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Fixed Interval Smoothing for Nonlinear Continuous Time Systems*

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An equation is derived for the probability density of the state of a nonlinear dynamical system, conditioned on measurements over a fixed interval. In deriving the equation, the conditional Fokker Planck equation yielding the probability density of the filtering problem is used several times in a novel way.

1. INTRODUCTION

We consider the nonlinear system

$$dx = f(x_t, t)\, dt + G(x_t, t)\, dv, \qquad t \geqslant 0, \tag{1}$$

with measurements

$$dz = h(x_t, t)\, dt + dw, \qquad t \geqslant 0. \tag{2}$$

Here, dv/dt and dw/dt are independent, zero mean, gaussian white noise processes, with covariances $I\delta(t - \tau)$ and $R(t)\,\delta(t - \tau)$, respectively. The matrix $R(t)$ is positive definite for all t. An a priori distribution for the density of x_0 is assumed known, and it is assumed that f, G, h and R all have sufficient smoothness properties to guarantee the usual existence and uniqueness requirements on solutions of (1) and (2), together with such other quantities as will be introduced. In particular, we assume that the conditional density $p(x_t \mid Z_{[0,t]})$ of x_t, given the measurements z_t over $[0, t]$, exists and satisfies the conditional Fokker–Planck equation; see, e.g., Jazwinski (1970).

In this paper, we aim to give a differential equation for the probability density $p(x_t \mid Z_{[0,T]})$, with T fixed. This is the probability density associated with the fixed-interval smoothing problem.

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Earlier results on fixed-interval smoothing may be found in Striebel (1965) (which are less complete than our results, and a good deal more formal), and Leondes *et al.* (1970). We derive the same basic result as Leondes *et al.* (1970), with, however, much greater economy. Part of this economy is acheived through use of the conditional Fokker–Planck equation, for the conditional filter density $p(x_t \mid Z_{[0,t]})$. Fixed point smoothing is discussed in Lee (1971).

In Section 2, we review the conditional filtering equation, and use it to prove two helpful lemmas. In Section 3, the main result is proved, and we also indicate an equation for the evolution of the mean of an arbitrary function of x_t. Section 4 contains some concluding remarks.

2. The Conditional Fokker–Planck Equation

The conditional Fokker–Planck equation is derived in Jazwinski (1970); for an original reference; see, e.g., Kushner (1962). Let us define the operator $\mathscr{L}(\cdot)$ by

$$\mathscr{L}[\phi(x_t, t)] = -\sum_i \frac{\partial}{\partial x_i}(\phi f) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}[\phi(GG')_{ij}] \tag{3}$$

with the superscript prime denoting matrix transposition. Also, henceforth let us write

$$p_f(\nu) = p_{x_t}(\nu \mid Z_{[0,t]}) \tag{4}$$

and

$$p_s(\nu) = p_{x_t}(\nu \mid Z_{[0,T]}). \tag{5}$$

Quantities such as $p_{x_{t+dt}}(\eta \mid Z_{[0,t+dt]})$ will not be abbreviated. Finally, for arbitrary $\phi(x_t, t)$, let

$$\bar{\phi}_f(t) = E[\phi \mid p_f]$$

with $\bar{\phi}_s$ defined obviously.

With this notation, the conditional Fokker–Planck equation then becomes

$$dp_f = \mathscr{L}(p_f)\,dt + (h - \bar{h}_f)'\,R^{-1}(dz - \bar{h}_f\,dt)\,p_f\,, \tag{7}$$

where dp_f and dz have obvious interpretations.

