span}(p_j; \deg p_j < \deg p_{d+1}) = \text{Span}(p_0, \ldots, p_d) \) in view of our ordering on the \( P_j \)'s. Therefore we have \( \text{Span}(q_0, \ldots, q_{d+1}) = \text{Span}(p_0, \ldots, p_{d+1}) \).

**Remark.** The polynomials \( q_\lambda \) are normalized so that \( q_\lambda(0) = 1 \). It follows in particular that since each polynomial \( p_\lambda \), as well as the measure \( d\overline{z} \), is real, each \( q_\lambda \) is real. Combining this observation with Eqs. (2.10) and (2.11) shows that for \( \lambda \neq 0 \),

\[
\phi_\lambda(z, t) = e^{i\lambda t} e^{-(1/4)i|\lambda| |t|^2} q_\lambda(|\lambda|^{1/2}z).
\]

Proposition 4.2 shows that the polynomials \( q_\lambda \), which determine the spherical functions \( \phi_\lambda \), are obtained from the invariants \( p_\lambda \) by applying the Gram-Schmidt algorithm. This involves computing certain integrals. We now present an alternative method which requires only the taking of derivatives!

Given \( p(z, \overline{z}) = p(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) \in C[z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n] \), let \( D_p \) denote the constant coefficient differential operator

\[
D_p := p\left(2 \frac{\partial}{\partial \overline{z}}, -2 \frac{\partial}{\partial z}\right).
\]

That is, replace each occurrence of \( z_j \) in \( p \) by \( 2(\partial/\partial \overline{z}_j) = \partial/\partial x_j + i(\partial/\partial y_j) \) and each occurrence of \( \overline{z}_j \) by \( -2(\partial/\partial z_j) = -\partial/\partial x_j + i(\partial/\partial y_j) \).

The map \( p \mapsto D_p \) can be given a coordinate free description. Let \( (V, \langle \cdot, \cdot \rangle) \) be a finite dimensional Hermitian vector space. Then \( \omega(\cdot, \cdot):= \text{Im}\langle \cdot, \cdot \rangle \) is a symplectic structure on the underlying real vector space \( V_R \). The pairing \( \omega \) sets up an isomorphism \( V_R \cong V_R^* \) and we obtain an induced isomorphism

\[
\Omega: \mathcal{P}(V_R) \rightarrow \mathcal{P}(V_R^*).
\]
Lemma 4.6. For $p \in \mathcal{P}(V_\mathbb{R})$ homogeneous of degree $m$, $i^m D_p = (-1)^m \lambda_V(\Omega(p))$ where $\lambda_V$ denotes the symmetrization map (3.7) for the abelian group $V_\mathbb{R}$ (see (3.7)).

Proof. Let $\mathcal{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ be a basis for $V_\mathbb{R}$ which is both $\omega$-symplectic (i.e., $\omega(X_i, Y_j) = \delta_{ij}$, etc.) and orthonormal with respect to the real inner product $(\cdot, \cdot) = \Re(\cdot, \cdot)$. Setting $Z_j = X_j + iY_j$, $\{Z_1, \ldots, Z_n\}$ is a $(\cdot, \cdot)$-orthonormal basis for $V$. Let $\mathcal{B}' = \{X_1', \ldots, X_n', Y_1', \ldots, Y_n'\}$ be the dual basis for $V_\mathbb{R}^*$ and let $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $(x', y') = (x_1', \ldots, x_n', y_1', \ldots, y_n')$ be coordinates on $V_\mathbb{R}$ and $V_\mathbb{R}^*$ with respect to $\mathcal{B}$ and $\mathcal{B}'$.

The isomorphism $V_\mathbb{R} \cong V_\mathbb{R}^*$ given by $\omega$ is $v \mapsto v^* := \omega(v, \cdot)$. We have $X_j^* = Y_j'$ and $Y_j^* = -X_j'$. Regarding the coordinate functions $x_j, y_j$ and $x'_j, y'_j$ as polynomials on $V_\mathbb{R}$ and $V_\mathbb{R}^*$ one sees that $\Omega(x_j) = -y_j'$ and $\Omega(y_j) = x_j'$. Indeed,
\[
\Omega(x_j)(x', y') = \Omega(x_j)(x'_j X_j' + \cdots + x'_n X_n' + y'_j Y_j' + \cdots + y'_n Y_n') = \Omega(x_j)(-y'_j x_j' + \cdots + y'_n X_n' + x'_1 Y_1' + \cdots + x'_n Y_n')
\]
and the second identity is similar.

