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On some numerical characteristics of operators

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Abstract. We investigate some numerical characteristics of Toeplitz operators including the numerical range, maximal numerical range and maximal Berezin set. Further, we establish an inequality for the Berezin number of an arbitrary operator on the Hardy–Hilbert space of the unit disc.

Keywords: Berezin symbol; Berezin number; Maximal numerical range; Maximal Berezin set; Toeplitz operator; Numerical range; Normal operator

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1. INTRODUCTION

In this article we investigate the so-called maximal numerical range in the sense of Stampfli [14] for some Toeplitz operators. We introduce the notion of maximal Berezin set for operators on a reproducing kernel Hilbert space (RKHS) and study some of its properties for the Toeplitz operators on the Hardy space $H^2(\mathbb{D})$. In particular, we focus on the model case of Toeplitz operators on the Hardy-Hilbert space on the unit disc. The Berezin number of an operator is also discussed.

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

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$$\|f\|_2^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < +\infty.$$

The symbol $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach algebra of functions bounded and analytic on the unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$. A function $\theta \in H^\infty$ for which $|\theta(\xi)| = 1$ almost everywhere in the unit circle $\mathbb{T}$ is called an inner function. It is convenient to establish a natural embedding of the space H^2 in the space $L^2 = L^2(\mathbb{T})$ by associating to each function $f \in H^2$ its radial boundary values $(bf)(\xi) := \lim_{r\to 1^-} f(r\xi)$, which exist for m-almost all $\xi \in \mathbb{T}$; where m is the normalized Lebesgue measure on $\mathbb{T}$. Then we have

$$H^2 = \left\{ f \in L^2 : \hat{f}(n) = 0,\ n < 0 \right\},$$

where $\hat{f}(n) := \int_{\mathbb{T}} \bar{\xi}^n f(\xi) dm(\xi)$ is the Fourier coefficient of the function f. For $\varphi \in L^\infty = L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by $T_\varphi f = P_+(\varphi f)$; here $P_+ : L^2(\mathbb{T}) \to H^2$ is the orthogonal projection (Riesz projector).

We shall use repeatedly the easy but useful fact that $T_\varphi^* \hat{k}_\lambda = \overline{\varphi(\lambda)}\hat{k}_\lambda$ for $\varphi \in H^\infty$; here $\hat{k}_\lambda$ is the normalized reproducing kernel for the Hardy space $H^2(\mathbb{D})$ (see Section 2).

2. On the maximal numerical range and maximal Berezin set

Recall that for the operator $T \in \mathcal{B}(H)$, (Banach algebra of all bounded linear operators on the Hilbert space H), Stampfli [14] defined the maximal numerical range as follows:

$$W_0(T) := \{\lambda \in \mathbb{C} : \langle Tx_n, x_n\rangle \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

When H is finite dimensional, it is easy to see that $W_0(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$). It is well known [14, Lemma 2] that the set $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the usual numerical range

$$W(T) := \left\{ \langle Tx, x\rangle : \|x\|_H = 1 \right\}.$$

It is well known (see [7]) that $W(A)$ is a convex set whose closure contains the spectrum $\sigma(A)$ of A. If A is a normal operator, then the closure of $W(A)$ is the convex hull of $\sigma(A)$. Furthermore, it is also known that each extreme point of $W(A)$ is an eigenvalue of A.

Let $\mathcal{B}$ be a Banach algebra with the norm $\|.\|_{\mathcal{B}}$. A derivation on $\mathcal{B}$ is a linear map $\mathcal{D} : \mathcal{B} \to \mathcal{B}$ which satisfies

