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Inverse Boundary Value Problems and a Theorem of Gel'fand and Levitan

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This report concerns two so-called inverse problems of mathematical physics. These are: (i) the problem of determining a second-order differential operator (in a normal form) on the half-axis from its spectral function; and, (ii) the problem of determining a hyperbolic boundary value problem of a special form in a (non-characteristic) half-plane from its response on the boundary to a unit impulse at some reference time $t = 0$ (boundary value of the Riemann function). We solve problem (ii) by a natural approach, and then indicate how the solution of problem (i) follows from the solution of problem (ii). Our solution of problem (ii) is constructive, and we obtain stability of the solution under perturbation of the data, in a well-defined sense. For problem (i), we obtain the well-known result of Gel'fand and Levitan, in the sharp formulation given by Levitan and Gasymov ([6]).

1. INTRODUCTION

This report concerns two so-called inverse problems of mathematical physics. These are: (i) the problem of determining a second-order differential operator (in a normal form) on the half-axis from its spectral function; and, (ii) the problem of determining a hyperbolic boundary value problem of a special form in a (non-characteristic) half-plane from its response on the boundary to a unit impulse at some reference time $t = 0$ (boundary value of the Riemann function). We solve problem (ii) by a natural approach, and then indicate how the solution of problem (i) follows from the solution of problem (ii). Our solution of problem (ii) is constructive, and we obtain stability of the solution under perturbation of the data, in a well-defined sense.

For problem (i), we obtain the well-known result of Gel'fand and Levitan, in the sharp formulation given by Levitan and Gasymov ([6]).

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2. Notation and Statement of Results

We will write $\mathbf{R}^+ = [0, \infty) \subset \mathbf{R}$ throughout. We use the common notations $C^k(U), C^\infty(U), C_0^\infty(U), \mathscr{E}_k'(U), \mathscr{E}'(U), \mathscr{D}'(U)$ for the space of k-times, respectively infinitely differentiable functions, respectively those with compact support, with the usual Frechet topologies, and their strong duals. Here $U \subset \mathbf{R}^n$ may be closed or open.

We denote by $W_{\text{loc}}^{m,1}(U)$ the collection of functions in $C^{m-1}(U)$ whose mth partial derivatives, which *a priori* exist as distributions, may be identified with locally absolutely integrable functions on $U \subset \mathbf{R}^n$, i.e. lie in $L_{\text{loc}}^1(U)$. We give $W_{\text{loc}}^{m,1}(U)$ its usual Fréchet topology.

If $U, V \subset \mathbf{R}$ are open sets, denote by $\mathscr{W}_{\text{loc}}^{1,1}(U \times V)$ the space of continuous functions on $U \times V$ whose (distributional) partial derivative in the first (second) variable may be identified with a continuous function of the second (first) variable with values in $L_{\text{loc}}^1(U)$ $(L_{\text{loc}}^1(V))$. According to Fubini's theorem,

$$\mathscr{W}_{\text{loc}}^{1,1}(U \times V) \subset W_{\text{loc}}^{1,2}(U \times V).$$

It is clear how to define $\mathscr{W}_{\text{loc}}^{1,1}(Q)$ for an arbitrary open set $Q \subset \mathbf{R}^2$, since the definition is local, and any $p \in Q$ has a product neighborhood. On the other hand, suppose Q is closed with smooth boundary, and let $f \in C^0(Q) \cap \mathscr{W}_{\text{loc}}^{1,1}(\text{int}\, Q)$. Let $p \in \partial Q$, and select $U, V \subset \mathbf{R}$ open so that $U \times V$ is a neighborhood of p in $\mathbf{R}^2$. Then for each $x \in U (y \in V)$, extend the partial derivative $D_2 f(x, \cdot)$ $(D_1 f \cdot, y))$ to a distribution on $V(U)$ by requiring

$$\langle D_2 f(x, \cdot), \phi\rangle = \langle D_2 f(x, \cdot); \phi\,|_{V_x}\rangle \qquad \text{for } \phi \in C_0^\infty(V)$$

$$(\langle D_1 f(\cdot, y), \phi\rangle = \langle D_1 f(\cdot, y), \phi\,|_{U_y}\rangle \qquad \text{for } \phi \in C_0^\infty(U))$$

where $V_x = \{y \in V: (x, y) \in Q\}$ $(U_y = \{x \in U: (x, y) \in Q\})$. We declare that $f \in \mathscr{W}_{\text{loc}}^{1,1}(Q)$ if and only if, for each choice of p, U, V as above, each $x \in U (y \in V)$, the partial derivative $D_2 f(x, \cdot)$ $(D_1 f(\cdot, y))$ may be identified with a locally integrable function on $V(U)$, and the map $x \mapsto D_2 f(x, \cdot)$ $(y \mapsto D_1 f(\cdot, y))$ is continuous, i.e. lies in $C^0(U; L_{\text{loc}}^1(V))$ $(C^0(V; L_{\text{loc}}^1(U))$.

The topology on $\mathscr{W}_{\text{loc}}^{1,1}(Q)$ is given by the $C^0(Q)$-seminorm and the local norms of the form

$$\sup_{x \in K} \int_L dy \,|\, D_2 f(x, y)| \,, \qquad \sup_{y \in L} \int_K dx \,|\, D_1 f(x, y)|$$

with $K, L \subset \mathbf{R}$ compact.

Finally, $f \in \mathscr{W}_{\text{loc}}^{m,1}(Q)$, $m \geqslant 1$, if and only if $f \in C^{m-1}(Q)$ and all $m - 1$st order partial derivatives of f are in $\mathscr{W}_{\text{loc}}^{1,1}(Q)$. The topology on $\mathscr{W}_{\text{loc}}^{m,1}(Q)$ is defined in a similar way to that of $\mathscr{W}^{1,1}_{\mathrm{loc}}(Q)$. We note that $\mathscr{W}^{m,1}_{\mathrm{loc}}(Q) \subset W^{m,1}_{\mathrm{loc}}(Q)$, and that the topology on $\mathscr{W}^{m,1}_{\mathrm{loc}}$ is stronger than that of $W^{m,1}_{\mathrm{loc}}$.

Functions in $\mathscr{W}^{m,1}_{\mathrm{loc}}(Q)$ may be constructed in the following way: suppose $f \in W^{m,1}_{\mathrm{loc}}(\mathbf{R})$. Then, as is easily verified, the function $F: Q \to \mathbf{C}$ defined by

$$F(x, y) = f(ax + by)$$

for some $(a, b) \in \mathbf{R}^2$, lies in $\mathscr{W}^{m,1}_{\mathrm{loc}}(Q)$, for any closed Q with piecewise smooth boundary. In fact, the spaces $\mathscr{W}^{m,1}_{\mathrm{loc}}$ enter the theory developed here in precisely this way.

We shall use various common notations for derivatives and partial derivatives, such as primes, subscripts, D_ν, $\partial/\partial x$, etc., without comment.

Our results provide an alternate route for part of the proof of the following theorem, which is a sharp version, due to Levitan and Gasymov ([6]), of the celebrated theorem of Gel'fand and Levitan ([4]).