The abelian symmetrization map $\lambda_V: \mathcal{P}(V_\mathbb{R}^*) \to \mathcal{U}(V_\mathbb{R})$ is multiplicative and characterized by $\lambda_V(x_j) = \partial/\partial x_j$ and $\lambda_V(y_j) = \partial/\partial y_j$ on the generators. Thus,
\[
(\lambda_V \circ \Omega)(x_j) = -\frac{\partial}{\partial y_j} \quad \text{and} \quad (\lambda_V \circ \Omega)(y_j) = \frac{\partial}{\partial x_j}.
\]

Now let $z_j = x_j + iy_j$ so that $(z_1, \ldots, z_n)$ are coordinates for $V$ with respect to $\{Z_1, \ldots, Z_n\}$. We have
\[
(\lambda_V \circ \Omega)(z_j) = -\frac{\partial}{\partial y_j} + i \frac{\partial}{\partial x_j} = i \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right) = 2i \frac{\partial}{\partial z_j}
\]
and similarly
\[
(\lambda_V \circ \Omega)(z_j) = -2i \frac{\partial}{\partial z_j}.
\]
Since $\Omega$ and $\lambda_V$ are algebra maps ($V_\mathbb{R}$ is abelian), we conclude that

$$\lambda_V : \Omega)(p(z, \bar{z})) = p\left(2i \frac{\partial}{\partial z}, -2i \frac{\partial}{\partial z}\right)$$

$$= i^m p\left(2 \frac{\partial}{\partial z}, -2 \frac{\partial}{\partial z}\right)$$

when $p$ is homogeneous of degree $m$ in $(z, \bar{z})$.

Since $\lambda_V$ is $K$-equivariant and the action of $K$ on $V$ preserves $\omega$, Lemma 4.6 shows that $D_p$ is $K$-invariant for $p \in \mathcal{P}(V_\mathbb{R})^K$. In particular, the $D_p$'s are $K$-invariant differential operators on $V$.

With notation as in Proposition 4.2, one has

**Proposition 4.7 (Rodrigues’ Formula).**

$$D_p(e^{-(1/2)|z|^2}) = q_\omega(z, \bar{z}) e^{-(1/2)|z|^2}$$

and hence

$$\phi_\omega(z, t) = e^{it} D_p(e^{-(1/2)|z|^2}) e^{(1/4)|z|^2}.$$

Proof. We will identify $H_n$ with its Lie algebra $\mathfrak{h}_n$ via the exponential map. The Lie bracket is given by $[(z, t), (z', t')] = (0, \omega(z, z'))$. The left and right derivations given by $A \in \mathfrak{h}_n$ are

$$(L_A f)(z, t) := \frac{d}{ds}igg|_{s=0} f((-sA)(z, t))$$

and

$$(R_A f)(z, t) := \frac{d}{ds}igg|_{s=0} f((z, t)(sA)).$$

($R_A$ is the left invariant vector field given by $A$.) Let $\{e_1, \ldots, e_n\}$ be the usual basis for $\mathbb{C}^n$ and $X_j := (e_j, 0)$, $Y_j := (ie_j, 0)$, $Z_j := X_j + iY_j$, and $\bar{Z}_j := X_j - iY_j$. We compute

$$L_{Z_j} = -2 \frac{\partial}{\partial z_j} + \frac{iz_j}{2} \frac{\partial}{\partial t}$$

$$L_{\bar{Z}_j} = -2 \frac{\partial}{\partial \bar{z}_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t}$$

$$R_{Z_j} = 2 \frac{\partial}{\partial z_j} + \frac{iz_j}{2} \frac{\partial}{\partial t}$$

$$R_{\bar{Z}_j} = 2 \frac{\partial}{\partial \bar{z}_j} - \frac{i\bar{z}_j}{2} \frac{\partial}{\partial t}. \quad (4.8)$$
The Fourier transforms in the \( t \)-direction at 1 are given by replacing \( \partial/\partial t \) by \( i \) in these equations.

\[
\begin{align*}
L_{z_j} = -2 \frac{\partial}{\partial z_j} - \frac{z_j}{2} \\
L_{\bar{z}_j} = -2 \frac{\partial}{\partial \bar{z}_j} + \frac{\bar{z}_j}{2} \\
R_{z_j} = 2 \frac{\partial}{\partial z_j} \frac{z_j}{2} \\
R_{\bar{z}_j} = 2 \frac{\partial}{\partial \bar{z}_j} \frac{\bar{z}_j}{2}.
\end{align*}
\] (4.9)

Note that, for example, \( L_z(e^{it}\psi(z)) = e^{it}(L_{z_j}\psi)(z) \).

Let \( M: \mathcal{C}(\mathbb{C}^n) \rightarrow \mathcal{C}(\mathbb{C}^n) \) be given by \((Mf)(z) := e^{-|z|^2/A}f(z)\). One has,

\[
\begin{align*}
L_{z_j} = -2M^{-1} \frac{\partial}{\partial z_j} M \\
R_{z_j} = 2M^{-1} \frac{\partial}{\partial \bar{z}_j} M.
\end{align*}
\] (4.10)

We also obtain differential operators from the derived representation of \( \pi \) on Fock space \( \mathcal{F} \). For a smooth vector \( f \in \mathcal{F} \), one has

\[
(\pi(A)f)(w) = \frac{d}{ds} \bigg|_{s=0} (\pi(sA)f)(w).
\]

Using (1.4), one computes that

\[
(\pi(Z_j)f)(w) = -w_j f(w) \quad \text{and} \quad (\pi(\bar{Z}_j)f)(w) = 2 \frac{\partial f}{\partial w_j}.
\] (4.11)

Extending \( \pi \) to \( \mathcal{U}(\mathfrak{h}_n) \), we see that for a homogeneous polynomial \( v(z_1, \ldots, z_n) \) of degree \( m \),

\[
\pi(v(Z))1 := \pi(v(Z_1, \ldots, Z_n))1 = (-1)^m v(w_1, \ldots, v_n) = (-1)^m v(w). \] (4.12)

Moreover

\[
\begin{align*}
\pi(Z_j) &= \pi(X_j) - i\pi(Y_j) \\
&= -\pi(X_j) + i\pi(Y_j) \\
&= \pi(-\bar{Z}_j) = -\pi(\bar{Z}_j)
\end{align*}
\] (4.13)
so that

$$\pi(v(Z)) = \pi(\bar{v}(-Z)) = (-1)^m \pi(\bar{v}(\bar{Z})),$$

(4.14)

where $\bar{v}$ is the polynomial whose coefficients are the complex conjugates of those in $v$.