$$\mathcal{D}(ab) = a\mathcal{D}(b) + \mathcal{D}(a)b$$

for all $a, b \in \mathcal{B}$. If for a fixed a, $\mathcal{D}_a : b \to ab - ba$, then $\mathcal{D}_a$ is called an inner derivation. It is well known that every derivation on a von Neumann algebra or on a simple C^*-algebra is inner (see [8,12,13]). It is obvious that $\|\mathcal{D}_a\| \leq 2\|a\|_{\mathcal{B}}$. Stampfli proved that (see [14, Theorem 4]) if $\mathcal{D}_T$ is a derivation on $\mathcal{B}(H)$, then $\|\mathcal{D}_T\| = 2dist(T, \mathbb{C}I)$, where $\mathbb{C}I$ denotes the set of all scalar operators λI $(\lambda \in \mathbb{C})$ on H. Stampfli also proved in terms of the maximal numerical range of T that $\|\mathcal{D}_T\| = 2\|T\|$ if and only if $0 \in W_0(T)$ (see [14, Theorem 4]). When $T = T_\varphi$, the Toeplitz operator defined on

$H^2(\mathbb{D})$, Stampfli's assertion "$\|\mathcal{D}_{T_\varphi}\| = 2\|T_\varphi\| \Longleftrightarrow 0 \in W_0(T_\varphi)$" is equivalent to "$dist(\varphi, \mathcal{F}_{\text{const}}) = \|\varphi\|_{L^\infty} \Longleftrightarrow 0 \in W_0(T_\varphi)$" (because it is elementary to show that $dist(T_\varphi, \mathbb{C}I) = dist(\varphi, \mathcal{F}_{\text{const}})$, where $\mathcal{F}_{\text{const}}$ is the set of all constant functions). Thus, the condition $0 \in W_0(T_\varphi)$ is important in the approximation problem. So, the notion of maximal numerical range plays a key role in distance estimates.

In this section, we investigate the structure of the maximal numerical range and maximal Berezin set of some Toeplitz operators on the Hardy space H^2.

The following result gives an example of an operator containing 0 in its maximal numerical range.

Proposition 1. *Let $\theta(z) = \exp\left(\frac{z+1}{z-1}\right)$ be a singular inner function. Then $0 \in W_0(T_\theta)$.*

Proof. It is well known that (see, for example [4]) the nontangential limit at the point 1 of the function θ is equal to zero. Therefore, there exists a sequence $\{\lambda_n\}_{n\geqslant 1} \subset \mathbb{D}$ such that $\theta(\lambda_n) \to 0$ as λ_n tends the nontangentially to 1. Let us consider the sequence $\{k_{\lambda_n}(z)\}_{n\geqslant 1} = \left\{\frac{1}{1-\bar{\lambda}_n z}\right\}_{n\geqslant 1}$. Clearly, $k_{\lambda_n} \in H^2$ for all $n \geqslant 1$. Let us denote $\hat{k}_{\lambda_n}(z) := \frac{k_{\lambda_n}(z)}{\|k_{\lambda_n}(z)\|_2} = \frac{(1-|\lambda_n|^2)^{\frac{1}{2}}}{1-\bar{\lambda}_n z}$. Then, by considering that $T_\theta^* k_{\lambda_n} = \overline{\theta(\lambda_n)} k_{\lambda_n}$, we have

$$\lim_n \left\langle T_\theta \widehat{k}_{\lambda_n}, \hat{k}_{\lambda_n} \right\rangle = \lim_n \left\langle \hat{k}_{\lambda_n}, T_\theta^* \hat{k}_{\lambda_n} \right\rangle = \lim_n \left\langle \hat{k}_{\lambda_n}, \overline{\theta(\lambda_n)} \hat{k}_{\lambda_n} \right\rangle = \lim_n \theta(\lambda_n) = 0$$

because $\theta(\lambda_n) \to 0$ as $\lambda_n \to 1$ nontangentially. On the other hand, since T_θ is an isometry, we have $\lim_n \left\| T_\theta \widehat{k}_{\lambda_n} \right\|_2 = 1 = \|T_\theta\|$, and hence $\lim_n \left\| T_\theta \hat{k}_{\lambda_n} \right\|_2 = \|T_\theta\|$, which shows that $0 \in W_0(T_\theta)$, as desired. □

Remark 1. In general, it is easy to see that $W_0(V) = \overline{W(V)}$ for any isometry V on a Hilbert space H. On the other hand, since $\sigma(V) \subset \overline{W(V)}$, we have that $0 \in W_0(V)$ for any non-unitary isometry V.