For our derivation of the smoothing equation, we shall require knowledge of the density $p_{x_{t+dt}}(\eta \mid x_t = \nu, dz)$. As shown in the proof of the following lemma, this density follows from (7):

LEMMA 1. *With quantities as defined earlier,*

$$p_{x_{t+dt}}(\eta \mid x_t = \nu, dz) = \delta(\eta - \nu) + \mathscr{L}_\eta(\delta(\eta - \nu))\, dt \tag{8}$$

Proof. Let us apply (7), taking as the initial time t and initial density $p_{x_t}(\eta) = \delta(\eta - \nu)$. Then (7) gives immediately

$$\begin{aligned} p_{x_{t+dt}}(\eta \mid p_{x_t}(\eta) = \delta(\eta - \nu), dz) - p_{x_t}(\eta) \\ = \mathscr{L}_\eta(p_{x_t}(\eta))\, dt + [h(\eta) - E(h(\eta))]'\, R^{-1}[dz - E(h(\eta))]\, p_{x_t}(\eta), \end{aligned}$$

or

$$\begin{aligned} p_{x_{t+dt}}(\eta \mid x_t = \nu, dz) - \delta(\eta - \nu) \\ = \mathscr{L}_\eta[\delta(\eta - \nu)]\, dt + [h(\eta) - h(\nu)]' R^{-1}[dz - h(\nu)]\, \delta(\eta - \nu) \\ = \mathscr{L}_\eta[\delta(\eta - \nu)]\, dt. \end{aligned}$$

The second lemma is concerned with the density $p_{x_t}(\nu \mid Z_{[0,t]}, dz)$, which is a smoothed density, because of the appearance of dz in the conditioning variables.

LEMMA 2. *With quantities as defined earlier,*

$$p_{x_t}(\nu \mid Z_{[0,t]}, dz) = [1 + (h - \bar{h}_f)'\, R^{-1}(dz - \bar{h}_f\, dt)]\, p_{x_t}(\nu \mid Z_{[0,t]}). \tag{9}$$

Proof. Let t be a fixed variable and τ a temporary running variable, and suppose temporarily that for $\tau \geqslant t$, (1) is replaced by

$$dx_\tau = 0.$$

Then $x_{t+dt} = x_t$, and so $p_{x_t}(\nu \mid Z_{[0,t]}, dz)$ will be the same as $p_{x_{t+dt}}(\nu \mid Z_{[0,t]}, dz)$, which is a filtering density. The conditional Fokker–Planck equation then yields (recall that now $f(x_t, t)$ and $G(x_t, t)$ are zero)

$$p_{x_{t+dt}}(\nu \mid Z_{[0,t]}, dz) - p_{x_t}(\nu \mid Z_{[0,t]}) = (h - \bar{h}_f)\, R^{-1}(dz - \bar{h}_f\, dt)\, p_{x_t}(\nu \mid Z_{[0,t]}).$$

Equation (9) is immediate.

3. Derivation of Main Results

The first result which we prove is stated in the following theorem:

THEOREM. *With quantities as defined previously,*

$$dp_s = \left[\mathscr{L}(p_f)\frac{p_s}{p_f} - p_f\mathscr{L}^a\left(\frac{p_s}{p_f}\right)\right] dt, \tag{10}$$

where the operator $\mathscr{L}^a$ is the formal adjoint of $\mathscr{L}$, i.e.,

$$\mathscr{L}^a[\phi(x_t, t)] = \sum_i f_i \frac{\partial \phi}{\partial x_i} + \frac{1}{2}\sum_{i,j} (GG')_{ij} \frac{\partial^2 \phi}{\partial x_i \, \partial x_j}. \tag{11}$$

We shall prove this theorem in several stages.