THEOREM I. *A nondecreasing function $\rho: \mathbf{R} \to \mathbf{R}$ is the spectral function of a boundary value problem on $\mathbf{R}^+ = [0, \infty)$:*

$$-y'' + (q - \lambda) y = 0$$

$$y'(0) + hy(0) = 0, \qquad h \in \mathbf{R}$$

with $q \in W^m_{\mathrm{loc}}(\mathbf{R}^+)$, if and only if ρ satisfies the conditions:

(i) *the integrals*

$$\int_{-\infty}^{N} d\sigma(\lambda) \cos \lambda^{1/2} x = I_N(x)$$

converge boundedly to functions in $W^{m+1}_{\mathrm{loc}}(\mathbf{R}^+)$, where

$$\sigma(\lambda) = \rho(\lambda) - \frac{2}{\pi} \lambda^{1/2}, \qquad \lambda \geqslant 0$$

$$= \rho(\lambda), \qquad \lambda \leqslant 0.$$

Moreover, $\{I_N: N \in \mathbf{Z}\}$ converges as $N \to \infty$ in $W^{m+1}_{\mathrm{loc}}(\mathbf{R}^+)$ to a function $\tilde{f}$ with $\tilde{f}(0) = h$.

(ii) *Suppose $u \in L^2(\mathbf{R}^+)$ has compact support. Let*

$$\tilde{u}(\lambda) = \int_0^\infty dx\, u(x) \cos(\lambda)^{1/2} x.$$

Then $\tilde{u} \in L^2(\mathbf{R}; d\rho)$, *and*

$$\int_{-\infty}^{\infty} d\rho(\lambda) \mid \tilde{u}(\lambda) \mid^2 = 0$$

if and only if $u \equiv 0$ a.e.

Remark. In (ii) above and for the rest of this paper, "$\cos \lambda^{1/2} x$" denotes the entire function of λ whose value for $\lambda > 0$ is $\cos \lambda^{1/2} x$. In particular, for $\lambda < 0$, $\cos \lambda^{1/2} x \equiv \cosh \mid \lambda \mid^{1/2} x$.

The bulk of this paper is devoted to proving the following two theorems, from which Theorem I follows, and which are of interest in their own right:

THEOREM II. *An even function* $\tilde{f} : \mathbf{R} \to \mathbf{R}$ *is the boundary value of the Riemann function:*

$$\tilde{f}(t) = R(0, t, 0, 0), \qquad t \neq 0$$

for a hyperbolic boundary value problem:

$$u_{tt} - u_{xx} + qu = 0$$

$$u_x(0, t) + hu(0, t) = 0, \qquad t \in \mathbf{R}$$

with $q \in W^m_{\text{loc}}(\mathbf{R}^+)$ *if and only if*

(i) $\tilde{f} \in W^{m+1}_{\text{loc}}(\mathbf{R})$, $\tilde{f}(0) = h$

(ii) *the kernel* $f(s, t) = \frac{1}{2}(\tilde{f}(s + t) + \tilde{f}(s - t))$ *satisfies the condition: for any* $T > 0$, *there exists* $\epsilon(T) > 0$ *so that for all* $u \in L^2([0, T])$,

$$\int_0^T ds \int_0^T dt\, u(s)\, \bar{u}(t) f(s, t) + \int_0^T \mid u \mid^2 \geqslant \epsilon(T) \int_0^T \mid u \mid^2.$$

Remark. An easy compactness argument shows that the condition (ii) above is equivalent to the assertion that

$$\int_0^\infty ds \int_0^\infty dt\, u(s)\, \bar{u}(t) f(s, t) + \int_0^\infty \mid u \mid^2 > 0$$

for all $u \in L^2(\mathbf{R}^+)$ with compact support.

THEOREM III. (i) *The collection of even functions* $\tilde{f} \in W^{m+1,1}_{\text{loc}}(\mathbf{R})$ *defined by condition* (ii) *in the statement of Theorem* II *forms an open set in* $W^{m+1,1}_{\text{loc}}(\mathbf{R})_{\text{even}}$.

(ii) *for any even* $\tilde{f}$ *in* $W_{\mathrm{loc}}^{m+1,1}(\mathbf{R})$, *there are precisely one* $h \in \mathbf{R}$ *and one* $q \in W_{\mathrm{loc}}^{m,1}(\mathbf{R}^+)$ *so that* $\tilde{f}(t) = R(0, t, 0, 0)$, $t \neq 0$, *for the Riemann function* R *of the boundary value problem*

$$u_{tt} - u_{xx} - qu = 0$$
$$u_x(0, t) + hu(0, t) = 0.$$

(iii) *The map* $\tilde{f} \mapsto (h, q)$ *whose existence is implicit in statement* (ii), *is a continuous map from the open set described in statement* (i) *to* $\mathbf{R} \times W_{\mathrm{loc}}^{m,1}(\mathbf{R}^+)$.

Remark. Theorem III is a uniqueness and stability theorem. Theorem I is, of course, more-or-less well known, and Theorem II could be deduced from Theorem I. Our method of proof, however, proceeds by means of an iteration scheme, with error bounds given explicitly in terms of the numbers $\epsilon(T)$ mentioned in Theorem II, and various norms of $\tilde{f}$. In particular, we obtain the stability statement of Theorem III, which seems to be new.

3. Heuristic Discussion of Results

Problem (i) is concerned with the spectral function of a boundary value problem

$$\begin{aligned} -y''(x) + (q(x) - \lambda)\, y(x) = 0, \qquad x > 0, \quad \lambda \in \mathbf{R} \\ y'(0) + hy(0) = 0 \end{aligned} \tag{3.1}$$

where h is some real number. Let $\phi(x, \lambda)$ be the solution to (1) selected by the initial condition

$$\phi(0, \lambda) = 1.$$

The *spectral function* of (1) is a nondecreasing function whose associated Stieltjes measure properly weights the "eigenfunctions" $\phi(\cdot, \lambda)$ in the spectral resolution of the identity for (1), which is concisely written

$$\delta(x - y) = \int_{-\infty}^{\infty} d\rho(\lambda)\, \phi(x, \lambda)\, \phi(y, \lambda). \tag{3.2}$$

In this heuristic discussion, we shall not worry about making precise sense of divergent integrals such as (2); that is done, in any case, in standard textbooks on spectral theory of ordinary differential operators, e.g. [2], Ch. XIII. Nor shall we make precise smoothness assumptions on q.

Problem (i) is: given ρ, find the differential equation (that is, q) and the boundary condition (that is, h), which give rise to ρ. Of course, this involves describing those nondecreasing ρ that arise as spectral functions of problems of type (1).

Problem (i) is a refined version of the *inverse eigenvalue problem:* to construct a differential operator of some special type, *cum* boundary conditions, having a given spectrum. This problem admits a large amount of non-uniqueness in its solution. Since the points of increase of the spectral function of (1) exactly amount to its spectrum, a solution to problem (i) certainly solves the inverse eigenvalue problem. The spectral function also carries normalization information, however, and this additional information makes its solution unique.

For a history of these problems consult [3] and references cited therein.

Problem (ii) is concerned with a hyperbolic boundary value problem

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q(x)\right) u(x, t) = 0 \qquad (x, t) \in \mathbf{R}^+ \times \mathbf{R} \\ &\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0, \qquad t \in \mathbf{R} \end{aligned} \tag{3.3}$$

The *Riemann Function* $R(x, t; x_0, t_0)$ is the solution of the mixed problem obtained by adding to (3.3) the initial conditions

$$\begin{aligned} u(x, t_0) &= \delta(x - x_0) \\ \frac{\partial u}{\partial t}(x, t_0) &= 0. \end{aligned}$$

Then $u(x, t) = R(x, t; x_0, t_0)$. Again, q is "smooth enough", and we do not worry for the moment about the sense in which a distribution satisfies a mixed problem of this sort. In fact, as explained in [5], R is a special distribution with well-defined restrictions to vertical (and horizontal) lines such as $\{x = 0\}$.