Recall that by a Reproducing Kernel Hilbert Space (RKHS) we mean a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that evaluation at any point of Ω is a continuous linear functional on $\mathcal{H}$. The classical Riesz representation theorem ensures that a functional Hilbert space $\mathcal{H}$ has a reproducing kernel, that is, a function $k_{\mathcal{H}} : \Omega \times \Omega \to \mathbb{C}$ with defining property $\langle f, k_{\mathcal{H},\lambda} \rangle = f(\lambda)$ for all $f \in \mathcal{H}$ and $\lambda \in \Omega$, where $k_{\mathcal{H},\lambda} = k_{\mathcal{H}}(., \lambda) \in \mathcal{H}$. Let $\hat{k}_{\mathcal{H},\lambda} := \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}}$ be the normalized reproducing kernel of $\mathcal{H}$. For any bounded linear operator A on $\mathcal{H}$, its Berezin symbol $\widetilde{A}$ is defined by (see [17])

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_{\mathcal{H},\lambda}, \hat{k}_{\mathcal{H},\lambda} \right\rangle, \quad \lambda \in \Omega.$$

It is well known that $\hat{k}_{H^2,\lambda} = \frac{(1-|\lambda_n|^2)^{\frac{1}{2}}}{1-\bar{\lambda} z}$ and $\widetilde{T}_\varphi = \widetilde{\varphi}$ for any Toeplitz operator on the Hardy space H^2, where $\tilde{\varphi}$ is the harmonic (Poisson) extension of φ into the unit disc $\mathbb{D}$. The Berezin symbol of an operator provides important information about the operator. Namely, it is well known that on the most familiar RKHS, including the Hardy, Bergman and Fock spaces, the Berezin symbol uniquely determines the operator (i.e., $\tilde{A}_1(\lambda) = \tilde{A}_2(\lambda)$ for all $\lambda \in \Omega$ implies $A_1 = A_2$).

The RKHS is said to be standard (see [11]) if the underlying set Ω is a subset of a topological space and the boundary $\partial\Omega$ is non-empty and has the property that $\left\{\hat{k}_{\mathcal{H},\lambda_n}\right\}$ converges weakly to 0 whenever $\{\lambda_n\}$ is a sequence in Ω that converges to a point in $\partial\Omega$.

For a compact operator K on the standard RKHS $\mathcal{H}$, it is clear that $\lim_{n\to\infty}\tilde{K}(\lambda_n) = 0$ whenever $\{\lambda_n\}$ converges to a point of $\partial\Omega$ (since compact operators send weakly convergent sequences into strongly convergent ones). In this sense, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary.

For a bounded linear operator on a RKHS its Berezin set and Berezin number are defined, respectively, as follows:

$$Ber(A) := Range(\widetilde{A}) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}$$

$$ber(A) := \sup\{|\eta| : \eta \in Ber(A)\} = \left\|\widetilde{A}\right\|_{L^\infty(\Omega)}.$$

Observe that Engliś showed, in his thesis, that $\widetilde{A}$ is a C^∞ function.

For any operator $A \in \mathcal{B}(\mathcal{H})$, let us define also the following set, which we call as *maximal Berezin set* of A:

$$\widetilde{W_0}(A) := \left\{\lambda \in \mathbb{C} : \exists\{\lambda_n\} \subset \Omega \quad \text{such that} \quad \lambda = \lim_{\lambda_n\to\partial\Omega}\widetilde{A}(\lambda_n) \text{ and } \lim_{\lambda_n\to\partial\Omega}\left\|A\hat{k}_{\mathcal{H},\lambda_n}\right\|_{\mathcal{H}} = \|A\|\right\}$$

Clearly, $\widetilde{W_0}(A) \subset W_0(A)$. It is also obvious that $\widetilde{W_0}(A) = W_0(A) = \{\lambda\}$ for any scalar operator $A = \lambda I$, and $\widetilde{W_0}(K) = \emptyset$ for every non-trivial compact operator K on the standard RKHS $\mathcal{H}$. However, the situation is not much trivial for other operators and RKHS.