(1) *A Chapman–Kolmogorov equation.* It is clear that

$$p_{x_t}(\nu \mid Z_{[0,T]}) = \int p_{x_t}(\nu \mid x_{t+dt} = \eta, Z_{[0,T]}) p_{x_{t+dt}}(\eta \mid Z_{[0,T]}) \, d\eta,$$

or, using the Markov nature of the x process,

$$p_s(\nu) = \int \left[p_{x_t}(\nu \mid x_{t+dt} = \eta, Z_{[0,t]}, dz)\left(p_s(\eta) + \frac{\partial p_s(\eta)}{\partial t} dt\right)\right] d\eta. \tag{12}$$

(2) *Evaluation of the integrand in* (12). An expression for $p_{x_t}(\nu \mid x_{t+dt} = \eta, Z_{[0,t]}, dz)$ in terms of more readily manageable quantities follows straightforwardly via Bayes' rule. Evidently,

$$\begin{aligned} p_{x_t}(\nu \mid x_{t+dt} &= \eta, Z_{[0,t]}, dz) \\ &= \frac{p_{x_{t+dt},x_t}(\eta, \nu \mid Z_{[0,t]}, dz)}{p_{x_{t+dt}}(\eta \mid Z_{[0,t+dt]})} \\ &= \frac{p_{x_{t+dt}}(\eta \mid x_t = \nu, Z_{[0,t]}, dz) p_{x_t}(\nu \mid Z_{[0,t]}, dz)}{p_{x_{t+dt}}(\eta \mid Z_{[0,t+dt]})} \\ &= \frac{p_{x_{t+dt}}(\eta \mid x_t = \nu, dz) p_{x_t}(\nu \mid Z_{[0,t]}, dz)}{p_{x_{t+dt}}(\eta \mid Z_{[0,t+dt]})}. \end{aligned} \tag{13}$$

The third equality follows on using the Markovian nature of x. Notice that the two densities in the numerator are replaceable by certain expressions stated in the lemmas of the last section. Notice too that the denominator is expressible in terms of $p_f(\eta)$, using the conditional Fokker–Planck equation.

(3) *Evaluation of the right-hand side of* (12). The manipulations, though intricate, are straightforward.

We write for the various terms in (13) their expansions as given from Section 2, and insert (13) into (12). With suppression of the argument t, we obtain

$$
\begin{aligned}
p_s(\nu) &= \int \frac{\left\{ \begin{aligned} &[\delta(\eta-\nu) + \mathscr{L}_\eta(\delta(\eta-\nu))\,dt][1 + (h(\nu) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)] \\ &\qquad \times p_f(\nu)[p_s(\eta) + (\partial p_s(\eta)/\partial t)\,dt]\,d\eta \end{aligned} \right\}}{[1 + \mathscr{L}_\eta(\cdot)\,dt + (h(\eta) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)]\,p_f(\eta)} \\
&= \int \frac{\left\{ \begin{aligned} &\delta(\eta-\nu)[1 + (h(\nu) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)] \\ &\qquad \times p_f(\nu)[p_s(\eta) + (\partial p_s(\eta)/\partial t)\,dt]\,d\eta \end{aligned} \right\}}{[1 + \mathscr{L}_\eta(\cdot)\,dt + (h(\eta) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)]\,p_f(\eta)} \\
&\quad + p_f(\nu) \int \frac{\mathscr{L}_\eta(\delta(\eta-\nu))\,p_s(\eta)}{p_f(\eta)}\,d\eta\,dt + o(dt). \qquad (14)
\end{aligned}
$$

To evaluate the first integral, replace the term

$$[1 + (h(\nu) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)]\,p_f(\nu)$$

in the numerator of the integrand by the difference of the two terms $[1 + \mathscr{L}_\nu(\cdot)\,dt + (h(\nu) - \bar{h}_f)\,R^{-1}(dz - \bar{h}_f\,dt)]\,p_f(\nu)$ and $\mathscr{L}_\nu(p_f(\nu))\,dt$. The first integral then becomes the difference of two integrals, readily computable to $o(dt)$, and is

$$p_s(\nu) + \frac{\partial p_s(\nu)}{\partial t}\,dt - \frac{p_s(\nu)}{p_f(\nu)}\,\mathscr{L}_\nu(p_f(\nu))\,dt.$$