Let $\{v_1, \ldots, v_l\}$ be an orthonormal basis for a $K$-irreducible subspace $P_a \subset \mathcal{P}_m(C^n)$. We have

$$\phi_a(z, t) = e^{it} e^{-\frac{(1/4)|z|^2}} q_a(z)$$

$$= \frac{1}{l} \sum_{j=1}^{l} \langle \pi(z, t)v_j, v_j \rangle_{\mathcal{F}}$$

$$= \frac{1}{l} \sum_{j=1}^{l} \langle \pi(z, t)\pi(v_j(Z)), 1, \pi(v_j(Z))1 \rangle_{\mathcal{F}}$$

(4.12)

$$= \frac{1}{l} \sum_{j=1}^{l} \langle \pi(v_j(Z))^*\pi(z, t)\pi(v_j(Z)), 1, 1 \rangle_{\mathcal{F}}$$

(4.14)

$$= \frac{1}{l} \sum_{j=1}^{l} \langle \pi(\bar{v}_j(-\bar{Z}))\pi(z, t)\pi(v_j(Z)), 1, 1 \rangle_{\mathcal{F}}$$

by (4.14)

$$= \frac{1}{l} \sum_{j=1}^{l} v_j(R_Z) \bar{v}_j(L_Z)(\langle \pi(z, t), 1 \rangle_{\mathcal{F}}).$$

Here $\langle \pi(z, t), 1 \rangle_{\mathcal{F}} = e^{it} e^{-\frac{(1/4)|z|^2}$, so

$$e^{it} e^{-\frac{(1/4)|z|^2}} q_a(z)$$

$$= \frac{e^{it}}{l} \sum_{j=1}^{l} (v_j(R_Z) \bar{v}_j(L_Z))(e^{-\frac{(1/4)|z|^2})$$

$$= \frac{e^{it}}{l} \sum_{j=1}^{l} \left( v_j \left( 2M^{-1} \frac{\partial}{\partial z_j} M \right) \bar{v}_j \left( -2M^{-1} \frac{\partial}{\partial z_j} M \right) \right) (e^{-\frac{(1/4)|z|^2})$$

by (4.10).

Hence,

$$q_a(z) e^{-\frac{(1/4)|z|^2} = M^{-1} \left( \frac{1}{l} \sum_{j=1}^{l} v_j \left( 2\frac{\partial}{\partial z_j} \right) \bar{v}_j \left( -2\frac{\partial}{\partial z_j} \right) \right) (Me^{-\frac{(1/4)|z|^2}})$$

so that,

$$q_a(z) e^{-\frac{(1/2)|z|^2} = \left( \frac{1}{l} \sum_{j=1}^{l} v_j \left( 2\frac{\partial}{\partial z_j} \right) \bar{v}_j \left( -2\frac{\partial}{\partial z_j} \right) \right) (e^{-\frac{(1/2)|z|^2}) = D_{\rho_z}(e^{-\frac{(1/2)|z|^2})$$

as claimed.
Proposition 4.7 can be reformulated using the symplectic Fourier transform on $V \simeq \mathbb{C}^n$ defined by

$$\hat{f}^{\omega}(\zeta) = \int f(z) e^{-i\text{Im}\langle z, \zeta \rangle} \, dz$$

$$= \int f(x, y) e^{-i(x\eta - y\xi)} \, dx \, dy$$

(4.15)

for $\zeta = \xi + i\eta$, $z = x + iy$. One has that

$$z_j \hat{f}^{\omega} = 2 \frac{\partial}{\partial \zeta_j} \hat{f}^{\omega}$$

$$\bar{z}_j \hat{f}^{\omega} = -2 \frac{\partial}{\partial \zeta_j} \hat{f}^{\omega}$$

(4.16)

and that $e^{-(1/2)|z|^2}$ is fixed by $\hat{f}^{\omega}$.

**Corollary 4.17.** $(p_\omega(z, \bar{z}) e^{-(1/2)|z|^2})^{\wedge \omega} = q_\omega(\zeta, \bar{\zeta}) e^{-(1/2)|\zeta|^2}$.