Our next result concerns the structure of the maximal Berezin set $\widetilde{W_0}(T_\varphi)$ for some analytic Toeplitz operator T_φ on H^2. Before giving it, let us introduce the following notation:

$$H^{\infty,d} := \left\{f \in H^\infty : |f| = \|f\|_\infty \text{ on a subset of } \mathbb{T} \text{of positive measure}\right\}.$$

In other words, for every $f \in H^{\infty,d}$ there exists a set $E_f \subset \mathbb{T}$ with $measE_f > 0$ such that $|f(\xi)| = \|f\|_\infty$ for all $\xi \in E_f$. According to the result of Fisher [3] (see also [4]), the set $H^{\infty,d}$ is dense in H^∞.

Theorem 1. *Let $\varphi \in H^{\infty,d}$ be a nonconstant function. Then there exists a subset $E_\varphi \subset \mathbb{T}$ of positive measure such that $\left\{\varphi(\xi) : \xi \in E_\varphi\right\} \subset \widetilde{W_0}(T_\varphi)$.*

Proof. Since $\varphi \in H^{\infty,d}$, there exists a set $E_\varphi \subset \mathbb{T}$ such that $measE_\varphi > 0$ and $|\varphi(\xi)| = \|\varphi\|_\infty$ for all $\xi \in E_\varphi$. Let $\xi \in E_\varphi$ be an arbitrary fixed point. Then there exists a sequence $\{\lambda_n\} \subset \mathbb{D}$ such that $\lim_n \lambda_n = \xi$ and $\lim_{\lambda_n \to \xi} \varphi(\lambda_n) = \varphi(\xi)$. Then by considering that $\hat{k}_{\lambda_n} := \widehat{k}_{H^2,\lambda_n} = (1 - |\lambda_n|^2)^{\frac{1}{2}} (1 - \overline{\lambda}_n z)^{-1}$ and

$$|\varphi(\lambda_n)| = \left| \left\langle T_\varphi \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n} \right\rangle \right| \leqslant \left\| T_\varphi \widehat{k}_{\lambda_n} \right\|_2 \leqslant \| T_\varphi \| = \|\varphi\|_\infty,$$

we have

$$\|\varphi\|_\infty = |\varphi(\xi)| = \lim_{\lambda_n \to \xi} |\varphi(\lambda_n)| \leqslant \lim_{\lambda_n \to \xi} \left\| T_\varphi \widehat{k}_{\lambda_n} \right\|_2 \leqslant \|\varphi\|_\infty.$$

Hence

$$\lim_{\lambda_n \to \xi} \left\| T_\varphi \hat{k}_{\lambda_n} \right\|_2 = \|\varphi\|_\infty.$$

This shows that $\{\varphi(\xi) : \xi \in E_\varphi\} \subset \widetilde{W}_0(T_\varphi)$, as desired. The theorem is proved. □

As will be shown below, the notion of maximal Berezin set can be useful in the investigation of some problems for some special C^*-operator algebras on the Bergman space L_a^2, which we will call Engliś algebras (because, apparently, these algebras have been introduced for the first time by Engliś in [2]). Recall that the Bergman space $L_a^2 = L_a^2(\mathbb{D})$ is the Hilbert subspace of the Lebesgue space $L^2(\mathbb{D})$ consisting of analytic functions, with induced norm $\|.\|_{L_a^2}$, and that the reproducing kernel of L_a^2 reads as $K_{L_a^2,\lambda}(z) = \frac{1}{(1-\overline{\lambda}z)^2}$, see [17] for further details. In particular, the following Engliś algebra $\mathcal{A}_B$ (the subscript B stands for "Bergman") is defined in [2] as follows:

