
Remarks on families of singular curves with hyperelliptic normalizations [☆]

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ABSTRACT

We give restrictions on the existence of families of curves on smooth projective surfaces S of nonnegative Kodaira dimension all having constant geometric genus $p_g \geq 2$ and hyperelliptic normalizations. In particular, we prove a Reidier-like result that relies on deformation theory and bending-and-breaking of rational curves in $\text{Sym}^2(S)$. We also give examples of families of such curves.

1. INTRODUCTION

The object of study of this paper is families of irreducible curves with hyperelliptic normalizations (of genus ≥ 2) on a smooth surface S . Such families give rise to, because of the unique g_2^1 's on their normalizations, families of the same dimensions of irreducible rational curves in the Hilbert scheme $\text{Hilb}^2(S)$. Because of the importance of rational curves and the subvarieties they cover due to Mori theory, it is natural to try to check the existence of, or bound the dimensions of, families of such curves, or alternatively, their counterparts on S .

Let S be a smooth surface and $V \subset \text{Hilb } S$ be a reduced and irreducible scheme parametrizing a flat family of curves on S all having constant geometric genus $p_g \geq 2$ and hyperelliptic normalizations.

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It is easy to see (cf. Lemma 2.1) that if $|K_S|$ is birational, then $\dim V = 0$. This shows that the problem of bounding the dimension of a family of curves with hyperelliptic normalizations is solved for a large class of surfaces. At the same time, it is relatively easy to find obvious examples of surfaces with large families of curves with hyperelliptic normalizations: In fact, if S is any smooth surface admitting a rational $2 : 1$ map $f : S \rightarrow R$ onto a rational surface, then we can just pull back families of rational curves on R . There are several examples of such double covers, even for $p_g(S) > 0$, see for instance the works of Horikawa [19–22] for surfaces of general type, Saint-Donat [35] for $K3$ surfaces, and [6,37,38] for surfaces with smooth hyperelliptic hyperplane sections.

We note that *smooth* hyperelliptic curves on surfaces have been extensively studied by means of adjunction theory (see [6,37,38] to mention a few). Of course Reider’s famous result [34] can be used to prove that if $C \subset S$ is a smooth hyperelliptic curve and $C^2 \geq 9$, then there is a pencil $|E|$ such that either $E^2 = 0$ and $E.C = 2$, or $E^2 = 1$, $C \equiv 3E$ and $|E|$ has one base point x lying on C (making the obvious modifications in the proof of [34, Corollary 2]). Unfortunately no such results seem to be available, at least as far as we know, in the case of singular curves.

In this paper we prove some results bounding dimensions of families of irreducible curves with hyperelliptic normalizations on smooth surfaces S in Section 2. In particular, we show that the dimension is bounded by one if S is fibered over a smooth nonhyperelliptic curve of genus ≥ 3 (Lemma 2.3) and by two if S has maximal albanese dimension (Proposition 2.5). We also give several examples of families of irreducible curves with hyperelliptic normalizations. Then, in Section 3, we prove a Reider-like result, cf. Theorem 3.3, stating that any family of dimension ≥ 3 (resp. ≥ 5) of curves with hyperelliptic normalizations on a smooth surface S with $p_g(S) > 0$ (resp. $\text{kod}(S) \geq 0$ and $p_g(S) = 0$) forces the existence of some special divisors enjoying some particular intersection properties. Moreover, these divisors “cut out” the g_2^1 ’s on the normalizations of the curves in the family.

We hope the results will find more applications and also hope that the reader will find the method of proof of interest: in fact, unlike the results on smooth curves, which use adjunction theory and/or vector bundle methods, our method uses deformations of curves and bending and breaking of rational curves in $\text{Sym}^2(S)$, a method we also used in [15]. Thus, the special divisor occurring in Theorem 3.3 is obtained as a component of a degenerated member at the border of the family. In this sense our method is perhaps more geometric and intuitive than Reider’s method.

In Section 2 we state the general setting, show that the dimension of families of curves with hyperelliptic normalizations can be bounded in various cases and give some examples of such families.

Then, in Section 3 we prove the Reider-like result, Theorem 3.3, passing by Proposition 3.1 and Lemma 3.2. Finally, we make some remarks in Section 4, including writing out the results in the case of smooth curves in Section 4.1 recovering Reider’s result, and in the case of only one singular point of multiplicity one in Section 4.2.

Consider the following assumptions:

- S is a smooth projective surface with $\text{kod}(S) \geq 0$ and $V \subset \text{Hilb } S$ a reduced and irreducible scheme parametrizing a flat family of
- (1) irreducible curves on S of constant arithmetic and geometric genera p_a and $p_g \geq 2$, respectively, and hyperelliptic normalizations. We denote by C the algebraic equivalence class of the curves.

Note that V as in (1) is, by default, nonempty.

We have the following elementary result, already mentioned in the introduction:

Lemma 2.1. *Under the assumptions (1), if $|K_S|$ is birational, then $\dim V = 0$.*

Proof. We may assume that $\dim V = 1$. Then after compactifying and resolving the singularities of the universal family over V , we obtain a smooth surface T , fibered over a smooth curve, with general fiber F a smooth hyperelliptic curve of genus ≥ 2 , and a surjective morphism $f : T \rightarrow S$. By adjunction $|K_T|$ is not birational on the general fiber F . Since $K_T = f^*K_S + R$, where R is the (effective) ramification divisor of f , and f is generically $1 : 1$ on the fibers, we see that $|K_S|$ cannot be birational. \square

Note that, as mentioned in the Introduction, any irreducible curve C on a surface S with hyperelliptic normalization (of geometric genus ≥ 2) gives rise to a unique irreducible rational curve $R_C \subset \text{Hilb}^2(S)$. Precisely, this can be seen in the following way: Let $\nu : \tilde{C} \rightarrow C$ be the normalization. Then the unique g_2^1 on \tilde{C} induces a map $\mathbb{P}^1 \rightarrow \text{Sym}^2(\tilde{C})$ and this \mathbb{P}^1 is mapped to an irreducible rational curve Γ_C in $\text{Sym}^2(S)$ by the natural composed morphism

$$\text{Sym}^2(\tilde{C}) \xrightarrow{\tilde{\nu}^{(2)}} \text{Sym}^2(C) \hookrightarrow \text{Sym}^2(S).$$

The irreducible rational curve $R_C \subset \text{Hilb}^2(S)$ is the strict transform by the Hilbert–Chow morphism $\mu : \text{Hilb}^2(S) \rightarrow \text{Sym}^2(S)$ of this curve. Note that the Hilbert–Chow morphism resolves $\text{Sing}(\text{Sym}^2(S)) \simeq S$ and gives an obvious one-to-one correspondence between irreducible curves in $\text{Hilb}^2(S)$ not contained in the exceptional locus (which is a \mathbb{P}^1 -bundle over S) and irreducible curves in $\text{Sym}^2(S)$ not contained in $\text{Sing}(\text{Sym}^2(S))$.

The correspondence in the opposite direction, that is, from irreducible rational curves in $\text{Hilb}^2(S)$ to curves in S is more delicate and we refer to [15, Section 2] for details. Suffice it to say that irreducible rational curves in $\text{Hilb}^2(S)$ not contained in the exceptional locus give rise to curves on S with rational, elliptic or hyperelliptic normalizations, by taking the (one-dimensional component of the) union of the supports of the points of the curve in $\text{Hilb}^2(S)$ when we consider these as length-two schemes on S .

To extend the correspondence into families, we proceed as follows (cf. also [15]).

Given (1), let $\varphi: \mathcal{C} \rightarrow V$ be the universal family. Normalizing \mathcal{C} we obtain, possibly restricting to an open dense subscheme of V , a flat family $\tilde{\varphi}: \tilde{\mathcal{C}} \rightarrow V$ of smooth hyperelliptic curves of genus $p_g \geq 2$, cf. [40, Theorem 1.3.2]. Let $\omega_{\tilde{\mathcal{C}}/V}$ be the relative dualizing sheaf. The morphism $\gamma: \tilde{\mathcal{C}} \rightarrow \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}/V}))$ over V is finite and of relative degree two onto its image (cf. also [29, Theorem 5.5(iv)]), which we denote by \mathcal{P}_V . We now have a universal family $\psi: \mathcal{P}_V \rightarrow V$ of rational curves and since the points in these curves correspond to couples of points of S , possibly coinciding, we have a natural morphism $\Phi_V: \mathcal{P}_V \rightarrow \text{Sym}^2(S)$. We define

$$(2) \quad R_V := \overline{\text{im } \Phi_V} \quad (\text{the Zariski closure}).$$

We have

$$(3) \quad \begin{array}{ccccc} & & \tilde{\mathcal{C}} & \xrightarrow{\gamma} & \mathcal{P}_V & \xrightarrow{\Phi_V} & R_V \subset \text{Sym}^2(S) \\ & \swarrow \pi & & \searrow \tilde{\varphi} & \downarrow \psi & & \\ S & & & & V & & \end{array}$$

where π is the natural morphism. Note that Φ_V maps no curve in the family to $\text{Sing}(\text{Sym}^2(S)) \simeq S$ by construction. Also note that $\dim R_V \leq 3$, as $\text{Sym}^2(S)$ is not uniruled, since $\text{kod}(S) \geq 0$ (see e.g. [18, Proposition 2.1]).