5. The Value Space and Polar Representations

Suppose that $(K, H, \nu)$ is a Gelfand pair with associated fundamental invariants $\gamma_1, \ldots, \gamma_d$ as introduced in Section 3. By construction each $\gamma_i$ (and indeed each $p_\omega$) is a homogeneous polynomial on $V_R$ that takes on only non-negative real values. Thus the range of $\gamma_i$ is $\mathbb{R}^+$. Let $\gamma := \gamma_1 \times \cdots \times \gamma_d$ and

$$\Gamma = \gamma(V) \subset (\mathbb{R}^+)^d.$$

(5.1)

Any polynomial $q \in \mathcal{P}(V_R)^K$ will factor through the projection $\gamma: V \to \Gamma$, and we obtain a polynomial $\tilde{q}$ on $\Gamma$ defined by

$$\tilde{q} \circ \gamma = q.$$

(5.2)

We may also transplant $K$-invariant differential operators $L$ on $V$ to obtain operators $\tilde{L}$ on $\Gamma$ by the rule

$$\gamma^*(\tilde{L}) = L.$$

(5.3)

We see that the functions $p_\omega(z)$, $q_\omega(z)$, and $\psi_\omega(z)$ correspond to functions $\tilde{p}_\omega$, $\tilde{q}_\omega$, and $\tilde{\psi}_\omega$ on $\Gamma$. This provides a mechanism for reducing the analysis
of $K$-spherical functions to "radial directions." In particular, the $\tilde{\psi}_s$'s are precisely the bounded simultaneous eigenfunctions for $L_1^1, \ldots, L_d^1$ which satisfy $\tilde{\psi}_s(0) = 1$ (where $L_j^1$ is given by (3.16)).

Thus, $K$-spherical functions on $H_n$ can be obtained from solutions to eigenvalue problems on $\Gamma$. The orthogonalization procedure (Proposition 4.2) and Rodrigues' formula (Proposition 4.7) both descend to $\Gamma$. In principle, these results provide algorithms for identifying solutions.

**Theorem 5.4.** Let $\nu$ be the measure on $\Gamma$ defined by $\gamma^*(\nu) = e^{-|z|^2/2} \, dz \, d\bar{z}$. The polynomials $\tilde{q}_s$ on $\Gamma$ are obtained by applying the Gram-Schmidt algorithm using $dv$ to the $\tilde{p}_s$'s (ordered as before) and normalizing so that $\tilde{q}_s(0) = 1$.

The Rodrigues' formula can be recast on $\Gamma$ most clearly when $K$ acts irreducibly on $V$. Recall that in this case, the first fundamental invariant is $\gamma_1(z) = |z|^2/2n$. Also note that the operators $D_{p_s}$ are $K$-invariant.

**Theorem 5.5.** Let $K$ act irreducibly on $V$. The polynomials $\tilde{q}_s$ on $\Gamma$ are given by $\tilde{B}_{p_s}(e^{-nm}) = \tilde{q}_s(y) \, e^{-nm}$.

**Polar Representations**

There is an alternative and more geometric radial reduction which can sometimes be performed. Consider the Hermitian vector space $(V, \langle \cdot, \cdot \rangle)$ with $K \subset U(V)$. A point $v_0 \in V$ is called regular if the $K$-orbit $K \cdot v_0$ has maximal dimension. The real part $(\cdot, \cdot)$ of $\langle \cdot, \cdot \rangle$ is an inner product on the underlying real vector space $V_R$. We define

$$\mathcal{A}_{v_0} := (k \cdot v_0)^\perp = \{u \in V \mid \langle u, X \cdot v_0 \rangle = 0 \text{ for all } X \in k \}.$$  \hspace{1cm} (5.6)

$\mathcal{A} = \mathcal{A}_{v_0}$ meets every $K$-orbit in $V$ (possibly more than once) [Da]. The representation $V$ of $K$ is polar if

$$(k \cdot u, \mathcal{A}) = 0 \quad \text{for all } u \in \mathcal{A}.$$  \hspace{1cm} (5.7)

That is, each $K$-orbit meets the cross section $\mathcal{A}$ orthogonally at each point of intersection.

The representation $V$ of $K$ may or may not be polar. For example, direct computations show that the first ten entries in Table 1.8 are polar but that the next six are not. For the remainder of this section, we assume that the representation $V$ of $K$ is polar.

Polar representations of compact groups are studied in [Da]. It is shown there that
LEMMA 5.8. $V$ is a polar representation of $K$ if and only if the translated tangent spaces $k \cdot v_0$ and $k \cdot v_1$ to the $K$-orbits through any two regular points $v_0, v_1$ differ by the action of $K$.

It follows that the cross section $\mathcal{A}$ is unique modulo the action of $K$ on $V$.

As a $K$-orbit is compact and orthogonal to $\mathcal{A}$, it meets $\mathcal{A}$ in a finite set of points. The Weyl group for $\mathcal{A}$ is defined as

$$ W(\mathcal{A}) := N(\mathcal{A})/Z(\mathcal{A}), $$

where $N(\mathcal{A}) = \{ k \in K \mid k \cdot \mathcal{A} = \mathcal{A} \}$ (the normalizer of $\mathcal{A}$) and $Z(\mathcal{A}) = \{ k \in K \mid k|_{\mathcal{A}} = id \}$ (the centralizer of $\mathcal{A}$). It is shown in [Da] that

LEMMA 5.10. $W := W(\mathcal{A})$ is a finite group with $(K \cdot u) \cap \mathcal{A} = W \cdot u$ for all $u \in \mathcal{A}$.

Thus we obtain correspondences

$$ \mathcal{A}/W \simeq V/K \quad \text{and} \quad C^\infty(V)^K \simeq C^\infty(\mathcal{A})^W $$

(via the inclusion $\mathcal{A} \subset V$ and restriction).