$$\mathcal{A}_B := \left\{ T \in \mathcal{B}(L_a^2) : \left\| T\hat{k}_{L_a^2,\lambda} \right\|_{L_a^2}^2 - \left| \left\langle T\hat{k}_{L_a^2,\lambda}, \hat{k}_{L_a^2,\lambda} \right\rangle \right|^2 \text{ and } \left\| T^*\hat{k}_{L_a^2,\lambda} \right\|_{L_a^2}^2 - \left| \left\langle T^*\hat{k}_{L_a^2,\lambda}, \hat{k}_{L_a^2,\lambda} \right\rangle \right|^2 \to 0 \text{ radially} \right\},$$

where $\widehat{k}_{L_a^2,\lambda}$ is the normalized reproducing kernel of L_a^2.

It was shown in [2] that the same algebra in the Hardy space case contains all the Toeplitz operators $T_\varphi, \varphi \in L^\infty(\mathbb{T})$. However, the situation is not clear for the algebra $\mathcal{A}_B$, and the following question has been posed in this respect by Engliś in [2, Question 1].

Question 1. *Is it true that $T_\phi \in \mathcal{A}_B$ for all $\phi \in L^\infty(\mathbb{D})$?*

Here, we characterize in terms of maximal Berezin set and Berezin number the Engliś algebra $\mathcal{A}_B$, which in particular sheds some light on the solution of Question 1.

Proposition 2. *Let $T \in \mathcal{B}(L_a^2(\mathbb{D}))$ be an operator on the Bergman space $L_a^2(\mathbb{D})$ such that $ber(T) \in \widetilde{W}_0(T) \cap \widetilde{W}_0(T^*)$. Then $T \in \mathcal{A}_B$ if and only if $\|T\| = ber(T)$.*

Proof. Since $ber(T) \in \widetilde{W}_0(T) \cap \widetilde{W}_0(T^*)$, there exists a sequence $\{\lambda_n\}$ and $\{\mu_n\} \subset \mathbb{D}$ tending to the boundary points ξ and $\eta \in \mathbb{T}$, respectively, such that

$$ber(T) = \lim_{\lambda_n \to \xi} \widetilde{T}(\lambda_n) \quad \text{and} \quad \lim_{\lambda_n \to \xi} \left\| T\hat{k}_{L^2_a, \lambda_n} \right\|_{L^2_a} = \|T\|$$

and

$$ber(T^*) = \lim_{\mu_n \to \eta} \widetilde{T^*}(\mu_n) \quad \text{and} \quad \lim_{\mu_n \to \eta} \left\| T^* \hat{k}_{L^2_a, \mu_n} \right\|_{L^2_a} = \|T^*\|.$$

If it is necessary, by choosing other points ξ and $\eta \in \mathbb{T}$, we can assume that at least one of the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ tends radially to a boundary point in $\mathbb{T}$. Then, by considering the obvious facts that $\|T^*\| = \|T\|$, $\left|\widetilde{T}(z)\right| = \left|\widetilde{T^*}(z)\right|$, $ber(T^*) = ber(T)$, it follows from the latter equalities that $T \in \mathcal{A}_B$ if and only if $\|T\| = ber(T)$. This proves the proposition. $\square$

The following question arises: is it true that there exists a Toeplitz operator T_φ on the Bergman space $L^2_a(\mathbb{D})$ with symbol $\varphi \in L^\infty(\mathbb{D})$ satisfying $ber(T_\varphi) \in \widetilde{W_0}(T_\varphi)$ and $ber(T_\varphi) < \|T_\varphi\|$?

By using Proposition 2, it is easy to see that a positive answer to this question would give a negative answer to the above mentioned Question 1.

3. Normality and the numerical range of Toeplitz operators

Our next result is the following characterization of normal operators in terms of Berezin symbols.