Given V as in (1), we will call $\mu_*^{-1} R_V \subset \text{Hilb}^2(S)$, the strict transform of R_V by the Hilbert–Chow morphism, the *locus covered by the associated rational curves in $\text{Hilb}^2(S)$* .

We will make use of the following consequence of Mumford’s well-known theorem on 0-cycles on surfaces [32, Corollary, p. 203], as generalized in [15, Corollary 3.2]:

Proposition 2.2. *Assume that S is a smooth surface with $p_g(S) > 0$ and $R \subset \text{Sym}^2(S)$ is a subvariety that is covered by a family of rational curves of dimension ≥ 3 .*

Then R is a surface with rational desingularization.

Proof. This is [15, Proposition 3.6]. \square

We will now give some results bounding the dimension of V as in (1) in various situations.

Lemma 2.3. *Assume (1) and that $f: S \rightarrow B$ is a fibration over a nonhyperelliptic smooth curve B of genus ≥ 3 .*

Then $\dim V \leq 1$ with equality holding if and only if the general fiber of f is a (smooth) hyperelliptic curve and V parametrizes a subset of the fibers.

Proof. Let $\{C_v\}_{v \in V}$ be the family on S given by V and $\{\Gamma_v = \Gamma_{C_v}\}_{v \in V}$ be the associated family of rational curves in $\text{Sym}^2(S)$, given by ψ and Φ_V as in (3).

Consider $f^{(2)}: \text{Sym}^2(S) \rightarrow \text{Sym}^2(B)$. Since B is neither elliptic nor hyperelliptic, $\text{Sym}^2(B)$ does not contain rational curves. Therefore, $f^{(2)}$ must contract every Γ_v to a point, say $b_v + b'_v \in \text{Sym}^2(B)$, with $b_v, b'_v \in B$. Letting $F_v := f^{-1}b_v$ and $F'_v := f^{-1}b'_v$ denote the two fibers of f , we get that Γ_v is contained in the surface $F_v + F'_v \subset \text{Sym}^2(S)$. Hence $\text{Supp}(C_v) = F_v \cup F'_v \subset S$. Since each C_v is irreducible, the result follows. \square

The following example shows that the condition that B is neither elliptic nor hyperelliptic is in fact necessary:

Example 2.4. Start with a Hirzebruch surface \mathbb{F}_e , with $e \geq 0$, with $\text{Pic} \mathbb{F}_e \simeq \mathbb{Z}[\Sigma] \oplus \mathbb{Z}[F]$, where $\Sigma^2 = -e$, $F^2 = 0$ and $\Sigma \cdot F = 1$. Choose integers α and β such that

$$(4) \quad \alpha \geq 2 \quad \text{and} \quad \beta \geq \{2, \alpha e + 1\}$$

and take a general pencil in $|\alpha\Sigma + \beta F|$. Note that the conditions (4) guarantee that the general element in the general such pencil is in fact irreducible. Now take the blow up $\pi: \tilde{\mathbb{F}}_e \rightarrow \mathbb{F}_e$ along the $(\alpha\Sigma + \beta F)^2 = \alpha(2\beta - e\alpha)$ base points of the pencil and denote the exceptional curves by E_i . Then

$$K_{\tilde{\mathbb{F}}_e} \sim -2\pi^*\Sigma - (e+2)\pi^*F + \Sigma E_i.$$

Set

$$\tilde{D} := \pi^*(\alpha\Sigma + \beta F) - \Sigma E_i.$$

Then $\tilde{D}^2 = 0$ and $|\tilde{D}|$ is a pencil defining a fibration $g: \tilde{\mathbb{F}}_e \rightarrow \mathbb{P}^1$. For any integer $l \geq 2$, choose a general $\Delta_l \in |\mathcal{O}_{\mathbb{P}^1}(2l)|$ consisting of distinct points and let $R_l \in |2l\tilde{D}|$ be the corresponding (smooth) divisor. Then Δ_l and R_l define two double covers ν and μ , respectively, that are compatible, in the sense that we have a commutative diagram:

$$(5) \quad \begin{array}{ccc} T & \xrightarrow{\mu} & \tilde{\mathbb{F}}_e \\ f \downarrow & & \downarrow g \\ B & \xrightarrow{\nu} & \mathbb{P}^1, \end{array}$$

where T is a smooth surface, B is a smooth curve and f is induced by g in the natural way.

By Riemann–Hurwitz, the genus of B is $g(B) = l - 1 \geq 1$ and B is either elliptic or hyperelliptic. As

$$K_T \sim \mu^*(K_{\tilde{\mathbb{F}}_e} + l\tilde{D}) = \mu^*((l-1)\tilde{D} + \pi^*((\alpha-2)\Sigma + (\beta-e-2)F)),$$

the conditions (4) imply $\text{kod}(S) \geq 0$, in fact even $p_g(S) > 0$.

The surface $\tilde{\mathbb{F}}_e$, being rational, contains families of irreducible (smooth) rational curves of arbitrarily high dimensions. Pulling them back on T yields families of (smooth) hyperelliptic curves of arbitrarily high dimensions. \square

Note that the surfaces T in the example have irregularity $q(T) = h^1(\mathcal{O}_T) = g(B) = l - 1 \geq 1$. Thus the surfaces have arbitrarily high irregularity. Nevertheless, using the albanese map (cf. e.g. [3, pp. 80–88]), we can prove the following bounds.

Proposition 2.5. *Assume (1) with $q(S) := h^1(\mathcal{O}_S) \geq 2$ and let $\alpha : S \rightarrow \text{Alb } S$ be the albanese map.*

If $\text{im } \alpha$ is a curve B , then $\dim V \leq 1$ unless B is hyperelliptic.

If $\text{im } \alpha$ is a surface (i.e., S is of maximal albanese dimension), then $\dim V \leq 2$.

Proof. If $\text{im } \alpha$ is a curve B , then B is necessarily smooth of genus $q(S) \geq 2$ (cf. e.g. [3, Proposition V.15]). Then we apply Lemma 2.3.

If $\text{im } \alpha$ is a surface T , we must have $p_g(S) > 0$ by [3, Lemme V.18]. Assume now, to get a contradiction, that $\dim V \geq 3$. Then R_V , defined in (2), is a surface with rational desingularization, by Proposition 2.2.

As above, let $\{C_v\}_{v \in V}$ be the family on S given by V and $\{\Gamma_v = \Gamma_{C_v}\}_{v \in V}$ be the associated family of rational curves in $\text{Sym}^2(S)$, given by ψ and Φ_V as in (3).

Consider $\alpha^{(2)} : \text{Sym}^2(S) \rightarrow \text{Sym}^2(T) \subset \text{Sym}^2(\text{Alb } S)$. As α does not contract C_v , for general $v \in V$, the surface R_V is mapped by $\alpha^{(2)}$ to a surface $R'_V \subset \text{Sym}^2(T)$. Let $\Sigma : \text{Sym}^2(\text{Alb } S) \rightarrow \text{Alb } S$ be the summation morphism. As $\text{Alb } S$, being abelian, cannot contain rational curves, cf. e.g. [7, Proposition 4.9.5], each rational curve $\Gamma \subset R'_V \subset \text{Sym}^2(T)$ must be contracted to a point by $\Sigma|_{R'_V}$, say $\Sigma(\Gamma) = p_\Gamma \in \text{Alb } S$. Now all fibers $\Sigma^{-1}p$, for $p \in \text{Alb } S$, are isomorphic to the Kummer variety of $\text{Alb } S$, cf. e.g. [7, Section 4.8] for the definition. As rational curves on Kummer varieties cannot move, by [33, Theorem 1], we must have that any family of rational curves on R'_V has dimension $\leq \dim \Sigma(R'_V) \leq 2$. But this contradicts the fact that R_V has rational desingularization.

Therefore $\dim V \leq 2$, as desired. \square

The following result shows that equality $\dim V = 2$ is in fact attained on abelian surfaces, which have maximal albanese dimension.

Lemma–Example 2.6. *Assume (1) with S abelian and that V is not contained in a scheme of larger dimension satisfying (1).*

Then $\dim V = 2$ and the locus covered by the associated rational curves in $\text{Hilb}^2(S)$ is a threefold birational to a \mathbb{P}^1 -bundle over S .

Furthermore, such families exist if and only if S is simple (i.e., not the product of two elliptic curves).

In particular, a bielliptic surface does not contain families as in (1).