We also note that our Riemann Function is a derivative of the object usually called by that name; see [1], Ch. VI, Section 15.

Note that the solution of the inhomogeneous mixed problem

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + q(x)\right) u(x, t) = g(x, t), \qquad x \geqslant 0 \\ &\frac{\partial u}{\partial x}(0, t) - hu(0, t) = 0 \\ &u(x, t_0) = \frac{\partial u}{\partial t}(u, t_0) = 0, \qquad x \geqslant 0 \end{aligned} \tag{3.4}$$

can be represented in the form

$$u(x, t) = \int_{t_0}^{t} d\sigma \int_{t_0}^{\sigma} d\tau \int_{0}^{x} dy\, R(x, t; y, \sigma)\, g(y, \tau) \tag{3.5}$$

(Duhamel's integral). The system (3.4) models certain processes (e.g. nonuniform transmission lines) subject to an imposed force g, in which signals, represented by u, propagate with unit speed, and some boundary conditions are imposed at the "surface" $x = 0$. Then (3.5) represents the "forward" ($t > t_0$) response of the system to the impulse g. If we choose in particular $g(x, t) = \delta(x - x_0, t - t_0)$, then

$$u(x, t) = \int_{t_0}^{t} R(x, t, x_0, \sigma)\, d\sigma, \qquad t > t_0 .$$

We conclude that

$$\int_0^t ds\, \tilde{f}(s) = \int_0^t ds\, R(0, s, 0, 0), \qquad t > 0$$

represents the response at time $t > 0$ at the surface $x = 0$, to a unit impulse applied at $t = 0$, $x = 0$.

Problem (ii) is: given $\tilde{f}$, find the boundary value problem (3.3), that is, find the function q and the number h. Otherwise put, we are to recover the dynamics of the system from a knowledge of its response along the "surface" $x = 0$ to a unit impulse, also applied at the surface.

Problem (ii) is prototypical of a variety of inverse wave propagation problems of applied mathematics. We refer the reader to [3, 8], for instances.

We now observe that problems (i) and (ii) are equivalent. In fact, if we denote by L the self-adjoint ordinary differential operator defined by the boundary value problem (3.1) (with a boundary condition at $x = \infty$ supplied, if necessary), then the Riemann function is just the distribution kernel of $\cos t(L^{1/2})$, and admits the spectral representation

$$R(x, t, x_0, t_0) = \int_{-\infty}^{\infty} d\rho(\lambda) \cos \lambda^{1/2} (t - t_0)\, \phi(x, \lambda)\, \phi(x_0, \lambda).$$

In particular

$$\tilde{f}(t) = R(0, t, 0, 0) = \int_{-\infty}^{\infty} d\rho(\lambda) \cos \lambda^{1/2}\, t$$

so that $\tilde{f}$ is the Fourier transform of ρ. Thus knowledge of f and knowledge of ρ are equivalent, so problems (i) and (ii) are equivalent.

Since problems (i) and (ii) are equivalent, the solution by Gel'fand and Levitan [4] of problem (i) also solves problem (ii). We prefer, however, to deduce the solution of problem (i) from that of problem (ii), and in so doing present a natural interpretation of the machinery in [4] in the context of hyperbolic p.d.e. Besides a better understanding of the well-known results and methods of [4], we are immediately led to the correct stability result for problem (ii) (Theorem

III), and to the solution of a number of other inverse boundary value problems (see [9]).

We should point out that the equivalence of problems (i) and (ii), and the hyperbolic interpretation of the ideas in [4], are more-or-less well known. However, no careful statements on the lines of Theorem II have appeared in the literature, nor has a stability result of the type of Theorem III been previously asserted, to our knowledge.

Our approach to problem (ii) is based on several elementary properties of the Riemann function, especially:

(i) (*Group property*). Let $U(t)$ be the operator which maps Cauchy data $(u(x, t_0), u_t(x, t_0))$ for a solution of (3.3) at time t_0 to the Cauchy data for the same solution $(u(x, t + t_0), u_t(x, t + t_0))$ at time $t + t_0$. Then

$$U(s)\,U(t) = U(s + t). \tag{3.6}$$

(ii) (*Progressing Wave Expansion*). The distribution $\tilde{R}(y, t) = R(y, t, 0, 0)$ can be decomposed:

$$\tilde{R}(y, t) = \delta(y + t) + \delta(y - t) + K(y, t)$$

where $K(y, t) = K(y, -t)$ has one more derivative than the coefficient q in the region $|y| \leqslant |t|$, vanishes identically outside that region, and on the boundary satisfies the *transport equation*

$$K(t, t) = -\tfrac{1}{2}\int_0^t q + h. \tag{3.7}$$

These assertions will be made precise in Section 4. They are essentially classical results for which methods of proof are to be found for instance in Chs. V and VI of [1].

Since the operator U is implemented by the Riemann function, (i) and (ii) plus some symmetry properties of R, together imply an integral equation for $\tilde{R}$:

$$f(s, t) = \tfrac{1}{2}(f(s + t) + f(s - t)) = \int_0^\infty dy\, \tilde{R}(y, t)\, \tilde{R}(y, s). \tag{3.8}$$

According to property (ii), we may replace the upper limit of integration on the right-hand side of (3.8) by $\max(|t|, |s|)$, and we obtain, for instance, for $t \geqslant s > 0$

$$f(s, t) = K(s, t) + \int_0^s dy\, K(y, t)\, K(y, s). \tag{3.9}$$

We base our solution of problem (ii) on this nonlinear Volterra equation, derived in a different way by Gel'fand and Levitan. The hypotheses of Theorem II are precisely what is necessary to ensure that (3.9) has a global solution.

Further, our method of solving (3.9) for K is manifestly stable against small changes in the data f (Theorem III). Finally, explicit error bounds appear (Section 5) which allow one to estimate the efficiency of our approach for numerical computation. This circumstance should be compared with most other treatments of problem (i) along the lines of [4] (see especially [6, 3]), which turn on the solution of a linear integral equation of Fredholm type related to (3.9). By contrast, our error estimates involve only a lower bound for this Fredholm operator.

Having solved (3.9) for K, we show that K is the appropriate piece of the Riemann Function for a boundary value problem (3.3), with coefficient function q related to K by the transport equation (3.7). This is accomplished in Section 7. Finally, Theorem II is used in Section 8 to supply the key ingredients of the proof given in [6] of Theorem I, and we leave the matter there.

4. Properties of the Riemann Function

This section is devoted to the necessity part of Theorem II, that is, the proof that the boundary value of the Riemann function of a problem of form (4.1) below must satisfy the conditions (i) and (ii) in Theorem II.

Most of the following assertions are standard and can be found, for instance, in [1, Chaps. V and VI], in one form or another. In a few cases, our finite differentiability hypotheses and imposition of boundary conditions in the progressing wave construction of R are incompatible with readily available results; however, the proofs are the appropriate modifications of the available ones, and we omit them.