We see that the $p_s(z)$'s and the polynomial components $q_s(z)$ of the spherical functions can be viewed as $W$-invariant polynomials on $\mathcal{A}$ via restriction.

The radial part $D_{\mathcal{A}}$ of a $K$-invariant differential operator $D$ on $V$ is characterized by

$$ D_{\mathcal{A}}(f|_{\mathcal{A}}) = (Df)|_{\mathcal{A}} \quad \text{for all} \quad f \in C^\infty(V)^K. $$

(See [He].) As an immediate consequence of Theorem 3.16, the functions $\psi_s(a) = e^{-|a|^{1/2}}q_s(a)$ on $\mathcal{A}$ are precisely the bounded $W$-invariant simultaneous eigenfunctions for $(L_1^1)^\mathcal{A}, \ldots, (L_d^1)^\mathcal{A}$ that satisfy $\psi_s(0) = 1$. Similarly, the $K$-spherical functions $\eta_\omega(a)$ on $\mathcal{A}$ are the bounded $W$-invariant simultaneous eigenfunctions for $(L_1^0)^\mathcal{A}, \ldots, (L_d^0)^\mathcal{A}$ satisfying $\eta_\omega(0) = 1$.

The orthogonalization procedure can be formulated explicitly in this setting. For $p \in \mathcal{P}(V_K)^K$,

$$ \int_\mathcal{A} p(z) e^{-|z|^{1/2}} dz d\bar{z} = \frac{1}{|W|} \int_\mathcal{A} p(a) e^{-|a|^{1/2}} J_K(a) da, $$

where $J_K(a) := \text{Vol}(K \cdot a)$. Propositions 4.2 and 4.7 yield the following two theorems.
Theorem 5.14. The polynomials $q_a \in \mathcal{P}(A)^W$ are obtained by applying the Gram-Schmidt algorithm to the polynomials $p_a |_{A} \in \mathcal{P}(A)^W$ using the measure

$$d\mu(a) = \frac{1}{|W|} e^{-|a|^2/2} J_K(a) \, da$$

and normalizing so that $q_a(0) = 1$. The $p_a$'s are ordered as in Proposition 4.2.

Theorem 5.15. $(D, e)(e^{-|a|^2/2}) = q_\alpha(a) e^{-|a|^2/2}$.

Remarks. 1. In practice, the operators $L^n_j$ on the value space $\Gamma$ can be computed directly, using the chain rule for $\gamma: V \to \Gamma$. In the first ten cases in Table 1.8, where $V$ is a polar representation of $K$, $\gamma|_{A}$ is a local diffeomorphism, so the operators $(L^n_j)|_{\mathcal{A}}$ are pull-backs of the $L^n_j$'s via $\gamma$.

2. As the spherical functions are normalized by their values at 0, one can ignore any constant factor in $J_K(a) = \text{Vol}(K, a)$. Again, if $\gamma: A \to \Gamma$ is a local diffeomorphism, the measure $d\mu$ can be pushed down to $\Gamma$.

In the next section, we provide explicit computations for the groups $K = T(n)$ and $K = U(n)$ on $H_n$ to illustrate these results. Further examples from Table 1.8 will appear elsewhere.

6. The Gelfand Pairs $(T(n), H_n)$ and $(U(n), H_n)$

We begin with the pair $(T(n), H_n)$, where $T(n)$ is the $n$-torus,

$$\{ e^{i\theta} = (e^{i\theta_1}, ..., e^{i\theta_n}) : \theta = (\theta_1, ..., \theta_n) \in \mathbb{R}^n \}.$$  

It is an easy consequence of (1.7) that $(T(n), H_n)$ is a Gelfand pair. One also has a straightforward proof, found in [HR, FS], of this fact: Given any $T(n)$-invariant function $f$ on $H_n$, and any representation $\pi_\lambda$ of $H_n$, we see that $\pi_\lambda(f)$ commutes with the action of $T(n)$ on Fock space $\mathcal{F}_\lambda$. Since the eigenvalues for the action of $T(n)$ are distinct on each monomial in $\mathcal{F}_\lambda$, $\pi_\lambda(f)$ must be diagonalized by the monomials. Thus $\pi_\lambda(f)$ and $\pi_\lambda(g)$ commute for all $f, g \in L^2_{T(n)}(H_n)$, and all $0 \neq \lambda \in \mathbb{R}$. By the Plancherel theorem, one concludes that $f \ast h = g \ast f$.

Let $w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n}$ for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$. Then $\mathcal{F}$ decomposes into the $T(n)$-invariant subspaces $Cw^\alpha$. The corresponding invariant polynomials are $p_\alpha = |z|^{2\alpha}/2^{|\alpha|}$. The spherical functions may be found by direct computation as in [HR], or by using Theorem (4.2). One has
**Proposition 6.1.** The bounded spherical functions for the Gelfand pair \((T(n), H_n)\) associated to \(\pi\) are

\[
\phi(z, t) = e^{it} e^{-\frac{1}{4}|z|^2} \prod_{j=1}^{n} L_{j/2}(\frac{1}{2}|z|^2),
\]

where

\[
I_k(x) = \sum_{j=0}^{k} \binom{k}{j} \left( \frac{-x}{j!} \right)^j
\]

is the \(k\)th Laguerre polynomial (of order 0).

