Harmonic Polynomials
and Springer's Representation for $SL(n, \mathbb{C})$

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INTRODUCTION

Let $G$ be a connected reductive complex algebraic group, $\mathcal{N}$ the set of all nilpotent elements in its Lie algebra $\mathfrak{g}$, $W$ its Weyl group, and $\mathcal{B}$ the variety of all Borel subgroups of $G$. Given a nilpotent element $v \in \mathcal{N}$ we denote by $O_v$ the $G$-orbit of $v$ under the adjoint action of $G$ on $\mathfrak{g}$ and by $\mathcal{B}_v$, the closed subvariety of $\mathcal{B}$ consisting of all Borel subgroups whose Lie algebra contains $v$. Springer ([10, 11], cf. also [4]) defines a $W$-module structure on the rational cohomology $H^*(\mathcal{B}_v, \mathbb{Q})$, compatible with the action of the isotropy group $G_v$, which yields a $W$-irreducible representation on the top cohomology of the $G_v$-fixed subspace $H^*(\mathcal{B}_v, \mathbb{Q})^{G_v}$.

Consider now an arbitrary Weyl group $W$ acting faithfully in a real space $E$ and identity $S(E^*) = \text{Sym}(E)$ with the space of constant coefficient differential operators on $E$; $S^*_+$ will denote the augmentation ideal of polynomial algebra $S(E^*)^W$. Then $g \in D(E)$ is said to be $W$-harmonic if $Dg = 0$ for all $D \in S^*_+$. If $\mathcal{H}$ denotes the augmentation ideal of $S(E)^W$, then the space $\mathcal{H}$ of $W$-harmonic polynomials is a $W$-stable complement to $S(E)\mathcal{H}$ in $S(E)$ and hence as $W$-module is isomorphic to the regular representation of $W$. Let $\Delta$ be the root system to which $W$ corresponds and let the $\delta_i$'s be the fundamental weights determined by $\Delta$. We recall that the Weyl dimension polynomial

$$P_{\Delta}(x) = \prod_{x \in \Delta_+} \frac{(x, \alpha')}{(\delta, \alpha')}$$

is $W$-harmonic and that after Steinberg [12] one has $\mathcal{H} = \{D(P_{\Delta}) | D \in S(E^*)\}$. Here $\Delta_+$ is the set of positive roots, $\delta = \sum \delta_i$, $(\cdot, \cdot)$ is a $W$-invariant symmetric bilinear form on $E$, and $\alpha' = 2\alpha/(\alpha, \alpha)$.

Let $W'$ be the group of affine transformations generated by $W$ and by the group of translations with respect to the elements of the weight-lattice.

444
Hulsurkar [5], proving a conjecture of Verma [13], gets a basis of \( H \) derived from \( P_\mathcal{A} \) by certain transformations of \( \mathcal{W} \).

This paper only deals with the case of the symmetric group or its Young subgroups. In the special case \( g = \mathfrak{sl}(n, \mathbb{C}) \), \( \mathcal{F} \) becomes isomorphic to the variety \( \mathcal{F} = \mathcal{F}(V) \) of all complete flags in an \( n \)-dimensional complex space \( V \) and the nilpotent orbits are in bijective correspondence with the partitions of \( n \).

Let \( u \) be a unipotent linear transformation of \( V \) of Jordan partition \( \sigma = (\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m) \). Then the Springer variety, denoted by \( \mathcal{F}_\sigma \), is the fixed point subvariety of \( \mathcal{F} \) with respect to the action of \( u \) and its rational cohomology, as an \( S_\sigma \)-module, is the induced representation from the identity representation of the Young subgroup \( S_\sigma = S_{\sigma_1} \times \cdots \times S_{\sigma_m} \) (MacDonald; cf. [3]).

We shall prove the following:

**Theorem (3.3).** Under the identification of \( H_*(\mathcal{F}, \mathbb{Q}) \) with the \( \mathbb{Q} \)-space of \( S_\sigma \)-harmonic polynomials, the homology \( H_*(\mathcal{F}_\sigma, \mathbb{Q}) \), which injects in \( H_*(\mathcal{F}, \mathbb{Q}) \), becomes the cyclic \( S_\sigma \)-submodule generated by \( P_\mu \), where \( \mu \) is the root system of the Weyl group \( S_{\mu} = S_{\mu_1} \times \cdots \times S_{\mu_r} \) and \( \mu = (\mu_1 \geq \cdots \geq \mu_r) \) is the dual partition of \( \sigma \).

In particular we get an action of the group \( S_\sigma \) on the rational (and integer) homology of \( \mathcal{F}_\sigma \) extending the group action of \( S_n \).

Moreover our approach may be used to prove in a new way Springer's theorem on the action of \( S_n \) in \( H^*(\mathcal{F}_\sigma, \mathbb{Q}) \).