We denote by $R(\cdot, \cdot; x_0, t_0)$ the solution of the boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} - qu &= 0 \\ u_x(0, t) - hu(0, t) &= 0. \end{aligned} \tag{4.1}$$

with initial conditions

$$\begin{aligned} R(x, t_0; x_0, t_0) &= \delta(x - x_0) \\ \frac{\partial}{\partial t} R(x, t, x_0, t_0)|_{t=t_0} &= 0. \end{aligned} \tag{4.2}$$

We assume that $q \in W^{m,1}_{\mathrm{loc}}(\mathbf{R}^+)$, $m \geqslant 1$. Then:

I. (Regularity)

$$R \in C^m(\mathbf{R}_t \times \mathbf{R}^+_{x_0} \times \mathbf{R}_{t_0}; \mathscr{E}'(\mathbf{R}_x{}^+)) \cap C^m(\mathbf{R}_x{}^+ \times \mathbf{R}^+_{x_0} \times \mathbf{R}_{t_0}; \mathscr{D}'(\mathbf{R}_t)).$$

Property I follows from a theorem on solutions of (4.1) with smooth Cauchy data and the Schwarz kernel theorem, via standard arguments.

II. (Symmetry)

$$\begin{aligned} R(x, t; x_0, t_0) &= R(x_0, t_0; x, t) \\ &= R(x, t_0; x_0, t) \\ &= R(x, t-s; x_0, t_0 - s) \end{aligned}$$

for $s \in \mathbf{R}$; in particular

$$R(x, t; x_0, 0) = R(x, -t; x_0, 0).$$

Note that I and II combine to yield further regularity properties.

Denote by $\mathscr{R}$ the matrix of distributions

$$\mathscr{R}(x, t; x_0, t_0) = \begin{pmatrix} R(x, t; x_0, t_0) & \int_{t_0}^{t} d\sigma\, R(x, t; x_0, \sigma) \\ \frac{\partial}{\partial t} R(x, t; x_0, t_0) & \frac{\partial}{\partial t} \int_{t_0}^{t} d\sigma\, R(x, t; x_0, \sigma) \end{pmatrix}$$

Then the solution of (4.1) with smooth Cauchy data

$$u(x, t_0) = u_0(x), \qquad u_t(x, t_0) = v_0(x) \in C^\infty(\mathbf{R}^+)$$

is given by

$$\begin{pmatrix} u(x, t) \\ u_t(x, t) \end{pmatrix} = \left\langle \mathscr{R}(x, t, t_0), \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\rangle \tag{4.3}$$

where we have written $\langle S, \phi \rangle$ for the evaluation of a distribution S on a test function ϕ, and regarded $\mathscr{R}$ as a matrix-valued distribution in the third variable. We shall find it convenient to write

$$\begin{pmatrix} u(x, t) \\ u_t(x, t) \end{pmatrix} = \int_0^\infty dx_0\, \mathscr{R}(x, t; x_0, t_0) \begin{pmatrix} u_0(x_0) \\ v_0(x_0) \end{pmatrix} \tag{4.4}$$

with the integration "in the sense of distributions", i.e. (4.4) means (4.3). We shall also find it convenient to indicate the composition of distribution kernels such as R, $\mathscr{R}$ by integrals (see [10] Part III; note that compositions of the type below make sense because R is *semiregular*, in the language of [10]).

III. (Group Law)

$$\mathscr{R}(x, s + t, x_0, t_0) = \int_0^\infty dy\, \mathscr{R}(x, s + t; y, s)\, \mathscr{R}(y, s; x_0, t_0).$$

This is just (3.6) written in terms of distribution kernels. It is a consequence of the independence of time of the coefficient and boundary condition of the problem (4.1).

As in Section 3, we write $\tilde{R}$ for the distribution kernel on $\mathbf{R}^+ \times \mathbf{R}$:

$$\tilde{R}(y, t) = R(y, t; 0, 0).$$

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$$\tfrac{1}{2}(\tilde{R}(0, t - s) + \tilde{R}(0, t + s)) = \int_0^\infty dy\, \tilde{R}(y, s)\, \tilde{R}(y, t). \tag{4.5}$$

Proof. By III,

$$\begin{aligned} R(0, t + s) = \int_0^\infty dy\, R(0, t + s, y, s)\, R(y, s, 0, 0) \\ + \int_0^\infty dy \left(\int_s^{t+s} d\sigma\, R(0, t + s, y, \sigma)\right) D_2R(y, s, 0, 0). \end{aligned} \tag{4.6}$$

However, using the symmetry properties (II),

$$\int_s^{t+s} d\sigma\, R(0, t + s, y, \sigma) = \int_0^t d\sigma\, R(0, t, y, \sigma)$$

$$D_2R(y, s, 0, 0) = -D_2R(y, -s, 0, 0).$$

Thus the second integral (composition of kernels) is odd in the parameter s whereas the first is even. Replacing s by $-s$ in (4.6), adding, and multiplying by $\frac{1}{2}$ gives the result.

IV. (Progressing Wave Expansion)

$$\tilde{R}(y, t) = \delta(y + t) + \delta(y - t) + K(y, t) \tag{4.7}$$

where $K \in \mathscr{W}^{m+1,1}_{\text{loc}}(Q)$,

$$Q = \{(y, t) \in \mathbf{R}^+ \times \mathbf{R} : y \leqslant |t|\}, \qquad \text{and} \qquad K(y, t) = 0, \quad y > |t|.$$

Also the *transport equation* holds: for $t \geqslant 0$,

$$K(t, \pm t) = -\tfrac{1}{2} \int_0^t q + h. \tag{4.8}$$

In the readily accessible literature, the progressing wave expansion is usually developed for equations with C^∞ coefficients and no boundary conditions. In our case, we just barely get away with it, because of the finite differentiability assumptions.

Slightly altering the notation used in Section 3, write

$$\tilde{R}(0, t) = 2\delta(t) + \tilde{f}(t)$$

$$\tfrac{1}{2}(\tilde{R}(0, s + t) + \tilde{R}(0, s - t)) = \delta(s + t) + \delta(s - t) + f(s, t)$$

$$f(s, t) = \tfrac{1}{2}(\tilde{f}(s + t) + \tilde{f}(s - t)).$$

Then $\tilde{f} \in W^{m+1,1}_{\text{loc}}(\mathbf{R})$, $f \in \mathscr{W}^{m+1,1}_{\text{loc}}(\mathbf{R} \times \mathbf{R})$ according to IV; according to II, $\tilde{f}$ is even and f is symmetric.

In view of all this symmetry, we consider the integral relation (4.5) when t, $s \geq 0$. Then the progressing wave expansion reads

$$\tilde{R}(y, t) = \delta(y - t) + K(y, t)$$

and (4.5) becomes, for $t > 0$,

$$\delta(s - t) + f(s, t) = \int_0^\infty dy(\delta(y - t) + K(y, t))(\delta(y - s) + K(y, s))$$

$$= \delta(s - t) + K(s, t) + \int_0^s dyK(y, t)\, K(y, s)$$

or

$$f(s, t) = K(s, t) + \int_0^s dyK(y, t)\, K(y, s). \tag{4.9}$$

Since both sides of (4.9) are continuous, it holds for $t = 0$ also. Restricted to $0 \leq t \leq T$, some $T > 0$, the kernel $\tilde{R}$ defines an operator of the form $I + K_T$, where I is the identity operator on $L^2([0, T])$, and K_T is the integral operator

$$K_T g(y) = \int_0^T dtK(y, t)\, g(t), \qquad 0 \leq y \leq T.$$

Thus $I + K_T$ is a bounded (Volterra) operator on $L^2([0, T])$, and in particular is invertible.