Proof. Assume that S is abelian. Consider the natural composed morphism

$$\alpha : \text{Hilb}^2(S) \xrightarrow{\mu} \text{Sym}^2(S) \xrightarrow{\Sigma} S,$$

where μ is the Hilbert–Chow morphism and Σ is the summation morphism. As above, let $\{\Gamma_v = \Gamma_{C_v}\}_{v \in V}$ be the associated family of rational curves in $\text{Sym}^2(S)$ given by V . As S cannot contain rational curves (see e.g. [7, Proposition 4.9.5]), Σ must contract each Γ_v to a point p_v . Therefore, the strict transform $R_v = R_{C_v} := \mu_*^{-1}(\Gamma_v) \subset \text{Hilb}^2(S)$ is contained in the surface $\alpha^{-1}(p_v) \subset \text{Hilb}^2(S)$. Now all such fibers of α over points in S are isomorphic to the (desingularized) Kummer surface of S (cf. [5, Section 7], [4] or [24, 2.3] and e.g. [7, Section 10.2] and [2, V.16] for the definition). Since a Kummer surface is K3, rational curves do not move inside it (this also follows from Lemma 4.4). Therefore, the family is given by $\{R_v\}_{p \in S}$, proving the first assertion.

If S is simple, then it contains irreducible curves of geometric genus two, see e.g. [28, Corollary 2.2].

Assume that $S = E_1 \times E_2$, with each E_i a smooth elliptic curve. Then each $\text{Sym}^2(E_i)$ is an elliptic ruled surface. Any rational curve in $\text{Sym}^2(S)$ not lying in $\text{Sing}(\text{Sym}^2(S))$ is mapped by the projections $\text{Sym}^2(S) \rightarrow \text{Sym}^2(E_i)$, $i = 1, 2$, to a rational curve in either $\text{Sym}^2(E_1)$ or $\text{Sym}^2(E_2)$, which has to be a fiber of the ruling. Therefore, the rational curve in $\text{Sym}^2(S)$ corresponds to a g_2^1 on one of the elliptic fibers of S , proving that there is no irreducible curve on S with hyperelliptic normalization of geometric genus ≥ 2 .

If S is bielliptic, there is a finite morphism $f : T \rightarrow S$ where T is a product of two elliptic curves, cf. e.g. [3, Definition VI.19] or [2, p. 199], whence T is abelian. Clearly, f is unramified, as $K_T \sim \mathcal{O}_T$, whence so is $f^{(2)} : \text{Sym}^2(T) \rightarrow \text{Sym}^2(S)$. Therefore, the family of rational curves $\psi : \mathcal{P}_V \rightarrow V$ as in (3) is pulled back, via $f^{(2)}$, to two copies of the family in $\text{Sym}^2(T)$. By what we proved above, the corresponding families of curves on T consist of elliptic curves, whence the same holds on S . \square

We conclude this section by giving some examples of families as in (1) of high dimensions.

Example 2.7. Let $W(n) \subset |\mathcal{O}_{\mathbb{P}^2}(n)|$ denote the Severi variety of nodal, irreducible rational curves in $|\mathcal{O}_{\mathbb{P}^2}(n)|$. Then $W(n)$ is irreducible of dimension $3n - 1$, by a well-known result of Severi and Harris, cf. [11, Theorem 1.1] and [17]. For any integer $b \geq 3$, take a general smooth $B \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$, so that, for any n , the general curve in $W(n)$ intersects B transversally.

Let $f : S \rightarrow \mathbb{P}^2$ be the double cover defined by B , so that S is a smooth surface and f is branched along B . Setting $H := f^*\mathcal{O}_{\mathbb{P}^2}(1)$, we have $K_S \sim (b - 3)H$. Let $V(n) \subset |nH|$ be the subscheme parametrizing the inverse images of the curves in $W(n)$ that intersect B transversally and let $p_a(n) = p_a(nH)$ and $p_g(n)$ denote the arithmetic and geometric genera of the curves in $V(n)$. Then $V(n)$ satisfies the conditions in (1) and

$$\dim V(n) = 3n - 1,$$

$$p_a(n) = n(n + b - 3) + 1 \quad \text{and} \quad p_g(n) = bn - 1,$$

the second equality by adjunction and the third by Riemann–Hurwitz. Note that the elements of $V(n)$ have $\delta(n) := p_a(n) - p_g(n) = n(n - 3) + 2$ nodes (two over each of the nodes of the corresponding curves in $W(n)$).

Also note that curves in $W(n)$ that are tangent to B yield families in $|nH|$ with lower geometric genera and lower dimensions.

Of course $\text{Hilb}^2(S)$ contains a copy of the \mathbb{P}^2 , which is precisely the locus in $\text{Hilb}^2(S)$ covered by the rational curves associated to the curves in $V(n)$.

Example 2.8. Let S be an Enriques surface. Then S contains several elliptic pencils and we can always pick (at least) two such, $|2E_1|$ and $|2E_2|$, with $E_1 \cdot E_2 = 1$, cf. [12, Theorems 3 and 3.2]. (By adjunction $E_1^2 = E_2^2 = 0$ and it is well known that $2E_i$ and $2E'_i$, where E'_i denotes the unique element of $|E_i + K_S|$, are the two multiple fibers of the elliptic pencils.) Consider $H := 3E_1 + E_2$; then $H^2 = 6$ and the base scheme of $|H|$ consists of two distinct points x and y , where

$$x = E_1 \cap E'_2 \quad \text{and} \quad y = E'_1 \cap E_2$$

(see [13, Proposition 3.1.6 and Theorem 4.4.1]). Let $f: \tilde{S} \rightarrow S$ be the blow up along x and y and E_x and E_y the two exceptional divisors. Set $L := f^*H - E_x - E_y$. Then $L^2 = 4$ and $|L|$ is base point free and, by [13, Theorem 4.5.2], defines a morphism of degree two $\varphi_L: \tilde{S} \rightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}^3$, which can be seen as the embedding of $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ by the complete linear system $|\ell_1 + \ell_2|$, where ℓ_1 and ℓ_2 are the two rulings. In particular, by construction, $L \sim \varphi_L^*(\mathcal{O}_Q(1)) \sim \varphi_L^*(\ell_1 + \ell_2)$. Furthermore, by [13, Remark 4.5.1 and Theorem 4.5.2] the pencil $|f^*(2E_1)|$ on \tilde{S} is mapped by φ_L to $|\ell_1|$, so that $\varphi_L^*\ell_1 \sim f^*(2E_1)$. We therefore have $\varphi_L^*\ell_2 \sim L - \varphi_L^*\ell_1 \sim f^*(E_1 + E_2) - E_x - E_y$.

It follows that, for any $n \geq 1$, the general smooth rational curve in $|\ell_1 + n\ell_2| \simeq \mathbb{P}^{2n+1}$ yields by pullback by φ_L a smooth hyperelliptic curve in $|f^*((n+2)E_1 + nE_2) - n(E_x + E_y)|$ on \tilde{S} of genus $3n + 1$ by adjunction (or by Riemann–Hurwitz and the description of the ramification in [13, Theorem 4.5.2]).

Pushing down to S we thus obtain subschemes, for each $n \in \mathbb{N}$,

$$V(n) \subset |(n+2)E_1 + nE_2|, \quad \text{such that } \dim V(n) = 2n + 1,$$

parametrizing irreducible curves with hyperelliptic normalizations of geometric genera $p_g(n)$ and arithmetic genera $p_a(n)$, where

$$p_g(n) = 3n + 1 \quad \text{and} \\ p_a(n) = \frac{1}{2}((n+2)E_1 + nE_2)^2 + 1 = n(n+2) + 1.$$

Note that for each $n \geq 2$ all the curves in the family have precisely two singular points, located at x and y , both of multiplicity n .

Of course $\text{Hilb}^2(S)$ contains a rational surface birational to $\mathbb{P}^1 \times \mathbb{P}^1$, which is precisely the locus in $\text{Hilb}^2(S)$ covered by the rational curves associated to the curves in $V(n)$.

One can repeat the construction with E_1 and E_2 interchanged, or with other elliptic pencils on the surface. Moreover, choosing smooth rational curves that are tangent to the branch divisor of φ_L , we can obtain families with lower geometric genera, that is, with more singularities.

Moreover, one can also repeat the process for $H := 2E_1 + E_2$, which defines a rational $2 : 1$ map onto \mathbb{P}^2 , following the lines of the previous example. Note that the smooth curves in $|2E_1 + E_2|$ form a two-dimensional family of *smooth* hyperelliptic curves on S , by [13, Corollary 4.5.1].

Example 2.9. Let S be a $K3$ surface. Then $\text{Hilb}^2(S)$ is a *hyperkähler fourfold*, also called an *irreducible symplectic fourfold*, and rational curves and uniruled subvarieties are central in the study of the (birational) geometry of $\text{Hilb}^2(S)$.

For example, a result of Huybrechts and Boucksom [8,25] implies that, if the Mori cone of $\text{Hilb}^2(S)$ is closed, then the boundaries are generated by classes of rational curves. Precise numerical and geometric properties of the rational curves that are extremal in the Mori cone have been conjectured by Hassett and Tschinkel [18].