Proposition 3. *Let A be a bounded operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ and let $\widetilde{A}(\lambda) = \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle$ be its Berezin symbol, where $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$ is a normalized reproducing kernel of $\mathcal{H}$. Then A is a normal operator on $\mathcal{H}$ if and only if*

$$\left\| \left(A - \widetilde{A}(\lambda)I\right)\widehat{k}_\lambda \right\|_{\mathcal{H}} = \left\| \left(A - \widetilde{A}(\lambda)I\right)^* \widehat{k}_\lambda \right\|_{\mathcal{H}}$$

for all $\lambda \in \Omega$.

Proof. By considering that $A\hat{k}_\lambda - \widetilde{A}(\lambda)\hat{k}_\lambda \perp \widetilde{A}(\lambda)\hat{k}_\lambda$, we have

$$\left\| A\hat{k}_\lambda \right\|^2_{\mathcal{H}} = \left\| A\hat{k}_\lambda - \widetilde{A}(\lambda)\hat{k}_\lambda \right\|^2_{\mathcal{H}} + \left|\widetilde{A}(\lambda)\right|^2 \tag{1}$$

for each $\lambda \in \Omega$. Since $\left\| A\widehat{k}_\lambda \right\|^2_{\mathcal{H}} = \left\langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle = \widetilde{A^*A}(\lambda)$ and $\left\| A^*\widehat{k}_\lambda \right\|^2_{\mathcal{H}} = \widetilde{AA^*}(\lambda)$, it follows immediately from formula (1) that

$$\left\| A\hat{k}_\lambda - \widetilde{A}(\lambda)\widehat{k}_\lambda \right\|^2_{\mathcal{H}} = \widetilde{A^*A}(\lambda) - \left|\widetilde{A}(\lambda)\right|^2$$

and

$$\left\| A^*\hat{k}_\lambda - \widetilde{A^*}(\lambda)\hat{k}_\lambda \right\|^2_{\mathcal{H}} = \widetilde{AA^*}(\lambda) - \left|\widetilde{A}(\lambda)\right|^2.$$

Since the Berezin symbol uniquely determines the operator, it follows from the last two formulas that $A^*A = AA^*$ if and only if

$$\left\| \left(A - \widetilde{A}(\lambda)I\right)\hat{k}_\lambda \right\|_{\mathcal{H}} = \left\| \left(A - \widetilde{A}(\lambda)I\right)^* \hat{k}_\lambda \right\|_{\mathcal{H}}, \quad \lambda \in \mathbb{D},$$

which proves the proposition, since $\widetilde{A^*}(\lambda) = \overline{\widetilde{A}(\lambda)}$. □

In the present, section we will give some applications of Proposition 2 in the study of some topological properties of the numerical range of Toeplitz operators acting on the Hardy space.

The description of the numerical range of an arbitrary Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ of the unit disc $\mathbb{D}$ was given in [10]. The Bergman space case was considered by Thukral [15] in case of bounded harmonic symbols. More recently Choe and Lee [1], as well as Gu [5], treat higher-dimensional Bergman space analogues. The case of Bergman space Toeplitz operators with bounded radial symbols has been considered very recently by Wang and Wu [16]. (For characterization of numerical ranges of certain classes of so-called dual Toeplitz operators, see, for instance, Guediri [6].)

Theorem 2 ([10]). *Let $u \in L^\infty(\mathbb{T})$. If $W(T_u)$ is not open in $\mathbb{C}$, then T_u is normal on H^2.*

Corollary 1 ([10]). *Let $u \in L^\infty(\mathbb{T})$. If T_u is not normal on H^2, then $W(T_u)$ is the interior of its closure.*

It is known (see [6, Thoerem 1.4-1]) that for any bounded linear operator T on a Hilbert space, if $W(T)$ is a part of a line segment, then T must be normal. In [10], the author considers the problem of when the converse of this fact is also true for Toeplitz operators. Namely, he proved the following result.