We now turn to the pair \((U(n), H_n)\). The irreducible subspaces in \(\mathcal{F}\) are the full subspaces \(\mathcal{P}_m(\mathbb{C}^n)\), with orthonormal basis \(\{w^m / \sqrt{2^m m!} : |m| = m\}\).

**Proposition 6.2.** The invariant polynomials for the pair \((U(n), H_n)\) are spanned by

\[
\left\{ p_m(w) = \frac{(n-1)!}{2^m(m+n-1)!} |\xi|^2^n : m \in \mathbb{Z}^+ \right\}.
\]

The corresponding bounded spherical functions associated to \(\pi\) are

\[
\phi_m(z, t) = e^{it} L_m^{(n-1)}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2},
\]

where \(L_m^{(n-1)}\) is the generalized Laguerre polynomial of order \(n-1\) (normalized to be 1 at 0), namely

\[
L_m^{(n-1)}(x) = (n-1)! \sum_{j=0}^{m} \binom{m}{j} \left( \frac{-x}{j+n-1} \right)^j.
\]

**Proof.** First note that

\[
p_m(z) = \frac{m! (n-1)!}{(m+n-1)!} \sum_{|\alpha| = m} \frac{1}{2^m \alpha!} z^{\alpha_1} \ldots z^{\alpha_n}
\]

\[
= \frac{(n-1)!}{2^m(m+n-1)!} \sum_{\alpha_1 + \ldots + \alpha_n = m} \frac{m!}{\alpha_1! \ldots \alpha_n!} |z_1|^{2\alpha_1} \ldots |z_n|^{2\alpha_n}
\]

\[
= \frac{(n-1)!}{2^m(m+n-1)!} |z|^{2m}.
\]

Now we need to obtain an orthonormal sequence from the \(p_m\)'s with respect to the measure \(dz \tilde{d}z = e^{-\frac{1}{4}|z|^2} dz d\tilde{z}\). If \(f\) is a function of \(|z|^2\), then

\[
\int f(z) e^{-\frac{1}{4}|z|^2} dz d\tilde{z} = c_n \int f(r^2) e^{-\frac{1}{2}r^2} r^{2m-1} dr,
\]

(6.3)
where \( c_n \) is the volume of the sphere \( S^{2n-1} \). Substituting \( s = r^2/2 \), (6.3) becomes

\[
\int f(2s) e^{-s} s^{n-1} ds.
\]

Let \( \| \cdot \| \) denote the \( L^2 \)-norm on \( C^n \) with respect to the measure \( d\bar{z} \), and let \( \| \cdot \|_1 \) denote the \( L^2 \)-norm on \([0, \infty)\) with respect to the measure \( d\bar{s} := e^{-s} s^{n-1} ds \). For \( f \in L^2([0, \infty), d\bar{s}) \) define \( \tilde{f} \) on \( C^n \) by \( \tilde{f}(z) = f(|z|^2/2) \).

Let \( r \) be the distance from \( z \) to \( 0 \), and for \( m \in \mathbb{Z}^+ \) set \( e_m(s) = s^m \) for \( s \in [0, \infty) \). Then \( \tilde{e}_m = m! p_m \). The generalised Laguerre polynomials, \( \{L_m^{(n-1)}\} \), are obtained by applying the Gram–Schmidt process to \( \{e_m\} \) with respect to \( d\bar{s} \) (cf., e.g., [Sz]). Thus \( \{L_m^{(n-1)}\} \) gives the desired polynomials, as claimed.

Consider now the bounded \( K \)-spherical functions associated to the one-dimensional representations. For \( K = U(n) \) the distinct \( K \)-orbits are parametrized by real \( \tau \geq 0 \). For \( \tau = 0 \), one has the trivial character and the \( K \)-spherical function \( \phi_0(z, t) \equiv 1 \). For each \( \tau > 0 \), the sphere \( S_\tau \) of radius \( \tau \) in \( C^n \) is a \( K \)-orbit and we obtain from (2.13) the \( K \)-spherical function \( \eta_\tau = \hat{\mu}_{S_\tau} \). The Fourier transform of the unit mass on the \((2n-1)\)-sphere is given in terms of the Bessel function \( J_{n-1} \). (See, e.g., [Ey].) After normalization one has

\[
\eta_\tau(z, t) = \frac{2^{n-1}(n-1)!}{(\tau |z|)^{n-1}} J_{n-1}(\tau |z|),
\]

for each \( \tau > 0 \).

Alternatively, by (3.19), \( 2nL_1^0 = 4A \) where \( A \) is the usual Euclidean Laplace operator. Eigenfunctions of the form \( e^{i\omega \cdot z} \) with \( |\omega| = 1 \) give eigenvalues \(-\tau^2\). Averaging these functions over \( K \)-orbits \( |z| = \text{const} \), we obtain (6.4).

For \( K = T(n) = \prod_{i=1}^n U(1) \), the \( K \)-orbits in \( C^n \) are parametrized by real, non-negative \( n \)-tuples, \( \tau = (\tau_1, \ldots, \tau_n) \). By (2.14) and the above result, the associated \( K \)-spherical functions on \( H_n \) are given by \( \eta_\tau(z, t) = \prod_{i=1}^n J_0(\tau_i |z_i|) \).