Uniruled subvarieties of $\text{Hilb}^2(S)$ are important in several aspects: The presence of a $\mathbb{P}^2 \subset \text{Hilb}^2(S)$ gives rise to a birational map (the so-called *Mukai flop*, cf. [31]) to another hyperkähler fourfold and all birational maps between hyperkähler fourfolds factor through a sequence of Mukai flops (see [9,23,41,42]). Moreover, uniruled threefolds in $\text{Hilb}^2(S)$ are central in the study of the birational Kähler cone of $\text{Hilb}^2(S)$ [25].

In the particular case of $K3$ surfaces, the study of families of irreducible curves with hyperelliptic normalizations and the loci the corresponding rational curves cover in $\text{Hilb}^2(S)$ is therefore of particular importance. In [15] we study such families.

Let now H be a globally generated line bundle on S and denote by $|H|^{hyper}$ the subscheme of $|H|$ parametrizing curves with hyperelliptic normalizations. Then, any component of $|H|^{hyper}$ has dimension ≥ 2 with equality holding if H has no decompositions into moving classes, e.g. if $\text{Pic } S \simeq \mathbb{Z}[H]$, by [15, Lemma 5.1].

As for concrete examples of such families on a *general* (in the moduli space) primitively polarized $K3$ surface (S, H) , here are the ones that are known to us:

(i) $|H|$ contains a two-dimensional family of irreducible curves of geometric genus $p_g = 2$, whose general element is nodal, by the nonemptiness of Severi varieties on $K3$ surfaces as a direct consequence of Mumford's theorem on the existence of nodal rational curves on $K3$ surfaces (cf. [30, pp. 351–352] or [2, pp. 365–367]) and standard results on Severi varieties (cf. [39, Lemma 2.4 and Theorem 2.6]; see also e.g. [11,14]). In the particular case $p_a(H) = 3$, i.e. when S is a smooth quartic in \mathbb{P}^3 , the locus in $\text{Hilb}^2(S)$ covered by the associated rational curves is a \mathbb{P}^1 -bundle over S , by [15, Example 7.6].

(ii) $|H|$ contains a two-dimensional family of irreducible nodal curves of geometric genus $p_g = 3$ with hyperelliptic normalizations, by [15, Theorem 5.2]. The locus in $\text{Hilb}^2(S)$ covered by the associated rational curves is birational to a \mathbb{P}^1 -bundle, by [15, Corollary 5.3 and Proposition 3.6(ii)].

(iii) [15, Proposition 7.7] If $H^2 = 2(d^2 - 1)$ for some integer $d \geq 1$, then $\text{Hilb}^2(S)$ contains a uniruled 3-fold that is birational to a \mathbb{P}^1 -bundle. The fibers give rise to a two-dimensional family of curves in $|H|$ with hyperelliptic normalizations of arithmetic genus $p_a = p_a(H) = d^2$ and geometric genus $p_g = 2d - 1$.

(iv) [15, Proposition 7.2] If $H^2 = 2(m^2 - 9m + 19)$ for some integer $m \geq 6$, then $\text{Hilb}^2(S)$ contains a \mathbb{P}^2 and the Severi varieties of rational curves in $|\mathcal{O}_{\mathbb{P}^2}(n)|$, for any $n \geq 1$, give rise to $(3n - 1)$ -dimensional subschemes $V(n) \subset |nH|$ parametrizing irreducible curves with hyperelliptic normalizations of arithmetic genus $p_a(n) = p_a(nH) = n^2(m^2 - 9m + 19)$ and geometric genus $p_g(n) = 2n - 9$.

We have now seen several examples of families as in (1) of dimension 2 on surfaces with $p_g(S) > 0$ (in Examples 2.7 and 2.9(iv) with $n = 1$, Examples 2.9(i)–(iii) and the abelian surfaces in Lemma–Example 2.6) and on Enriques surfaces (the case mentioned in the last lines of Example 2.8).

At the same time we have seen infinite series of examples of families as in (1) of arbitrarily high dimensions ≥ 3 (Examples 2.7 and 2.9(iv) with $n \geq 2$, and Examples 2.4 and 2.8).

In the next section we will see the difference between those “small” and “big” families.

3. A REIDER-LIKE RESULT

Consider the additional assumptions

$$(6) \quad \dim V \geq \begin{cases} 3, & \text{if } p_g(S) > 0 \text{ or } S \text{ is Enriques,} \\ 5, & \text{otherwise.} \end{cases}$$

The following result is an improvement of [15, Proposition 4.2]. In fact, the idea of the proof is essentially the same.

Proposition 3.1. *Assume (1) and (6). Then there is a decomposition into two effective, algebraically moving classes*

$$[C] = [D_1] + [D_2]$$

such that, for general $\xi, \eta \in R_V$ (cf. (2)), each with support at two distinct points of S , there are effective divisors $D'_1 \sim_{\text{alg}} D_1$ and $D'_2 \sim_{\text{alg}} D_2$ such that $\xi \in D'_1$ and $\eta \in D'_2$ and $[D'_1 + D'_2] \in \bar{V}$, where \bar{V} is the closure of V in the component of the Hilbert scheme of S containing V .

Proof. We must have $\dim R_V = 2$ or 3 by (6). If $p_g(S) > 0$, then $\dim V \geq 3$ by (6), whence R_V is a surface by Proposition 2.2. If S is Enriques, then there is an

unramified double cover $f: T \rightarrow S$ such that T is a smooth $K3$ surface, cf. e.g. [2, VIII, Lemma 15.1(ii)]. Therefore, also $f^{(2)}: \text{Sym}^2(T) \rightarrow \text{Sym}^2(S)$ is unramified. Hence, the family of rational curves in $\text{Sym}^2(S)$ given by Φ_V and ψ as in (3) is pulled back to two copies in $\text{Sym}^2(T)$. Since $\dim V \geq 3$ by (6), we conclude by Proposition 2.2 that these families only cover a surface in $\text{Sym}^2(T)$. Hence $R_V \subset \text{Sym}^2(S)$ is a surface as well.

Therefore, in any case, the assumptions (6) guarantee that, for general $\xi, \eta \in R_V$, the locus of points in V parametrizing curves in \mathcal{P}_V passing through ξ and η in R_V is at least one-dimensional. For general $\xi, \eta \in R_V$, let $B = B_{\xi, \eta} \subset V$ be a smooth curve (not necessarily complete) parametrizing such curves and

$$(7) \quad \begin{array}{ccccc} & & \tilde{\mathcal{C}}_B & \xrightarrow{\gamma_B} & \mathcal{P}_B & \xrightarrow{\Phi_B} & R_V \subset \text{Sym}^2(S) \\ & \swarrow \pi_B & & \searrow \tilde{\varphi}_B & \downarrow \psi_B & & \\ & S & & & B & & \end{array}$$

the corresponding restriction of (3) over B . Let \bar{B} be any smooth compactification of B . By Mori's bend-and-break technique, as in [27, Lemma 1.9] or [26, Corollary II.5.5] (see also [15, Lemma 2.10] for the precise statement we need), there is an extension of the right hand part of (7)

$$\begin{array}{ccc} \bar{\mathcal{P}}_B & \xrightarrow{\bar{\Phi}_B} & R_V \\ \downarrow \bar{\psi}_B & & \\ \bar{B} & & \end{array}$$

such that, for some $b_0 \in \bar{B} \setminus B$ we have $(\bar{\Phi}_B)_*(\bar{\psi}_B^{-1}b_0) \supseteq \Gamma_\xi + \Gamma_\eta$, where Γ_ξ and Γ_η are irreducible rational curves on R_V (possibly coinciding) such that $\xi \in \Gamma_\xi$ and $\eta \in \Gamma_\eta$. Let now

$$I := \{(x, \xi) \in S \times R_V \mid x \in \text{Supp}(\xi)\} \subset S \times R_V$$

be the incidence variety with projection morphisms $p: I \rightarrow S$ and $q: I \rightarrow R_V$. Then $\dim I = \dim R_V = 2$ or 3 and q is finite of degree two. Consider the commutative diagram

$$(8) \quad \begin{array}{ccccc} \bar{\mathcal{P}}_B \times_{R_V} I & =: & \bar{\mathcal{P}}'_B & \xrightarrow{\bar{\Phi}'_B} & I & \xrightarrow{p} & S \\ & & \downarrow q' & & \downarrow q & & \\ & & \bar{\mathcal{P}}_B & \xrightarrow{\bar{\Phi}_B} & R_V & & \\ & & \downarrow \bar{\psi}_B & & & & \\ & & \bar{B} & & & & \end{array}$$

where the square is Cartesian. Define $\bar{\pi}_B := p \circ \bar{\Phi}'_B$. Note that for $b \in B$ we have $\bar{\pi}_B(q'^{-1}\bar{\psi}_B^{-1}b) = \pi_B(\tilde{\varphi}_B^{-1}b)$. In particular, $\bar{\pi}_B$ is dominant and generically one-to-one on the fibers over B . Therefore p must also be dominant.