Theorem 3 [10]. *Let $u \in L^\infty(\mathbb{T})$. Then T_u is normal on H^2 if and only if $W(T_u)$ is an open line segment.*

The following results are immediate from Theorems 2 and 3, Corollary 1 and Proposition 2.

Proposition 4. *Let $u \in L^\infty(\mathbb{T})$. If there exists a $\lambda_0 \in \mathbb{D}$ such that*

$$\left\| (T_u - \widetilde{u}(\lambda_0)I)\hat{k}_{\lambda_0} \right\|_2 \neq \left\| (T_u - \tilde{u}(\lambda_0)I)^* \hat{k}_{\lambda_0} \right\|_2,$$

then the numerical range $W(T_u)$ of T_u is open in $\mathbb{C}$.

Proposition 5. *Let $u \in L^\infty(\mathbb{T})$. Then, the numerical range $W(T_u)$ is an open line segment if and only if $\left\| (T_u - \tilde{u}(\lambda)I)\widehat{k}_\lambda \right\|_2 = \left\| (T_u - \tilde{u}(\lambda)I)^* \widehat{k}_\lambda \right\|_2$, $\forall \lambda \in \mathbb{D}$.*

The same results can be similarly proved for Toeplitz operators on the analytic or harmonic Bergman spaces of the unit disc, on Hardy and Bergman (pluriharmonic Bergman) spaces of the unit ball or polydisc in $\mathbb{C}^n$.

4. An estimate for $ber(A)$

The next result slightly improves a result in [9, Theorem 1] and gives an inequality for the Berezin number of operators.

Theorem 4. *Let $\varphi \in H^\infty, \|\varphi\| \leq 1$, be any function, and θ be any nonconstant inner function. For any operator $A \in \mathcal{B}(H^2)$ we denote*

$$N_{\varphi,\theta,A} := T_\varphi\left(I - T_\theta A T_{\overline{\theta}}\right).$$

For any $\varepsilon \in (0,1)$, let $L_{\varepsilon,\theta} := \{z \in \mathbb{D} : |\theta(z)| \leqslant \varepsilon\}$ be an ε-level set of θ. Then

$$ber(A) \geqslant \sup_{0<\varepsilon<1} \frac{\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty\left(L_{\varepsilon,\theta}\right)}}{\varepsilon^2}.$$

Proof. Arguing in the same manner as in Theorem 1 of [9], we obtain

$$\widetilde{N}_{\varphi,\theta,A}(\lambda) = \left\langle N_{\varphi,\theta,A}(\lambda)\hat{k}_\lambda, \widehat{k}_\lambda\right\rangle = \varphi(\lambda)\left(1 - |\theta(\lambda)|^2\widetilde{A}(\lambda)\right), \quad \lambda \in \mathbb{D}$$

from which we obtain that

$$\left|\varphi(\lambda) - \widetilde{N}_{\varphi,\theta,A}(\lambda)\right| = \left|\widetilde{A}(\lambda)\right||\varphi(\lambda)||\theta(\lambda)|^2 \leqslant ber(A)|\theta(\lambda)|^2$$

for all $\lambda \in \mathbb{D}$. In particular,

$$\left|\varphi(\lambda) - \widetilde{N}_{\varphi,\theta,A}(\lambda)\right| \leqslant ber(A)\varepsilon^2$$

for all $\lambda \in L_{\varepsilon,\theta}$, and hence

$$\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty\left(L_{\varepsilon,\theta}\right)} \leqslant ber(A)\varepsilon^2, \quad 0 < \varepsilon < 1,$$

which implies that

$$\sup_{0<\varepsilon<1} \frac{\left\|\varphi - \widetilde{N}_{\varphi,\theta,A}\right\|_{L^\infty\left(L_{\varepsilon,\theta}\right)}}{\varepsilon^2} \leqslant ber(A)$$

as desired. The proof is completed. $\square$

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