Write F_T for the integral operator on $L^2([0, T])$ with kernel $f(s, t)$. Then (4.9) can be written

$$I + F_T = (I + K_T{}^\dagger)(I + K_T) \tag{4.10}$$

which shows that the left-hand side is a positive semidefinite symmetric Fredholm operator on $L^2([0, T])$. We claim that, in fact, it is positive definite. Indeed, is the product of an invertible operator and its adjoint, and therefore positive.

Denote by $\epsilon(T) > 0$ the smallest eigenvalue of $I + F_T$. Then for any $\psi \in L^2([0, T])$,

$$\langle \psi, (I + F_T) \psi \rangle_{L^2([0,T])} = \int_0^T \int_0^T ds\, dt\, \bar{\psi}(s)\, \psi(t) f(s, t) + \int_0^T |\psi|^2 \geqslant \epsilon(T) \int_0^T |\psi|^2. \tag{4.11}$$

This completes the demonstration of the necessity of hypotheses (i) and (ii) in Theorem II. The next three sections are devoted to their sufficiency.

5. Solution of a Nonlinear Volterra Equation

Let $T > 0$, and set $Q_T = \{(s, t): 0 \leqslant s \leqslant t \leqslant T\}$. In this and the next section, we shall show that the equation

$$f(s, t) = K(s, t) + \int_0^s dy K(y, s)\, K(y, t) \quad (s, t) \in Q_T \tag{5.1}$$

has a solution $K \in \mathscr{W}_{\mathrm{loc}}^{m,1}(Q^T)$ for every symmetric $f \in \mathscr{W}_{\mathrm{loc}}^{m,1}([0, T] \times [0, T])$ which satisfies condition (4.11) for some $\epsilon(T) > 0$.

We first show that (5.1) has a continuous solution for each continuous symmetric f satisfying (4.11). We note in passing that continuous solutions are necessarily unique, since (5.1) is a Volterra equation.

The existence proof consists of three steps.

Step 1. There exists some $\bar{t} > 0$ so that (5.1) has a continuous solution in $Q_{\bar{t}}$.

This is true, in fact, independently of the hypothesis (4.11). We denote by V the Volterra operator on $C^0(Q_{\bar{t}})$

$$VK(s, t) = f(s, t) - \int_0^t dy K(y, s)\, K(y, t) \quad (s, t) \in Q_{\bar{t}}.$$

Then a standard contraction mapping argument shows that, provided $\bar{t}$ is small enough, the operator V has a fixed point in $C^0(Q_{\bar{t}})$.

Precisely, we obtain

A. Let $\| \|_{\bar{t}}$ denote the sup norm in $C^0(Q_{\bar{t}})$. Then the ball

$$B_{\epsilon,\bar{t}}(f) = \{g \in C^0(Q_{\bar{t}}): \| g - f \|_{\bar{t}} \leqslant \epsilon\}$$

is invariant under V, provided

$$2\bar{t}(\epsilon + \| f \|_{\bar{t}})^2 \leqslant \epsilon.$$

B. For $g_1, g_2 \in B_{\epsilon,\bar{t}}(f)$, ϵ as above,

$$\| Vg_1 - Vg_2 \|_{\bar{t}} \leqslant 2\bar{t}(\epsilon + \| f \|_{\bar{t}}) \| g_1 - g_2 \|_{\bar{t}} .$$

Thus V is a contraction operator for $\bar{t}$ small enough.

C. Denote by K the fixed point of V in $C^0(Q_{\bar{t}})$, and set

$$K^0 = f, \qquad K^n = V^n K^0$$

then

$$\| K - K^n \|_{\bar{t}} \leqslant \{2\bar{t}(\epsilon + \| f \|_{\bar{t}})\}^n \epsilon. \tag{5.2}$$

These conclusions are completely straightforward. We exhibit them only to make explicit the dependence of the error estimate (5.2) on the size of f.

Step 2. We suppose that (5.1) has been solved in $Q_{\bar{t}}$ for *some* $\bar{t} > 0$. We write

$$\Phi(s, t) = K(s, t), \qquad 0 \leqslant s \leqslant \bar{t} \leqslant t \leqslant T$$

and note that (5.1) becomes, in the region $0 \leqslant s \leqslant \bar{t} \leqslant t \leqslant T$,

$$f(s, t) = \Phi(s, t) + \int_0^s dy K(y, s)\, \Phi(y, t) \tag{5.3}$$

which for each $t \in [\bar{t}, T]$, is a linear Volterra equation with continuous kernel, hence has a continuous solution, which also depends continuously on the parameter t, since the inhomogeneous term does.

Set $f_t(s) = f(s, t)$, $\Phi_t(s) = \Phi(s, t)$. Then, in the notation of (4.10, 11), replacing T by $\bar{t}$, (5.3) becomes

$$f_t = (I + K_{\bar{t}})^\dagger \Phi_t$$

and (4.11), implies

$$\begin{aligned} \| f_t \|^2_{L^2([0,\bar{t}])} &= \langle \Phi_t, (I + K_{\bar{t}})(I + K_{\bar{t}}^\dagger)\, \Phi_t \rangle \\ &\geqslant \langle \Phi_t, (I + F_{\bar{t}})\, \Phi_t \rangle \geqslant \epsilon(\bar{t}) \, \| \Phi_t \|^2_{L^2([0,\bar{t}])}, \end{aligned} \tag{5.4}$$

since $(I + K_{\bar{t}})(I + K_{\bar{t}}^\dagger)$ has the same lower bound as $(I + K_{\bar{t}}^\dagger)(I + K_{\bar{t}}) = I + F_{\bar{t}}$. We observe that we can replace $\epsilon(\bar{t})$ by $\epsilon(t')$, $t' \geqslant \bar{t}$, in (5.4), in particular we can replace $\epsilon(\bar{t})$ by $\epsilon(T)$. Together with the Schwarz inequality, this implies

$$\int_0^{\bar{t}} dy \, |\Phi(y, t_1)| \, |\Phi(y, t_2)| \leqslant \frac{A(T)}{\epsilon(T)} . \tag{5.5}$$

Here

$$A(T) = \sup_{0 \leqslant s,t \leqslant T} \{\| f_s \|_{L^2([0,T])} \| f_t \|_{L^2([0,T])}\}.$$

The important point, of course, is that the right-hand side of (5.5) is independent of $\bar{t}$.

Step 3. For $\bar{t} \leqslant s, t$, (5.1) reads, in the notation of *Step* 2

$$f(s, t) = K(s, t) + \int_0^{\bar{t}} dy \Phi(y, s)\, \Phi(y, t) + \int_{\bar{t}}^{s} dy K(y, s)\, K(y, t). \qquad (5.6)$$

This problem is of the same form as that dealt with in Step 1. Therefore we can solve (5.6) in a region of the form $\{(s, t)\colon \bar{t} \leqslant s \leqslant t \leqslant \bar{t} + \delta\}$, where δ depends as in Step 1 on the uniform norm of the inhomogeneous term

$$f_{\bar{t}}(s, t) = f(s, t) - \int_0^{\bar{t}} dy \Phi(y, s)\, \Phi(y, t).$$

Since (5.5) provides us with an estimate of the size of $f_{\bar{t}}$ which may be taken *independent of* $\bar{t}$, the increment δ may also be chosen independently of $\bar{t}$.