We have

$$(\bar{\Phi}'_B)_*(q'^{-1}\bar{\psi}_B^{-1}b_0) \supseteq q^{-1}\Gamma_\xi + q^{-1}\Gamma_\eta.$$

Denoting by $b \in B$ a general point and recalling that $\bar{\pi}_B$ is generically one-to-one on the fibers over B , we have

$$\begin{aligned} C \sim_{\text{alg}} (\bar{\pi}_B)_*(q'^{-1}\bar{\psi}_B^{-1}b) &\sim_{\text{alg}} (\bar{\pi}_B)_*(q'^{-1}\bar{\psi}_B^{-1}b_0) \\ &\supseteq p_*(q^{-1}\Gamma_\xi) + p_*(q^{-1}\Gamma_\eta) \supseteq D_\xi + D_\eta, \end{aligned}$$

where $D_\xi := p(q^{-1}\Gamma_\xi)$ and $D_\eta := p(q^{-1}\Gamma_\eta)$. Now the curves contracted by p are precisely the curves of type $\{x, x + D\}$, for a point $x \in S$ and a curve $D \subset S$. Since S is not covered by rational curves, and ξ and η are general, their support on S does not intersect any of the finitely many rational curves γ on S with $\gamma \cdot C \leq C^2$. If $q^{-1}\Gamma_\xi$ contained a component of the form $\{x, x + D\}$, then, by definition of q , we would have

$$q^{-1}\Gamma_\xi = \{x, x + y\}_{y \in D} \cup \{y, x + y\}_{y \in D},$$

a union of two irreducible rational curves, each being mapped isomorphically by p to Γ_ξ . Then $p(\{y, x + y\}_{y \in D}) = D \subset S$ would be an irreducible rational curve intersecting $\text{Supp } \xi$, a contradiction. The same argument works for $q^{-1}\Gamma_\eta$. Therefore, none of the components of $q^{-1}\Gamma_\xi$ nor $q^{-1}\Gamma_\eta$ are contracted by p . We therefore have $D_\xi \supset \xi$ and $D_\eta \supset \eta$, viewing ξ and η as length-two subschemes of S . (Note that D_ξ and D_η are not necessarily distinct.) Moreover,

$$C \sim_{\text{alg}} (\bar{\pi}_B)_*(q'^{-1}\bar{\psi}_B^{-1}b) = D_\xi + D_\eta + E_{\xi, \eta}$$

with $E_{\xi, \eta} \geq 0$, and by construction, $D_\xi + D_\eta + E_{\xi, \eta}$ lies in the border of the family $\varphi: \mathcal{C} \rightarrow V$ of curves on S , and as such, $[D_\xi + D_\eta + E_{\xi, \eta}] \in \bar{V}$, where \bar{V} is the closure of V in the component of the Hilbert scheme of S containing V . Moreover, as the number of such effective decompositions of $[C]$ is finite (as S is projective), we can find one decomposition $[C] = [D_1] + [D_2]$ holding for general $\xi, \eta \in R_V$. Since this construction can be repeated for general $\xi, \eta \in R_V$ and the set $\{x \in S \mid x \in \text{Supp}(\xi) \text{ for some } \xi \in R_V\}$ is dense in S , as the curves parametrized by V cover the whole surface S , the obtained classes D_1 and D_2 must move in an algebraic system of dimension at least one. \square

We now prove an additional, more precise result:

Lemma 3.2. *Assume (1) and (6). Then we can find a decomposition as in Proposition 3.1 satisfying the additional properties*

- (a) $D_1.D_2 \leq p_a - p_g + 2$; and
 (b) there is a reduced and irreducible component of D'_1 (resp. D'_2) containing ξ (resp. η).

Proof. Let ξ and $\eta \in R_V$ be general and $[D'_1 + D'_2] \in \bar{V}$ such that $\xi \subset D'_1$ and $\eta \subset D'_2$ as in Proposition 3.1. Let $D_\xi \subseteq D'_1$, $D_\eta \subseteq D'_2$, $\Gamma_\xi \subset R_V$ and $\Gamma_\eta \subset R_V$ be as in the proof of Proposition 3.1, that is, $D_\xi := p(q^{-1}\Gamma_\xi)$ and $D_\eta := p(q^{-1}\Gamma_\eta)$.

If $q^{-1}\Gamma_\xi$ were reducible, it would consist of two rational components, each being mapped isomorphically to Γ_ξ by q . Therefore, ξ , viewed as a length-two scheme on S , would intersect a rational curve $\gamma \subset S$ satisfying $\gamma.C \leq C^2$. As in the proof of Proposition 3.1, for ξ general this cannot happen. Hence $q^{-1}\Gamma_\xi$ is reduced and irreducible and so is $D_\xi = p(q^{-1}\Gamma_\xi)$ as well. Of course the same reasoning also works to show that $D_\eta = p(q^{-1}\Gamma_\eta)$ is irreducible. This proves (b).

Set $D := D'_1 + D'_2$. We now want to show that there is an effective decomposition $D = D_1 + D_2$ with $D_\xi \subseteq D_1$, $D_\eta \subseteq D_2$ and $D_1.D_2 \leq p_a - p_g + 2$.

We know that a partial desingularization of D , say \tilde{D} , which can be obtained by a succession of blowups $f: \tilde{S} \rightarrow S$, is a limit of smooth hyperelliptic curves, as $[D] \in \bar{V}$ by Proposition 3.1. Let \tilde{D}_ξ and \tilde{D}_η denote the strict transforms of D_ξ and D_η , respectively. We now claim that there is an effective decomposition

$$(9) \quad \tilde{D} = \tilde{D}_1 + \tilde{D}_2 \quad \text{with } \tilde{D}_\xi \subseteq \tilde{D}_1, \tilde{D}_\eta \subseteq \tilde{D}_2 \text{ and } \tilde{D}_1.\tilde{D}_2 \leq 2.$$

To show (9), we first write $\tilde{D} = \tilde{D}_\xi + \tilde{D}'$. We have a short exact sequence

$$0 \rightarrow \omega_{\tilde{D}_\xi} \rightarrow \omega_{\tilde{D}} \rightarrow \omega_{\tilde{D}'}(\tilde{D}_\xi) \rightarrow 0.$$

Since $H^1(\omega_{\tilde{D}_\xi}) \simeq H^1(\omega_{\tilde{D}})$ by Serre duality, the map $H^0(\omega_{\tilde{D}}) \rightarrow H^0(\omega_{\tilde{D}'}(\tilde{D}_\xi))$ is surjective, whence $|\omega_{\tilde{D}'}(\tilde{D}_\xi)|$ is not birational on \tilde{D}_η , since $|\omega_{\tilde{D}}|$ is 2 : 1 on every nonrational component, as \tilde{D} is a limit of hyperelliptic curves.

Let now $\tilde{D}_1 \subseteq \tilde{D}$ be maximal with respect to the properties that $\tilde{D}_\xi \subseteq \tilde{D}_1$, $\tilde{D}_2 := \tilde{D} - \tilde{D}_1 \supseteq \tilde{D}_\eta$ and $|\omega_{\tilde{D}_2}(\tilde{D}_1)|$ is not birational on \tilde{D}_η .

If $\tilde{D}_2 = \tilde{D}_\eta$, then $\tilde{D}_1.\tilde{D}_2 \leq 2$ by [10, Proposition 2.3], and (9) is proved.

Now assume that $\tilde{D}_2 \not\supseteq \tilde{D}_\eta$.

If there is a reduced and irreducible component $\tilde{D}_{22} \subseteq \tilde{D}_2 - \tilde{D}_\eta$ such that $\tilde{D}_{22}.\tilde{D}_1 > 0$, set $\tilde{D}_{21} := \tilde{D}_2 - \tilde{D}_{22} \supseteq \tilde{D}_\eta$. Then from

$$0 \rightarrow \omega_{\tilde{D}_{22}}(\tilde{D}_1) \rightarrow \omega_{\tilde{D}_2}(\tilde{D}_1) \rightarrow \omega_{\tilde{D}_{21}}(\tilde{D}_1 + \tilde{D}_{22}) \rightarrow 0$$

and the fact that $h^1(\omega_{\tilde{D}_{22}}(\tilde{D}_1)) = 0$, we see, as above, that $|\omega_{\tilde{D}_{21}}(\tilde{D}_1 + \tilde{D}_{22})|$ is not birational on \tilde{D}_η , contradicting the maximality of \tilde{D}_1 .

Therefore $\tilde{D}_1.\tilde{D}_2 = \tilde{D}_1.\tilde{D}_\eta$ and since $|\omega_{\tilde{D}_2}(\tilde{D}_1)|$ is not birational on \tilde{D}_η , and $H^0(\omega_{\tilde{D}_\eta}(\tilde{D}_1)) \subseteq H^0(\omega_{\tilde{D}_2}(\tilde{D}_1))$, then $|\omega_{\tilde{D}_\eta}(\tilde{D}_1)|$ is not birational on \tilde{D}_η either. It follows, using [10, Proposition 2.3] again, that

$$\tilde{D}_1.\tilde{D}_2 = \tilde{D}_1.\tilde{D}_\eta \leq 2,$$

and (9) is proved.