1. **The Deformation**

1.1. Let \( \sigma = (\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m) \) be a partition and \( \mathcal{F}_\sigma \) the fixed point variety defined in the introduction. These varieties \( \mathcal{F}_\sigma \) have been studied by several authors [3, 4, 8–11] but for the following we need only to recall the analysis given by Spaltenstein [8, 9].

Let \( m_1 > m_2 > \cdots > m_h \) be the distinct lengths of rows of \( \sigma \), each appearing \( s_i \) times and set \( W_i = \text{Ker}(v) \cap \text{Im}(v)^m-1 \) i = 1, ..., h, where \( v \) is a nilpotent linear transformation of Jordan partition \( \sigma \). \( W_i \) is a space of dimension \( q_i = s_1 + \cdots + s_i \).

Consider now the canonical projection \( p: \mathcal{F} \to \mathbb{P}(V) \) on the projective space of lines in \( V \). We have:

1. \( p(\mathcal{F}_\sigma) = \mathbb{P}(\text{Ker}(v)) = \mathbb{P}(W_h) \),

2. setting \( U_i = \mathbb{P}(W_i) - \mathbb{P}(W_{i-1}) \) the map \( p(p^{-1}(U_i)) \cap \mathcal{F}_\sigma \) is a Zariski locally trivial fibration over \( U_i \) with fibre the variety \( \mathcal{F}_{\sigma_i} \).
From now on if $\sigma$ is a partition of $n$, $\sigma^i$ will denote the partition of $n-1$ defined as follows: if $t$ is the largest row index for which $\sigma_i = \sigma_t$ we set $\sigma^i = (\sigma_1^i \geq \sigma_2^i \geq \cdots \sigma^i_m)$ where $\sigma^i_j = \sigma_j$ if $j \neq t$ and $\sigma^i_j = \sigma_j - 1$.

1.2. Let $D = D(1, \varepsilon)$ be the open disc of $\mathbb{C}$ with centre 1 and radius $\varepsilon$ where $\varepsilon$ is such that $D$ does not contain any $n$th root of unity. We want to define a map $\phi$ from this open disc to the general linear group $GL(n, \mathbb{C})$. We choose a Jordan basis of $V$ for $u = v + 1$. We shall denote it by $\{e_{i,k_i} \mid i = 1, \ldots, m \ k_i = 1, \ldots, \sigma_i\}$ so that we have $u(e_{i,k_i}) = e_{i,k_i-1} + e_{i,k_i}$ if $k_i > 1$ and $u(e_{i,k_i}) = e_{i,k_i}$ if $k_i = 1$. Then we define $\phi$ by the following position: $\phi(1) = u$ and $\phi(t)$ is the linear transformation of $V$ given by

$$
\phi(t)(e_{i,k_i}) = e_{i,k_i-1} + t e_{i,k_i} \quad \text{if} \quad k_i > 1 \quad \text{and} \quad \phi(t)(e_{i,k_i}) = t e_{i,k_i} \quad \text{if} \quad k_i = 1.
$$

So in the chosen basis the matrix of $\phi(t)$ is

$$
A(t) = \begin{pmatrix}
A_1(t) & \cdots & A_m(t)
\end{pmatrix},
$$

where

$$
A_i(t) = \begin{pmatrix}
t & 0 & \cdots & 0 \\
-t^2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & -t^\sigma_i
\end{pmatrix}.
$$

Clearly $\phi(t)$ is semisimple for $t \neq 1$ and if we set $V_i(t) = \{v \in V \mid \phi(t)v = t^i v\}$, then the dimension of $V_i(t)$ is $\mu_i$ where $\mu$ is the dual partition of $\sigma$. The following holds:

1.3. **Lemma.** Given a Jordan basis $\{e_{i,k_i} \mid i = 1, \ldots, m \ k_i = 1, \ldots, \sigma_i\}$ as before, $H_i$ will denote the subspace of $V$ spanned by $\{e_{i,k_i} \mid k_i = 1, \ldots, \sigma_i\}$; then for each $v^i(t)$ in $V_j(t)$ we can write

$$
v^i(t) = \sum_{i \in I} a_{i,j} v^i_j(t),
$$

where $a_{i,j} \in \mathbb{C}$, $I = \{s \mid 1 \leq s \leq m, \sigma_s \geq j\}$, $v^i_j(t) \in H_i \cap V_j(t)$, $v^i_j(t) = e_{k_1} + (t^j - t) e_{i,2} + (t^j - t)(t^j - t^2) e_{i,3} + \cdots + (t^j - t)(t^j - t^2) \cdots (t^j - t^{j-1}) e_{i,s}$.

So there is no ambiguity if we write $V_i$ instead of $V_i(t)$.