Lemma 3.15 shows that the \( U(n) \)-invariant differential operators on \( H_n \) are generated by \( L_1 = \Delta(\gamma_1) \) and \( \partial/\partial t \). Here we consider the more usual

\[
2nL_1 = \frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j).
\]

We let \( L := 2nL_1^j \) where \( L_j^j \) is as in (3.16).
LEMMA 6.6. \( L = 4A - |z|^2/4 \).

Proof. We have

\[
L = \frac{1}{2} \sum_{j=1}^{n} \left\{ \left( 2 \frac{\partial}{\partial z_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \right) \left( 2 \frac{\partial}{\partial \bar{z}_j} - \frac{iz_j}{2} \frac{\partial}{\partial \bar{t}} \right) + \left( 2 \frac{\partial}{\partial \bar{z}_j} - \frac{iz_j}{2} \frac{\partial}{\partial \bar{t}} \right) \left( 2 \frac{\partial}{\partial z_j} + \frac{iz_j}{2} \frac{\partial}{\partial t} \right) \right\}.
\]

As \( \phi_z(z, t) = e^{zt} \psi_x(z) \), when applied to \( \phi_z \),

\[
L = 4 \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} + \frac{1}{2} \sum_{j=1}^{n} \left( \bar{z}_j \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_j} \right) - |z|^2/4.
\]

Note that \( \bar{z}_j (\partial/\partial \bar{z}_j) f(|z|^2) = z_j (\partial/\partial z_j) f(|z|^2) \).

PROPOSITION 6.7. The functions \( \psi_m(z) = L_m^{(n-1)}(|z|^2/2) e^{-(1/4)|z|^2} \) are “radial” eigenfunctions of the operator \( L = 4A - |z|^2/4 \), with eigenvalues \(- (2m + n)\).

Proof. This is a direct consequence of Lemma 3.15 together with an application of Proposition 3.20. Choose \( v_m = z_1^m \) in \( \mathcal{B}_m \). Then

\[
\pi \left( \frac{1}{2} \sum_{j=1}^{n} (Z_j Z_j + \bar{Z}_j Z_j) \right) v_m = \frac{1}{2} \sum_{j=1}^{n} \left( -2z_j \frac{\partial}{\partial z_j} - 2 \frac{\partial}{\partial z_j} z_j \right) z_1^m = -(2m + n) z_1^m.
\]

As the action of \( U(n) \) on \( \mathbb{C}^n \) is polar, we can apply the techniques of Section 5. For \( v_0 = e_1 = (1, 0, \ldots, 0) \),

\[
k v_0 = i \mathbf{Re}_1 + \mathbb{C}^{n-1}, \quad \text{and} \quad \gamma : (k v_0)^+ = \mathbf{Re}_1.
\]

The value space \( \Gamma \) is \( \mathbb{R}^+ \), with \( \gamma : \mathbb{C}^n \to \mathbb{R}^+ \) given by

\[
\gamma(z) = |z|^2/2.
\]

(Actually, \( p_1(z) = |z|^2/2n \), but the analysis is simplified by taking \( \gamma = np_1 \).

Note that \( \gamma : \mathcal{A} \to \Gamma \) is just

\[
\gamma(\alpha e_1) = a^2/2.
\]

The proof of Proposition 6.2 used reduction to the value space via \( \gamma \). The measure \( \mu \) on \( \mathbb{R}^+ \) appearing in Theorem 5.4 is \( \mu = dS = s^{-1} s^{n-1} ds \) in this case and the resulting orthogonal polynomials are \( \tilde{q}_m(s) = L_m^{(n-1)}(s) \).
An alternative proof for Proposition 6.2 uses Proposition 4.7 and Theorem 5.5. One needs to verify that

$$e^{(1/2)|z|^2} D_{pm}(e^{-(1/2)|z|^2}) = L_m^{(n-1)}(1/2|z|^2). \quad (6.11)$$

Using $\gamma$ to reduce to the value space requires making the change of variables $s = |z|^2/2$ in (6.11). An exercise with the chain rule yields

$$D_{pm}(e^{-s}) = \frac{(n-1)!}{(m+n-1)!} (-1)^m \left( s \frac{d^2}{ds^2} + n \frac{d}{ds} \right)^m e^{-s} = L_m^{(n-1)}(s) e^{-s}. \quad (6.12)$$

One can show by a painful induction on $m$ that (6.12) is equivalent to

$$\frac{(n-1)!}{(m+n-1)!} s^m e^{-s} \left( s \frac{d}{ds} \right)^m (s^{m+n-1} e^{-s}) = L_m^{(n-1)}(s), \quad (6.13)$$

which is the classical Rodrigues' formula.