It follows that finitely many repetitions of Steps 1–3 suffice to yield a continuous solution K of (5.1) in $Q(T)$. The construction proceeds by alternately solving nonlinear Volterra equations over small intervals, and linear Volterra equations over large intervals. The error in each step, hence the cumulative error, and the number of steps required, may be estimated in terms of the three numbers

$$\| f \|_T, \qquad A(T), \qquad \epsilon(T).$$

Suppose now that $f \in C^m(Q_T)$. By differentiating (5.1) formally, we obtain equations for the derivatives of K, which we solve in exactly the same way as we solved (5.1). Thus $K \in C^m(Q_T)$.

We remark with an eye to the proof of Theorem III that the following estimate shows that the condition (4.11) is stable against C^0-small perturbations of $\tilde{f}$:

$$\begin{aligned} &\left| \int_0^T ds \int_0^T dt\, \psi(s)\, \bar{\psi}(t)\, (f(s, t) - f'(s, t)) \right| \\ &\quad \leqslant \| \phi \|^2_{L^2[0,T]} \| \| f(s, \cdot) - f'(s, \cdot) \|_{L^2[0,T]} \|_{L_s^2[0,T]} \qquad (5.7) \\ &\quad \leqslant \| \psi \|^2_{L^2[0,T]}\, T \| f - f' \|_{C^0(Q_T)} \end{aligned}$$

for any $\psi \in L^2[0, T]$.

It follows that, for any $f \in C^m(Q_T)$ which satisfies condition (4.11), there exists a neighborhood of f in $C^m(Q_T)$ in which condition (4.11) is uniformly satisfied (for some smaller $\epsilon > 0$, perhaps). Easy perturbation arguments show that the solution $K \in C^m(Q_T)$ of (5.1) depends continuously on the left-hand side f, as f ranges over any such neighborhood.

6. Existence and Stability in $\mathscr{W}^{m,1}_{\mathrm{loc}}$

The existence and stability of solutions of (5.1) with locally integrable derivatives follow from the basic existence and stability theorem for linear Volterra equations. We state this elementary result for the reader's convenience, though we have used it tacitly several times in the previous section.

THEOREM. *Suppose $a < b$, $Q = \{(y, s): a \leqslant y \leqslant s \leqslant b\}$, E a Banach Space, $g \in C^m([a, b]; E)$, $V \in C^m(Q; \mathbf{R})$. Then there exists a unique $\varphi \in C^m([a, b]; E)$ with*

$$g(s) = \varphi(s) - \int_a^s dy V(y, s)\, \varphi(y).$$

Furthermore, the solution map:

$$C^m([a, b]; E) \times C^m(Q; \mathbf{R}) \to C^m([a, b], E)$$
$$(g, V) \mapsto \varphi$$

is continuous.

Finally, there exists a $W \in C^m(Q; \mathbf{R})$, called the resolvent kernel corresponding to V, in terms of which the solution φ may be written

$$\varphi(s) = g(s) + \int_a^s dy W(y, s)\, g(y).$$

Now suppose that $f \in \mathscr{W}^{2,1}_{\mathrm{loc}}(\mathbf{R}^2)$ satisfies (4.11) (The case $f \in \mathscr{W}^{m,1}_{\mathrm{loc}}(\mathbf{R}^2)$, $m > 2$' can be handled in exactly the same way so we give details only for $m = 2$.) Since $\mathscr{W}^{2,1}_{\mathrm{loc}} \subset C^1$, the previous section guarantees that the solution K of (5.1) is in $C^1(Q^T)$ for every $T > 0$, and depends continuously on $f \in C^1(Q^T)$.

We shall first show that K has a second derivative $D_2{}^2K$ with respect to its second argument in $C^0([0, \bar{t}]; L^1([\bar{t}, T]))$ for every $\bar{t} \in [0, T]$. We differentiate (5.1) formally twice with respect to t, and obtain

$$D_2{}^2 f(s, t) = D_2{}^2 K(s, t) + \int_0^s dy K(y, s)\, D_2{}^2 K(y, t). \tag{6.1}$$

Now $f \in \mathscr{W}^{2,1}_{\text{loc}}(\mathbf{R}^2)$ means that $D_2^2 f \in C^0([0, \bar{t}]_s; L^1([\bar{t}, T])_t)$, for each $\bar{t} \in [0, T]$. Let $g: [0, \bar{t}] \to L^1[\bar{t}, T]$ by $g(s) = D_2^2 f(s, \cdot)$. Then the basic existence theorem asserts that

$$g(s) = u(s) + \int_0^t dy K(y, t)\, u(y) \tag{6.2}$$

has a unique solution $u \in C^0([0, \bar{t}]; L_1[\bar{t}, T])$. It remains to show that u may be identified with $D_2^2 K$ as a distribution. In terms of the resolvent kernel G corresponding to K the solution u to (6.2) may be expressed:

$$u(s) = g(s) + \int_0^s dy G(y, s)\, g(y).$$

The kernel G is also continuous. If we regard u and g as $\mathscr{D}'([\bar{t}, T])$-valued functions of $s \in [0, \bar{t}]$, then for $\phi \in C_0^\infty((\bar{t}, T))$ we may write

$$\begin{aligned}\langle u(s), \phi\rangle &= \langle g(s), \phi\rangle + \int_0^s dy G(y, s)\, \langle g(y), \phi\rangle \\ &= \langle f(s, \cdot), \phi''\rangle + \int_0^s dy G(y, s)\, \langle f(s, \cdot)\, \phi''\rangle \\ &= \langle K(s, \cdot), \phi''\rangle\end{aligned}$$

where we have used the equation, equivalent to (5.1):

$$K(s, t) = f(s, t) + \int_0^s dy G(y, s) f(y, t).$$

Thus u is indeed the second derivative of K, and we conclude that

$$D_2^2 K \in C^0([0, \bar{t}]; L^1[\bar{t}, T])$$

for each $\bar{t} \in [0, T]$, $T > 0$. Finally, since $K \in C^1(Q^T)$ depends continuously on $f \in \mathscr{W}^{2,1}(Q^T)$, the stability part of the basic theorem implies that $D_2^2 K$ also depends continuously on f, in the sense of $C^0([0, \bar{t}]; L^1[\bar{t}, T])$, for each $\bar{t}$.

The other two formal derivatives of (5.1) read:

$$\begin{aligned}D_1^2 f(s, t) = D_1^2 K(s, t) &+ \left(\frac{d}{ds} K(s, s)\right) K(s, t) + K(s, s)\, D_1 K(s, t) \\ &+ D_2 K(s, s)\, K(s, t) + \int_0^s dy D_2^2 K(y, s)\, K(y, t)\end{aligned} \tag{6.3}$$

$$D_1 D_2 f(s, t) = D_1 D_2 K(s, t) + K(s, s)\, D_2 K(s, t) + \int_0^s dy D_2 K(y, s)\, D_2 K(y, t). \tag{6.4}$$

From these equations, which hold *a priori* in the sense of distributions, we can immediately deduce that $K \in \mathscr{W}^{2,1}_{\mathrm{loc}}(Q)$; we leave the details to the reader.