Further, the definition of $\mathscr{L}^a$ leads to ready evaluation of the second integral:

$$
\begin{aligned}
&p_f(\nu) \int \frac{\mathscr{L}_\eta(\delta(\eta-\nu))\,p_s(\eta)}{p_f(\eta)}\,d\eta\,dt \\
&\qquad = p_f(\nu) \int \delta(\eta-\nu)\,\mathscr{L}_\eta^a\left[\frac{p_s(\eta)}{p_f(\eta)}\right]d\eta\,dt = p_f(\nu)\,\mathscr{L}_\nu^a\left[\frac{p_s(\nu)}{p_f(\nu)}\right]dt.
\end{aligned}
$$

Accordingly, (14) becomes

$$p_s(\nu) = p_s(\nu) + \frac{\partial p_s(\nu)}{\partial t}\,dt - \frac{p_s(\nu)}{p_f(\nu)}\,\mathscr{L}_\nu(p_f(\nu)) + p_f(\nu)\,\mathscr{L}_\nu^a\left[\frac{p_s(\nu)}{p_f(\nu)}\right],$$

or

$$dp_s = \left[\frac{p_s}{p_f}\mathscr{L}(p_f) - p_f\mathscr{L}^a\left(\frac{p_s}{p_f}\right)\right] dt. \tag{10}$$

The boundary condition for this equation is obtained by observing that the smoothing density associated with x_T is the same as the filtered density. In other words, one must solve the conditional Eq. (7) forwards in time, using the measurements and with boundary condition the prescribed density of x_0, and then solve (10) backwards in time, using stored values of the filtered density. Solution of (10) does not however require re-use of the measurements.

From Eq. (10), it is straightforward to obtain an exact equation for the evolution of the conditional mean $\bar{\phi}_s$ of an arbitrary function $\phi(x)$. Multiply (10) by $\phi(x)$ and integrate over the whole of the x_t-space. The result is

$$\begin{aligned} d\bar{\phi}_s &= \left[\int \frac{\phi(x)\,p_s(x)}{p_f(x)}\mathscr{L}_x(p_f(x))\,dx - \int p_f(x)\,\phi(x)\,\mathscr{L}_x^a\left(\frac{p_s(x)}{p_f(x)}\right)dx\right] dt \\ &= \left[\overline{\frac{\phi}{p_f}\mathscr{L}(p_f)}_s - \overline{\phi\mathscr{L}^a\left(\frac{p_s}{p_f}\right)}_f\right] dt \end{aligned} \tag{15}$$

Alternative forms follow by noticing that

$$\overline{\frac{\phi}{p_f}\mathscr{L}(p_f)}_s = \int \mathscr{L}^a\left(\frac{\phi p_s}{p_f}\right)p_f\,dx = \overline{\mathscr{L}^a\left(\frac{\phi p_s}{p_s}\right)}_f \tag{16}$$

and

$$\overline{\phi\mathscr{L}^a\left(\frac{p_s}{p_f}\right)}_f = \int \mathscr{L}(p_f\phi)\frac{p_s}{p_f}\,dx = \overline{\frac{1}{p_f}\mathscr{L}(p_f\phi)}_s. \tag{17}$$

4. Concluding Remarks

As with nonlinear filtering, the exact equations of nonlinear smoothing are impractical to use. Storage of $p_f(x_t)$ for all x_t, together with its derivatives (which are required in (10)) is a task which would be well nigh impossible. In Leondes *et al.* (1970), approximate equations are derived for the mean and covariance of x, under the assumptions that (1) $f(x_t, t]$ can be represented by a Taylor series expansion around $E(x_t \mid Z_{[0,T]})$ up to the quadratic term, (2) the filtered density is Gaussian, with known mean and covariance and (3) third central moments associated with the smoothed estimate are negligible. These equations become exact for the standard linear-gaussian problem. A problem which suggests itself is application of the approximation technique of Kushner (1967), to the smoothing problem; the first two assumptions noted above are common in filtering problems and, as pointed out in Kushner (1967), may be quite inadequate.

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