A direct calculation shows:

**Lemma 6.14.**

$$\tilde{L}(q_m(y)) e^{-y/2} = -(2m+n) \tilde{q}_m(y) e^{-y/2},$$

As $\tilde{L}(\tilde{q}_m(y) e^{-y/2}) = -(2m+n) \tilde{q}_m(y) e^{-y/2}$, we can derive the equation

$$\left( 2\gamma \frac{d^2}{d\gamma^2} + (2n-2\gamma) \frac{d}{d\gamma} - n \right) \tilde{q}_m(\gamma) = -(2m+n) \tilde{q}_m(\gamma).$$

That is,

$$\left( \gamma \frac{d^2}{d\gamma^2} + (n-\gamma) \frac{d}{d\gamma} + m \right) \tilde{q}_m(\gamma) = 0 \quad (6.15)$$

which is the standard differential equation for the generalized Laguerre polynomial of order $n-1$ and degree $m$.

As $\mathcal{A} = \mathbb{R}^+ \setminus \{0\}$, $N(\mathcal{A}) = \{\pm 1\} \times U(n-1)$ and $Z(\mathcal{A}) = \{1\} \times U(n-1)$. Thus $W = \{\pm 1\}$, and hence the polynomials $q_m|_{\mathcal{A}}$ are even eigenfunctions.
of $L_\alpha$. If we take the map $a \mapsto a^2$ from $\mathcal{A}$ to $\Gamma$, (6.15) becomes the equation for Hermite polynomials of even order.

Finally, an application of Proposition 2.4 yields some rather nice results on Laguerre polynomials. Part (b) of Theorem 6.16 can be found in [St, Sc].

**Theorem 6.16.** Let $L_m^{(k)}$ be the generalized Laguerre polynomial of degree $m$ and order $k$.

(a) $\sum_{|z|=m} \prod_{j=1}^{m-1} L_{z_j}(u_j) = \begin{pmatrix} m+n-1 \\ m \end{pmatrix} L_m^{(n-1)}(u)$

(b) $u^{-n-1} L_m^{(n-1)}(u) = \int_0^u \cdots \int_0^{u_{n-1}} L_{z_1}(u_1) L_{z_2}(u_2-u_1) \cdots L_{z_{n-1}}(u-u_{n-1}) du_1 \cdots du_{n-1}$, where $L_k := L_k^{(0)}$ and $m = |z|$.

**Proof.** We apply Proposition 2.4 with $K' = T(n)$ and $K = U(n)$. Then $\mathcal{P}_m = \sum_{|z|=m} P_z$, where $P_z = Cw^z$. We have $q_m(z) = L_m^{(n-1)}(|z|^2/2)$ and $q_z(z) = \prod_{j=1}^{n-1} L_{z_j}(|z_j|^2/2)$. By (2.6),

$$\psi_m = \frac{1}{\dim \mathcal{P}_m} \sum_{|z|=m} \dim(P_z) \psi_z,$$

so

$$\begin{pmatrix} m+n-1 \\ m \end{pmatrix} q_m(z) e^{-|z|^2/4} = \sum_{|z|=m} q_z(z) e^{-|z|^2/4},$$

and (a) follows.

By (2.5), for any $\alpha$ with $|\alpha| = m$,

$$\psi_m(z) = \int_{U(n)/T(n)} \psi_z(k \cdot z) \, dk.$$

Thus, for $f \in L_{U(n)}(\mathbb{C}^n)$,

$$\int f(|z|^2) \psi_z(z) \, dz \, d\bar{z}$$

$$= \int f(|z|^2) \prod_{i=1}^{n} L_{x_i}(\frac{1}{2} |z_i|^2) e^{-\frac{1}{4}|z|^2} \, dz \, d\bar{z}$$

$$= a \int f \left( \sum_{i=1}^{n} r_i^2 \right) \prod_{i=1}^{n} L_{x_i}(\frac{1}{2} r_i^2) e^{-\frac{1}{4} \sum r_i^2 r_1 \cdots r_n} \, dr_1 \cdots dr_n$$

$$= b \int f \left( \sum_{i=1}^{n} 2s_i \right) \prod_{i=1}^{n} L_{x_i}(s_i) e^{-\frac{1}{2} \sum s_i} \, ds_1 \cdots ds_n.$$
\[
\begin{aligned}
= b \int_0^\infty \int_0^{u_1} \cdots \int_0^{u_2} f(2u_n) L_{21}(u_1) \\
\times L_{22}(u_2 - u_1) \cdots L_{2n}(u_n - u_{n-1}) e^{-(1/2)u_n} \, du_1 \cdots du_n \\
= \int f(|z|^2) \psi_m(z) \, dz \, d\bar{z} \\
= c \int_0^\infty f(r^2) L_m^{(n-1)}(1/2 r^2) e^{-(1/4) r^2 \, \rho^{2n-1}} \, dr \\
= d \int_0^\infty f(2u) L_m^{(n-1)}(u) e^{-(1/2)u} \, du.
\end{aligned}
\]

Comparing normalizations, one concludes that for \( |x| = m \),

\[
u_n^{m-1} L_m^{(n-1)}(u) = \int_0^{u_n-1} \cdots \int_0^{u_2} L_{21}(u_1) \\
\times L_{22}(u_2 - u_1) \cdots L_{2n}(u - u_{n-1}) \, du_1 \cdots du_{n-1}
\]

as claimed.

**ACKNOWLEDGMENTS**

The authors thank Tony Dooley and Roger Howe for many helpful discussions. Dooley conjectured the existence of a “Rodrigues’ formula” and Howe patiently guided us through some invariant theory. We also thank the referee who suggested many improvements to an earlier version of this paper.

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