Now let E_1, \dots, E_n be the (total transforms of the) exceptional divisors of $f: \tilde{S} \rightarrow S$, so that $K_{\tilde{S}} = f^*K_S + \sum E_i$, $\tilde{D}_1 = f^*D_1 - \sum \alpha_i E_i$ and $\tilde{D}_2 = f^*D_2 - \sum \beta_i E_i$, for $\alpha_i, \beta_i \geq 0$, where $D_1 \supseteq \xi$ and $D_2 \supseteq \eta$. We compute

$$\begin{aligned} 2p_g - 2 &= (\tilde{D}_1 + \tilde{D}_2) \cdot (\tilde{D}_1 + \tilde{D}_2 + K_{\tilde{S}}) \\ &= (D_1 + D_2) \cdot (D_1 + D_2 + K_S) \\ &\quad - 2 \sum \alpha_i \cdot \beta_i + \sum (\alpha_i(1 - \alpha_i) + \beta_i(1 - \beta_i)) \\ &\leq D \cdot (D + K_S) - 2 \sum \alpha_i \cdot \beta_i = 2p_a - 2 - 2 \sum \alpha_i \cdot \beta_i, \end{aligned}$$

whence $\sum \alpha_i \cdot \beta_i \leq p_a - p_g$. Inserting this into

$$\tilde{D}_1 \cdot \tilde{D}_2 = \left(f^*D_1 - \sum \alpha_i E_i \right) \cdot \left(f^*D_2 - \sum \beta_i E_i \right) = D_1 \cdot D_2 - \sum \alpha_i \cdot \beta_i$$

and using (9), we obtain the desired result $D_1 \cdot D_2 \leq p_a - p_g + 2$. \square

A consequence is the following Reider-like result. Note that when $p_a = p_g$, that is, the family consists of *smooth* hyperelliptic curves, we retrieve the results of Reider [34].

We make the following notation: if $C \subset S$ an irreducible curve with hyperelliptic normalization and $f: \tilde{S} \rightarrow S$ a birational morphism inducing the normalization $v: \tilde{C} \rightarrow C$, then we define

$$W_{[2]}(C) := \{ \xi \in C_{smooth} \mid \xi = f_*(Z) \text{ with } Z \in \mathfrak{g}_2^1(\tilde{C}) \} \subset \text{Hilb}^2(S).$$

Theorem 3.3. *Assume (1) and (6). Then there is an effective divisor D on S such that $h^0(D) \geq 2$, $h^0(C - D) \geq 2$, $D^2 \leq (C - D)^2$ and*

$$(10) \quad 0 \leq 2D^2 \stackrel{(i)}{\leq} D \cdot C \leq D^2 + p_a - p_g + 2 \stackrel{(ii)}{\leq} 2(p_a - p_g + 2),$$

with equalities in (i) or (ii) if and only if $C \equiv 2D$.

Furthermore there is a flat family parametrized by a reduced and irreducible complete subscheme V_D of the component of the Hilbert scheme of S containing $[D]$ with the following property: for general $[C] \in V$ there is a complete rational curve $V_D(C) \subseteq V_D$ such that for general $\xi \in W_{[2]}(C)$, there is a $[D_\xi] \in V_D(C)$ such that $\xi \subset D_\xi$.

Proof. We have, by Proposition 3.1 and Lemma 3.2, a decomposition into algebraically moving classes $C \sim_{alg} D_1 + D_2$ with $D_1 \cdot D_2 \leq p_a - p_g + 2$. Without loss of generality we can assume that $D_1^2 \leq D_2^2$, or equivalently $D_1 \cdot C \leq D_2 \cdot C$. We first show that we can assume that $D_1^2 \geq 0$.

Indeed, if $D_1^2 < 0$, then the algebraic system in which D_1 moves must have a base component $\Gamma > 0$. We can write $D_1 \sim_{alg} D_0 + \Gamma$, where D_0 moves in an algebraic system of dimension at least one, without base components. In particular $0 \leq D_0^2 =$

$D_1^2 - 2D_1 \cdot \Gamma + \Gamma^2 < \Gamma^2 - 2D_1 \cdot \Gamma$, so that $\Gamma^2 > 2D_1 \cdot \Gamma$. Moreover, we have $\Gamma \cdot D_2 \geq -\Gamma \cdot D_1$ as C is nef. Hence

$$\begin{aligned} D_0 \cdot (D_2 + \Gamma) &= D_1 \cdot D_2 - \Gamma \cdot D_2 + \Gamma \cdot D_1 - \Gamma^2 \\ &\leq D_1 \cdot D_2 + 2\Gamma \cdot D_1 - \Gamma^2 < D_1 \cdot D_2, \end{aligned}$$

and we can substitute D_1 with $D_0 \subseteq D_1$, as clearly $\text{Supp}(\xi) \cap \Gamma = \emptyset$ for general $\xi \in R_V$. Therefore, we can assume $D_1^2 \geq 0$.

Combining with the Hodge index theorem, we get $(2D_1 \cdot C) \cdot D_1^2 \leq C^2 \cdot D_1^2 \leq (D_1 \cdot C)^2$, so that $2D_1^2 \leq D_1 \cdot C$, with equality if and only if $C \equiv 2D_1$. Moreover, $D_1^2 \leq \frac{1}{2}D_1 \cdot C = \frac{1}{2}(D_1^2 + D_1 \cdot D_2) \leq \frac{1}{2}(D_1^2 + p_a - p_g + 2)$ yields $D_1^2 \leq p_a - p_g + 2$, again with equality if and only if $C \equiv 2D_1$.

Finally, from Proposition 3.1 and the fact that we have at most removed base components of the obtained family, it is clear that there is a reduced and irreducible complete scheme V_{D_1} parametrizing curves algebraically equivalent to D_1 with the property that for general $[C] \in V$ and general $\xi \in W_{[2]}(C)$, there is a $[D_\xi] \in V_{D_1}$ such that $\xi \subset D_\xi$. For fixed C this yields a complete curve $V_{D_1}(C) \subseteq V_{D_1}$ such that all $[D_\xi] \in V_{D_1}(C)$ for general $\xi \in W_{[2]}(C)$. This gives a natural rational map $C \dashrightarrow V_{D_1}(C)$ inducing a morphism between the normalizations $\tilde{C} \rightarrow \tilde{V}_{D_1}(C)$ that is composed with the hyperelliptic double cover $\tilde{C} \rightarrow \mathbb{P}^1$. Hence $V_{D_1}(C)$ admits a surjective map from \mathbb{P}^1 and is therefore rational.

If $h^0(D_1) = 1$, then the variety parametrizing curves algebraically equivalent to D_1 is abelian and therefore cannot contain rational curves, cf. e.g. [7, Proposition 4.9.5]. Hence $h^0(D_1) \geq 2$.

Substituting D_1 with D_2 we also obtain $h^0(D_2) \geq 2$, thus finishing the proof. \square

In particular, we have a slight improvement of [15, Corollary 4.7]:

Corollary 3.4. *Assume (1) and in addition that there is no decomposition $C \sim_{alg} C_1 + C_2$ such that $h^0(\mathcal{O}_S(C_i)) \geq 2$ for $i = 1, 2$.*

Then $\dim V \leq 2$ if $p_g(S) > 0$ and $\dim V \leq 4$ otherwise.

The conditions in Corollary 3.4 are for instance satisfied if $\text{NS}(S) \simeq \mathbb{Z}[C]$. Theorem 3.3 gives additional restrictions on the existence of such a family as in (1) and (6). In particular it shows that when the difference $\delta := p_a - p_g$ is “small”, then such a family cannot exist unless there are some quite special divisors on the surface (cf. also [16, Theorem 1], where we in fact show the nonexistence of curves with hyperelliptic normalizations in the primitive linear system $|H|$ with $\delta \leq \frac{p_a - 3}{2}$ on a general primitively polarized $K3$ surface (S, H)). Unlike the results of Reider, Theorem 3.3 cannot be used to say that if $[C] \equiv mL$ for some $m \gg 0$, then families as in (6) do not occur, as p_a grows quadratically with m , but it shows that the difference $\delta := p_a - p_g$ must get bigger as m grows. This was already seen in Examples 2.7 and 2.9(iv) above.

Note that the families in Examples 2.7 and 2.9(iv) for $n \geq 2$ and in Example 2.8 satisfy the conditions (1) and (6) and that the conditions in Theorem 3.3 are satisfied for $D = H$ in Examples 2.7 and 2.9(iv) and for $D = 2E_1$ in Example 2.8.