From the stability assertion about $D_2{}^2K$ and some obivous fiddling with equations (6.3) and (6.4), we can conclude that the map $f \mapsto K$ is continuous in $\mathscr{W}^{2,1}_{\mathrm{loc}}(Q)$, in a neighborhood of every f satisfying (4.11).

Finally, we point out that K inherits from f a property somewhat more special than that of being in $\mathscr{W}^{m,1}_{\mathrm{loc}}$, if f is of the special form

$$f(s, t) = \tfrac{1}{2}(\tilde{f}(s + t) + \tilde{f}(s - t)), \qquad \tilde{f} \in W^{m,1}_{\mathrm{loc}}(\mathbf{R}).$$

Indeed, it follows from a remark in Section 2 that for any $(a, b, c, d) \in \mathbf{R}^4$, the function

$$s \mapsto f(as + b, cs + d)$$

is then in $W^{m,1}_{\mathrm{loc}}(\mathbf{R})$, and a simple modification of the above argument shows that that the function

$$s \mapsto K(as + b, cs + d)$$

is then also in $W^{m,1}_{\mathrm{loc}}$ on its interval of definition. In particular,

$$s \mapsto K(s, s)$$

defines an element of $W^{m,1}_{\mathrm{loc}}(\mathbf{R}^+)$.

The details of this argument are very similar to that used to establish that $K \in \mathscr{W}^{m,1}_{\mathrm{loc}}(Q)$, and we omit them.

7. Recovery of the Bounary Value Problem

We shall show that the kernel K constructed in the last section is the regular part of the Riemann function of some boundary value problem (4.1), thus completing the proofs of Theorems II and III.

Suppose that $\tilde{f} \in \mathscr{W}^{m+1,1}_{\mathrm{loc}}(\mathbf{R})$ is even. Set

$$f(s, t) = \tfrac{1}{2}(\tilde{f}(s + t) + \tilde{f}(s - t))$$
$$\tilde{f}(0) = h.$$

Then $f \in \mathscr{W}^{m+1,1}_{\mathrm{loc}}(\mathbf{R}^2)$ is symmetric. Suppose that f satisfies (4.11) for each $T > 0$. Let K be the solution of (5.1) constructed in Sections 5, 6, with f as above. Then $K \in \mathscr{W}^{m+1,1}_{\mathrm{loc}}(Q)$, where $Q = \{(s, t): 0 \leqslant s \leqslant t\}$.

As remarked in Section 6, the functions

$$s \mapsto K(as + b, cs + d)$$

are in $W^{m+1,1}_{loc}$ of their appropriate intervals of definition. In particular, the function

$$q(s) = -2 \frac{d}{ds} K(s, s)$$

is in $W^{m,1}_{loc}(\mathbf{R}^+)$.

We claim that K solves the characteristic mixed problem

$$\begin{aligned} \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} + q(s)\right\} K(s, t) = 0, \qquad s, t \in Q \\ \frac{\partial K}{\partial s}(0, t) + hK(0, t) = 0 \\ K(s, s) = h - \tfrac{1}{2} \int_0^s q. \end{aligned} \tag{7.1}$$

Before showing that K solves (7.1), we remark that the problem (7.1) has a unique solution in $\mathscr{W}^{m+1,1}_{loc}(Q)$, as is shown by standard arguments. Since the results of Section 4 imply that the regular part of the Riemann function of the boundary value problem (4.1), with q as above, must solve (7.1) also, we conclude that K *is* the regular part of the Riemann function. We have therefore proven Theorem II if we show that K is a solution of (7.1).

To do this, note first that f solves

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) f = 0, \qquad \frac{\partial f}{\partial s}(0, t) \equiv 0, \qquad t \in \mathbf{R}.$$

(Note that f has second derivatives which are locally integrable in $\mathbf{R}^2$). According to the integral equation

$$f(s, t) = K(s, t) + \int_0^s dy K(y, s)\, K(y, t)$$

we have

(1) $f(0, 0) = K(0, 0) = h$

(2) $$\frac{\partial f}{\partial s}(0, t) = \frac{\partial K}{\partial s}(0, t) + K(0, 0)\, K(0, t)$$
$$= \frac{\partial K}{\partial s}(0, t) + hK(0, t) = 0, \qquad t > 0$$

(3) $$\begin{aligned} 0 &= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) f(s, t) \\ &= \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2}\right) K(s, t) - \left(\frac{d}{ds} K(s, s)\right) K(s, t) \\ &\quad - K(s, s)\, D_1 K(s, t) - D_2 K(s, s)\, K(s, t) \\ &\quad - \int_0^s dy D_2^2 K(y, s)\, K(y, t) + \int_0^s dy K(y, s)\, D_2^2 K(y, t). \end{aligned} \tag{7.2}$$

The differentiation under the integral sign above, and various integrations by parts below are justified (for the only questionable case, $m = 1$) by

LEMMA. *Suppose* $u \in C^0[t_1, t_2]$, $v = Du \in L^1[t_1, t_2]$. *Then*

$$\int_{t_1}^{t_2} v = u(t_2) - u(t_1).$$

We apply this Lemma to compute

$$\begin{aligned}
&\int_0^s dy D_1^2K(y, s)\, K(y, t) \\
&\quad = \int_0^s dy D_y(D_1K(y, s)\, K(y, t) - \int_0^s dy D_1K(y, s)\, D_1K(y, t) \\
&\quad = D_1K(s, s)\, K(s, t) - D_1K(0, s)\, K(0, t) \\
&\qquad + \int_0^s dy K(y, s)\, D_1^2K(y, t) - \int_0^s dy D_y(K(y, s)\, D_1K(y, t)) \\
&\quad = D_1K(s, s)\, K(s, t) - D_1K(0, s)\, K(0, t) - K(s, s)\, D_1K(s, t) \\
&\qquad + K(0, s)\, D_1K(0, t) + \int_0^s dy K(y, s)\, D_1^2K(y, t).
\end{aligned}$$

Hence for $0 < s \leqslant t$

$$\begin{aligned}
&K(s, s)\, D_1K(s, t) - D_1K(s, s)\, K(s, t) \\
&\quad = \int_0^s dy K(y, s)\, D_1^2K(y, t) - \int_0^s dy D_1^2K(y, s)\, K(y, t)
\end{aligned} \tag{7.3}$$

where we have made use of the boundary condition (2). By virtue of the identity

$$\frac{d}{ds} K(s, s) = D_1K(s, t) + D_2K(s, t)|_{s=t}$$

we can add (7.3) to (7.2) to obtain

$$\begin{aligned}
0 = {} & (D_2^2 - D_1^2)\, K(s, t) - 2 \left(\frac{d}{ds} K(s, s)\right) K(s, t) \\
& + \int_0^s dy K(y, s)\, (D_2^2 - D_1^2)\, K(y, t) - \int_0^s dy((D_2^2 - D_1^2)\, K(y, s))\, K(y, t).
\end{aligned} \tag{7.4}$$

Set

$$u(s, t) = (D_2{}^2 - D_1{}^2)\, K(s, t) - 2 \left(\frac{d}{ds} K(s, s)\right) K(s, t). \tag{7.5}$$

We can re-write (7.4) as

$$0 = u(s, t) + \int_0^s dyK(y, s)\, u(y, t) - \int_0^s dyu(y, s)\, K(y, t). \tag{7.6}$$

If $m > 0$, we can immediately conclude that $u = 0$, since (7.6) is a homogeneous equation of Volterra type, which finishes the proof of (7.1) in that case. If $m = 0$, we must be slightly more careful, since we know *a priori* only that u is locally integrable in $Q = \{(s, t): 0 \leqslant s \leqslant t\}$.

We reason as follows. Suppose that $f \in \mathscr{W}^{2,1}_{\text{loc}}$ is a proximated by $f_n \in \mathscr{W}^{3,1}_{\text{loc}}$. Then, according to Sections 5, 6, the corresponding solutions $K_n \in \mathscr{W}^{3,1}_{\text{loc}}(Q)$ converge in $\mathscr{W}^{2,1}_{\text{loc}}(Q)$ to K. As noted in Section 2, the topology of $\mathscr{W}^{m,1}_{\text{loc}}$ is stronger than that of $W^{m,1}_{\text{loc}}$. If we denote by u_n the expression (7.5) formed with K_n instead of K, the sequence u_n thus converges in $L^1_{\text{loc}}(Q)$ to u. However, the u_n all satisfy (7.6) and are continuous, hence vanish identically. Thus u vanishes almost everywhere, and the differential equation in (7.1) is satisfied in the sense of distributions.

This concludes the proof of sufficiency, hence the entire proof, of Theorem II.

To prove Theorem III, we note that the coefficient q is determined by the transport equation (characteristic boundary condition in (7.1)):

$$q(s) = -2 \frac{d}{ds} K(s, s).$$

Since it was shown in Section 6 that $K \in \mathscr{W}^{m+1,1}_{\text{loc}}(Q)$ depends continuously on $\tilde{f} \in W^{m+1,1}_{\text{loc}}(\mathbf{R}^+)$ it follows that the map

$$W^{m+1,1}([0, T]) \ni \tilde{f} \mapsto q \in W^{m,1}([0, T])$$

is (well-defined and) continuous in a neighborhood of every $\tilde{f}$ satisfying (4.11), which is exactly the assertion of Theorem III.

8. Proof of Theorem I

Suppose $\rho: \mathbf{R} \to \mathbf{R}$ is a nondecreasing function satisfying conditions (i) and (ii) of Theorem I.

Lemma. *$\tilde{f}$, defined as in Theorem* I, *condition* (i), *satisfies condition* (ii) *of Theorem* II.

Proof. Suppose $u \in L^2(\mathbf{R}^+)$ has support in $[0, T]$. Then

$$\begin{aligned}\int_{\mathbf{R}} d\rho(\lambda) \mid \tilde{u}(\lambda)|^2 &= \lim_{N\to\infty} \int_{-N}^{N} d\rho(\lambda) \mid \tilde{u}(\lambda)|^2 \\ &= \lim_{N\to\infty} \int_{-N}^{N} d\left(\frac{2}{\pi}\lambda^{1/2}\right) \mid \tilde{u}(\lambda)|^2 + \int_{-N}^{N} d\sigma(\lambda) \mid \tilde{u}(\lambda)|^2 \\ &= \int_0^\infty \mid u \mid^2 + \lim_{N\to\infty} \int_{-N}^{N} d\sigma(\lambda) \left(\int_0^T ds\, u(s) \cos \lambda^{1/2} s\right)\left(\int_0^T dt\, \bar{u}(t) \cos \lambda^{1/2} t\right)\end{aligned}$$

where we have used the ordinary Parseval formula for the (cosine) Fourier transform.

Now according to condition (i) of Theorem I,

$$\lim_{N\to\infty} \int_{-N}^{N} d\sigma(\lambda) \cos \lambda^{1/2} s \cos \lambda^{1/2} t$$

exists in $C^0([0, T] \times [0, T])$. Hence

$$u(s)\, \bar{u}(t) \int_{-\infty}^{\infty} d\sigma(\lambda) \cos \lambda^{1/2} s \cos \lambda^{1/2} t = u(s)\, \bar{u}(t) f(s, t)$$

is in $L^1([0, T] \times [0, T])$, and by Fubini's Theorem

$$\int_{-\infty}^{\infty} d\sigma(\lambda) \mid \tilde{u}(\lambda)|^2 = \int_0^T \int_0^T ds\, dt\, u(s)\, \bar{u}(t) f(s, t).$$

Thus

$$0 \leqslant \int_{-\infty}^{\infty} d\rho(\lambda) \mid \tilde{u}(\lambda)|^2 = \int_0^T \mid u \mid^2 + \int_0^T \int_0^T ds\, dt\, u(s)\, \bar{u}(t) f(s, t)$$

and condition (ii) say that the inequality is strict if u is not identically zero a.e. This however is equivalent to Condition (ii) of Theorem II, as remarked in Section 2.

We may therefore apply Theorem II to conclude the existence of a hyperbolic boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} + qu &= 0 \\ u_x(0, t) + hu(0, t) &= 0 \end{aligned} \tag{8.1}$$

for which f is the boundary value of the Riemann function. Here $q \in W^{m,1}_{\text{loc}}(\mathbf{R}^+)$.

Denote by $\phi(x, \lambda)$ the solution (in $C^{m+1}(\mathbf{R}^+)$, entire in $\lambda \in \mathbf{C}$) of

$$\begin{gathered} -\phi''(x, \lambda) + (q(x) - \lambda)\phi(x, \lambda) = 0, \qquad x \geqslant 0 \\ \phi(0, \lambda) = 1, \qquad \phi'(0, \lambda) = -h. \end{gathered} \tag{8.2}$$

One immediately verifies that the function

$$u^\lambda(x, t) \equiv \cos(\lambda)^{1/2}\, t\phi(x, \lambda)$$

solves (8.1) with initial values

$$\begin{aligned} u^\lambda(x, 0) &= \phi(x, \lambda) \\ u_t^\lambda(x, 0) &= 0. \end{aligned}$$

It follows that

$$u^\lambda(x, t) = \int_0^\infty dx_0 R(x, t, x_0, 0)\, \phi(x_0, \lambda).$$

In particular

$$\begin{aligned} u^\lambda(0, t) &= \cos \lambda^{1/2}\, t \\ &= \int_0^\infty dx_0 R(0, t, x_0, 0)\, \phi(x_0, \lambda) \\ &= \int_0^\infty dx_0 \tilde{R}(x_0, t)\, \phi(x_0, \lambda) \\ &= \phi(t, \lambda) + \int_0^t dx_0 K(x_0, t)\, \phi(x_0, \lambda) \end{aligned}$$

with $K \in \mathscr{W}_{\mathrm{loc}}^{m+1,1}(Q)$.

As in Section 6, denote by G the resolvent kernel for K. Then

$$\phi(t, \lambda) = \cos \lambda^{1/2}\, t + \int_0^t dx G(x, t) \cos \lambda^{1/2}\, x. \tag{8.3}$$

On the other hand, K satisfies equation (5.1). These two equations ((5.1) and (8.3)) constitute the input of the argument in [6], and we refer the reader to that excellent reference for the remainder of the proof of Theorem I.

We remark that equation (8.3) was first derived by Povzner [7] in exactly this way.

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