Also note that there is no divisor (in general) satisfying the conditions in Theorem 3.3 in the two-dimensional families given in Examples 2.7–2.9 and Lemma–Example 2.6. We therefore see that the conditions (6) cannot, in general, be weakened.

4. FURTHER REMARKS

We make the following observation.

Lemma 4.1. *The divisor class D in Theorem 3.3 can be chosen in such a way that if $D^2 = 0$ (resp. $D^2 = 1$ and $D.C$ is odd), then the general curve parametrized by V_D is reduced and irreducible and V_D is a base point free, complete linear pencil with $D.C$ even (resp. V_D is a complete linear pencil with one base point x that is a point of every $[C] \in V$).*

Proof. Assume that either $D^2 = 0$ or that $D^2 = 1$ and $D.C$ is odd. Also assume that the general $[D] \in V_D$ is of the form $D = D' + D''$ with $\xi \subset D'$. Since the number of effective decompositions of $[D]$ in $\text{NS}(S)$ is finite, we can in fact assume that all such D' and D'' are algebraically equivalent. For the same reason we can, possibly after moving base components from D' to D'' , assume that D' is nef, in particular that $D'^2 \geq 0$. Since $D'^2 = D^2 - 2D.D'' + D''^2$, we then get $D''^2 \geq 2D.D'' - D^2$. Moreover, we have $D''.(C - D) \geq -D''.D$ as C is nef. Hence

$$\begin{aligned} D'.(C - D + D'') &= D.(C - D) - D''.(C - D) + D''.D - D''^2 \\ &\leq D.(C - D) + 2D''.D - D''^2 \leq D.(C - D) + D^2, \end{aligned}$$

so that $D'.(C - D + D'') \leq D.(C - D)$ unless when $D^2 = 1$, $D''^2 = 2D.D'' - 1$ and $D''.(C - D) = -D''.D$ (whence $D''.C = 0$). It follows that $D'^2 = 0$, so that $D'.(C - D) \leq \delta + 2$, unless $D^2 = 1$ and $D.C = \delta + 3$, in which case $D'.C = \delta + 3$.

This proves that the general curve parametrized by V_D can be taken to be reduced and irreducible, possibly upon changing D , when $D^2 = 0$. By Theorem 3.3, $|D|$ is a pencil, and as $D^2 = 0$, we must have that $|D|$ is the whole component of the Hilbert scheme of S containing $[D]$. Therefore $V_D = |D| \simeq \mathbb{P}^1$. Moreover, $D.C$ is the degree of the morphism $\tilde{C} \rightarrow V_D \simeq \mathbb{P}^1$, which is composed with a g_2^1 , so it must be even.

We now treat the case $D^2 = 1$.

We have seen that if the general curve parametrized by V_D is not reduced and irreducible, then $D'^2 = 0$ and $D'.C = D.C = \delta + 3$, which is odd by assumption. But then, arguing as in the case $D^2 = 0$, we get that $D'.C$ must be even, a contradiction. Finally, as the degree of $\tilde{C} \rightarrow \tilde{V}_D(C)$ (where, as above, $\tilde{V}_D(C)$ is the normalization of $V_D(C)$) must be even, the family parametrized by V_D must have base points lying on every curve C parametrized by V . Of course there can only be one base point, as $D^2 = 1$, and the result follows by blowing up S at this base point and using the result for $D^2 = 0$. \square

An additional application of the Hodge index theorem yields the following, which says that in fact when $\delta := p_a - p_g$ is “small” the only cases of families of irreducible curves with hyperelliptic normalizations of large dimensions arise in fact from $2 : 1$ rational maps as explained in the introduction and seen in Example 2.7, or from elliptic fibrations on the surface (whence $\text{kod}(S) \leq 1$).

Corollary 4.2. *Assume (1) and (6), set $\delta := p_a - p_g$ and assume furthermore that $C^2 > (\delta + 3)^2$ or that $C^2 = (\delta + 3)^2$ and $C \not\equiv (\delta + 3)C_0$ for any divisor C_0 (resp. the intersection form on S is even, and either $C^2 > \frac{(\delta+4)^2}{2}$ or $C^2 = \frac{(\delta+4)^2}{2}$ and $C \not\equiv \frac{\delta+4}{2}C_0$ for any divisor C_0).*

Then there is a linear pencil $|D|$, whose general member is a smooth, irreducible elliptic or hyperelliptic curve such that $D^2 = 0$, $D.C$ is even and $D.C \leq \delta + 2$.

Furthermore, in the case when the general member of $|D|$ is hyperelliptic (which is the case if $\text{kod}(S) = 2$), there is a $2 : 1$ rational map $S \rightarrow R$ to a smooth rational surface and the family parametrized by V is the pullback of a family of irreducible, rational curves on R .

Proof. Let D be as in Theorem 3.3 and Lemma 4.1. If $D^2 > 0$, then the Hodge index theorem yields

$$C^2 \leq \frac{(D^2 + \delta + 2)^2}{D^2} \leq (\delta + 3)^2,$$

with equality implying $C \equiv (\delta + 3)D$, contradicting our hypotheses. The same reasoning works if the intersection form on S is even, so that $D^2 \geq 2$.

If $D^2 = 0$, then by Lemma 4.1 $|D|$ is a base point free complete pencil, and for general $[C] \in V$ and general $\xi \in W_{[2]}(C)$, there is a $D_\xi \in |D|$ such that $\xi \subset D_\xi$. Therefore, the induced morphism $\tilde{C} \rightarrow |D| \simeq \mathbb{P}^1$, where \tilde{C} denotes the normalization of C , of degree $D.C$, is composed with the hyperelliptic double cover $\tilde{C} \rightarrow \mathbb{P}^1$. In particular, $D.C$ is even.

By construction, the general element of $|D|$ has a partial desingularization that admits a $2 : 1$ map or $1 : 1$ map onto a \mathbb{P}^1 , as it is a component of a limit of smooth rational curves. Since $\text{kod}(S) \geq 0$, it cannot be rational, so it is a smooth, irreducible elliptic or hyperelliptic curve, as $|D|$ is base point free. In the latter case $S \rightarrow \mathbb{P}(\pi_*(\omega_{S/|D|}))$ is the desired $2 : 1$ rational map, since it maps all members of V generically $2 : 1$ onto irreducible, rational curves. \square

Example 4.3. *Assume (1) and (6) with $\text{NS}(S) \simeq \mathbb{Z}[H]$ such that $C \equiv mH$ with $m \geq 2$. We must have $(m - 1)H^2 \leq p_a - p_g + 2$ by Theorem 3.3, whence*

$$C^2 = m^2 H^2 \leq \frac{m^2(p_a - p_g + 2)}{m - 1}.$$

Note that if one can bound C^2 and the difference $\delta = p_a - p_g$ is given, one can obtain dimension bounds on V . This follows from the following general result, which is “folklore”. We include the proof, pointed out to us by C. Ciliberto, for

lack of a suitable reference. The bounds on $\dim V$ obtained by combining the next lemma and Corollary 4.2 are however probably far from being sharp.

Lemma 4.4. *Let $V \subset \text{Hilb } S$ be a reduced and irreducible scheme with $\dim V > 0$ parametrizing a flat family of curves on a smooth projective surface S with $\text{kod}(S) \geq 0$ of constant arithmetic and geometric genera p_a and $p_g \geq 2$, respectively, and algebraic equivalence class C . Then*

$$(11) \quad \dim V \leq \left\lfloor p_g - \frac{1}{2} K_S \cdot C \right\rfloor = \left\lfloor \frac{1}{2} C^2 + 1 - (p_a - p_g) \right\rfloor \leq p_g.$$

In particular, if $\dim V = p_g$, then either $\text{kod}(S) \leq 1$ and $\dim V = 1$, or $\text{kod}(S) = 0$.

Proof. Note that the equality in (11) follows from the adjunction formula.

Denote by C a general curve in the family, \tilde{C} its normalization and $f: \tilde{C} \rightarrow S$ the natural morphism. Then we have a short exact sequence

$$(12) \quad 0 \rightarrow \mathcal{T}_{\tilde{C}} \xrightarrow{df} f^* \mathcal{T}_S \rightarrow \mathcal{N}_f \rightarrow 0,$$

defining the *normal sheaf* \mathcal{N}_f to f . Let $T \subset \mathcal{N}_f$ be the torsion subsheaf and $\bar{\mathcal{N}}_f := \mathcal{N}_f/T$, which is locally free on \tilde{C} . The sections $H^0(T) \subset H^0(\mathcal{N}_f)$ vanish at the generic point of \tilde{C} , cf. [1, Section 6], whence the tangent space at $[C]$ of the equigeneric family in which C moves, which corresponds to infinitesimal deformations of f that do not vanish at the generic point of \tilde{C} , maps injectively to $H^0(\bar{\mathcal{N}}_f)$, cf. also [36, Section 3.4.3]. It follows that $\dim V \leq h^0(\bar{\mathcal{N}}_f)$.

From (12) we have $\deg \bar{\mathcal{N}}_f = \deg \mathcal{N}_f - \text{length } T = -K_S \cdot C + 2p_g - 2 - \text{length } T$.

If now $h^1(\mathcal{N}'_f) = 0$, then by Riemann and Roch

$$(13) \quad \begin{aligned} h^0(\bar{\mathcal{N}}_f) &= \deg \bar{\mathcal{N}}_f + 1 - p_g = -K_S \cdot C + p_g - 1 - \text{length } T \\ &\leq -K_S \cdot C + p_g - 1. \end{aligned}$$

If $h^1(\bar{\mathcal{N}}_f) > 0$, then by Clifford's theorem we have

$$(14) \quad \begin{aligned} h^0(\bar{\mathcal{N}}_f) &\leq \frac{1}{2} \deg \bar{\mathcal{N}}_f + 1 = \frac{1}{2} (-K_S \cdot C + 2p_g - 2 - \text{length } T) + 1 \\ &\leq -\frac{1}{2} K_S \cdot C + p_g. \end{aligned}$$

Let $\varphi: S \rightarrow S_0$ denote the morphism to the minimal model of S .

If $K_S \cdot C < 0$, then C must be an exceptional curve of φ , so that $K_S \cdot C = C^2 = -1$ and C is a smooth rational curve and $\dim V = 0$, a contradiction.

If $K_S \cdot C \geq 0$, then (11) follows from (13) and (14). Furthermore, if $\dim V = p_g$, then $K_S \cdot C = 0$, so that by adjunction we have $2p_g - 2 \leq 2p_a - 2 = C^2$. Moreover, we must have $C = \varphi^* C_0$ for an irreducible curve $C_0 \subset S_0$ and $K_{S_0} \cdot C_0 = 0$.

Since $\dim V > 0$ by assumption, we must have $C_0^2 \geq 0$ and the Hodge index theorem implies $K_{S_0}^2 = 0$, whence $\text{kod}(S) \leq 1$. Furthermore, if $\dim V \geq 2$, then $C_0^2 \geq 2$, so that $K_{S_0} \equiv 0$, whence $\text{kod}(S) = 0$. \square

We conclude the paper by writing out the results in the two simplest cases $\delta = p_a - p_g = 0$ and 1.

4.1. The case $\delta = p_a - p_g = 0$

Assume (1) and (6) with $p_a = p_g$. This means that V parametrizes a flat family of smooth irreducible curves. We get an effective divisor D as in Theorem 3.3 and Lemma 4.1 and we now consider the various possibilities occurring.

We first show that the case $D^2 = 1$ and $C \equiv 2D$ cannot happen.

Indeed, in this case we would have $\dim V = 3$ by Lemma 4.4, and by

$$(15) \quad 0 \rightarrow \mathcal{O}_S(D - C) \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0,$$

and the fact that $D - C \equiv -D$ and D is big and nef, as C is nef, we get $h^0(D) = h^0(\mathcal{O}_C(D)) = h^0(g_2^1) = 2$, so that $\bar{V}_D = |D|$ is a pencil. It has to have one base point, say x . But then $|D|$ cuts out a g_1^1 on any curve numerically equivalent to C passing through x . But, as curves algebraically equivalent to C form a family of dimension at least $\dim V = 3$, the family of such curves through x has dimension at least 2, and the surface is uniruled, a contradiction as $\text{kod}(S) \geq 0$.

Here is a list of the other possibilities.

Case $D^2 = 0$ and $D.C = 2$

Then $|D|$ is a linear pencil cutting out a g_2^1 on every smooth curve numerically equivalent to C . As in Corollary 4.2, either this is an elliptic pencil (whence $\text{kod}(S) \leq 1$), or there is a $2 : 1$ rational map $S \rightarrow R$ to a smooth rational surface and the family parametrized by V is the pullback of a family of irreducible, rational curves on R .

Case $D^2 = 1$ and $D.C = 3$

By Lemma 4.1, $|D|$ is a pencil with one base point x lying on every curve parametrized by V , and thus cuts out the g_2^1 on every member of V , and in fact on any smooth curve numerically equivalent to C passing through x .

As $(C - D)^2 \geq D^2$ by Theorem 3.3, we can only have

$$C^2 = 7, 8 \text{ or } 9, \quad \text{with } C \equiv 3D \text{ if } C^2 = 9,$$

the latter by the Hodge index theorem.

By Lemma 4.4 we have $\dim V \leq 4$ if $C^2 = 7$ and $\dim V \leq 5$ if $C^2 = 8$ or 9.

Blowing up at x , we reduce to the case above. As $K_S.D \geq 0$, since $\text{kod}(S) \geq 0$ and D moves, we must have $p_a(D) \geq 2$, so that we can conclude that there is a $2 : 1$ rational map $S \rightarrow R$ to a smooth rational surface and the family parametrized by V is the pullback of a family of irreducible, rational curves on R .

Case $D^2 = 2$ and $C \equiv 2D$

By Lemma 4.4 we have $\dim V \leq 5$. Moreover, by (15) we have $h^0(\mathcal{O}_S(D)) = h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_{C'}(D))$ for any $[C] \in V$ and any $C' \equiv C$.

Let $|M|$ be the moving part of $|D|$. Note that either $M.C = 4$ or $M.C = 2$, and the latter implies $M^2 = 0$ by the Hodge index theorem, so that we can reduce to the case treated above. We can and will therefore assume that $M.C = 4$. By the same reasoning, we can assume that $|M|$ is not composed with a pencil. By the Hodge index theorem, $M^2 \leq 2$ with equality implying $M \sim D$.

If $h^0(M) = 2$, then, by Theorem 3.3, $|M|$ is a pencil with two base points x and y (possibly infinitely near) lying on every curve parametrized by V , and thus cuts out the g_2^1 on every member of V , and in fact on any smooth curve numerically equivalent to C passing through x and y . It follows that $M^2 \geq 2$, whence $M \sim D$. Blowing up at x and y , we find as above that there is a $2 : 1$ rational map $S \rightarrow R$ to a smooth rational surface and the family parametrized by V is the pullback of a family of irreducible, rational curves on R .

If $h^0(M) \geq 3$, then we must in fact have $h^0(M) = 3$ and $M^2 = 2$ by Lemma 4.4, whence again $M \sim D$. We have $h^0(\mathcal{O}_{C'}(D)) = 3$ for any $C' \in |C|$. Therefore $|\mathcal{O}_{C'}(D)|$ is a g_4^2 , so that in fact every smooth curve in $|C|$ is hyperelliptic. It follows from Lemma 4.4 that $|D|$ is base point free, so that it defines a $2 : 1$ morphism $S \rightarrow \mathbb{P}^2$, such that V is the pullback of a subfamily of the 5-dimensional family of conics in \mathbb{P}^2 , as $C \equiv 2D$.

Of course, this is just Reider's result [34] (under stronger hypotheses) if $C^2 \geq 9$. For $C^2 < 9$ the results seem to be new with respect to the existing results in [6,37, 38] in the sense that these papers always assume ampleness or very ampleness of $\mathcal{O}_S(C)$. (But of course, we have the additional hypotheses on the dimension of the family and on $\text{kod}(S)$.)

We hope in any case that the reader may find this treatment of interest because of the completely different approach than the other papers. In particular, we have obtained a "vector bundle-free" Reider-like result.

4.2. The case $\delta = 1$

Assume (1) with $p_g = p_a - 1$.

We first note that an immediate consequence of Corollary 4.2 is that if $C^2 > 16$ or $C^2 = 16$ and C is not 4-divisible in $\text{NS}(S)$, and there is no pencil $|D|$ such that $D.C = 2$ (in which case *all* curves numerically equivalent to C , smooth or not, would carry a g_2^1 , meaning a $2 : 1$ finite map onto \mathbb{P}^1), then $\dim V \leq 2$ if $p_g(S) > 0$ or S is Enriques and $\dim V \leq 4$ otherwise.

Under the assumptions (6) we get an effective divisor D as in Theorem 3.3 and Lemma 4.1. We now consider the various possibilities occurring.

As above, the case $D^2 = 1$ and $C \equiv 2D$ cannot happen. Moreover, the case $D^2 = 0$ and $D.C = 3$ does not happen by Lemma 4.1.

The cases $(D^2, D.C) = (0, 2)$ and $(1, 3)$ can be reduced to the case $\delta = 0$, in the sense that the curves parametrized by V already have g_2^1 s.

The new cases are:

Case $D^2 = 2$ and $C \equiv 2D$

If $h^0(\mathcal{O}_C(D)) = 3$, then we can also reduce to the case $\delta = 0$. By Lemma 4.4 we have $\dim V \leq 4$.

Case $D^2 = 2$ and $D.C = 5$

By the Hodge index theorem, we have $C^2 \leq 12$ and by Lemma 4.4 we have $\dim V \leq 6$.

Case $D^2 = 3$ and $C \equiv 2D$

By Lemma 4.4 we have $\dim V \leq 6$.

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