1.4. We set $\mathcal{F}_{\phi(t)}$ the fixed point variety of $\phi(t)$ for $t \in D$. We have $\mathcal{F}_{\phi(1)} = \mathcal{F}_{\sigma}$.

**Proposition.** For every $t \neq 1$ there exists an isomorphism of algebraic varieties between $\mathcal{F}_{\phi(t)}$ and the disjoint union of $n! / \mu_1 ! \mu_2 ! \cdots \mu_r !$ copies of the product of flag varieties $\mathcal{F}(V_1) \times \cdots \times \mathcal{F}(V_r)$. 

Proof. If $F = (F_i)_{i=1}^n$ is a flag of $\mathcal{F}_{\psi(t)}$, $t \neq 1$, we can associate to it $(f_1, f_2, ..., f_i, i)$ where $f_j = (F_1 \cap V_j, F_2 \cap V_j, ..., F_n \cap V_j)$ is a flag of $\mathcal{F}(V_j)$ while $i$ determines a class in $S_n/S_{\mu}$, $S_{\mu} = S_{\mu_1} \times \cdots \times S_{\mu_s}$, and is inductively defined by: $i_1$ is such that $F_1 \subset V_{i_1}$; $i_j$ is such that $F_s \cap V_{i_j} \neq \{0\}$ for $j = 1, ..., s$ and $F_s \cap V_h = \{0\}$ for every $h \neq i_1, ..., i_s$.

1.5. Consider the closed subset $\Gamma = \{(t, F) | t \in D, F \in \mathcal{F}_{\psi(t)}\}$ of the product $D \times \mathcal{F}$. We set $\pi: \Gamma \to D$ the restriction to $\Gamma$ of the canonical projection of $D \times \mathcal{F}$ on $D$; for every $t \neq 1$ the fibres $\Gamma_t = \pi^{-1}(t)$ are all isomorphic. We have a trivialization of $\Gamma^* = \pi^{-1}(D^*)$ over $D = D - \{1\}$. In fact let $X_i$ be the matrix having for columns the vectors $v_i(t)$ defined as in the lemma; we have $X_i^{-1}A(t) X_i = D_i$ where $D_i$ is the diagonal matrix with diagonal $(t, t^2, ..., t^{q_i})$.

If we set $\mathcal{F}'$ the fixed point variety of $D_i$ (we can note that $\mathcal{F}'$ is independent on $t$), then the claimed trivialization is given by:

$$\psi: D^* \times \mathcal{F}' \to \Gamma^*, \quad (t, F) \to (t, X_iF),$$
$$\psi^{-1}: \Gamma^* \to D^* \times \mathcal{F}', \quad (t, F) \to (t, X_i^{-1}F).$$

1.6. After Spaltenstein we know that $\mathcal{F}_\sigma$ can be paved by locally closed cells isomorphic to affine spaces and that this cellular structure is compatible with the filtration $V^j_\sigma = p^{-1}(P(V_j)) \cap \mathcal{F}_\sigma$ where $0 \subset W_1 \subset W_2 \cdots \subset W_{\mu_1} = \text{Ker}(v)$ is a total flag of subspaces of $\text{Ker}(v)$ refining the partial flag $\text{Ker}(v) \cap \text{Im}(v)^{\mu_1-1}$.

More precisely $V^j_\sigma - V^{j-1}_\sigma$ is isomorphic to $\Delta_j \times \mathcal{F}_\sigma$, where $\Delta_j = P(W_j) - P(W_{j-1})$ and the open cells are naturally indexed by standard Young tableaux of shape $\sigma$ ([8, 9], cf. also [3, 4]).

In particular $\mathcal{F}_\sigma$ can be seen as a subcomplex of $\mathcal{F}$ (the numbers of cells of $\mathcal{F}_\sigma$ is $n! / \sigma_1! \sigma_m!$) and so there exists an open neighbourhood $U$ of $\mathcal{F}_\sigma$ in $\mathcal{F}$ such that $\mathcal{F}_\sigma$ is a deformation retract of $U$. We want to show that $\Gamma$ is homotopically equivalent to $\Gamma_1$ (it is the mean lemma of Morse theory applied to $|\pi|: \Gamma - \Gamma_1 \to R$; cf. [7]).

For an arbitrarily small real positive number $\tau$ the map $H_t: \Gamma \times [0, 1] \to \Gamma, \quad ((t, F), s) \to (t_s X_t, X_t^{-1}F)$, $t_s = \tau(st)/|t| + (1-s)t$ is a deformation retract of $\Gamma$ to $\pi^{-1}(D(1, \tau))$. Since $\pi$ is a proper map (hence closed) there exist positive numbers $\delta, \tau$ such that $D \times U \supset \pi^{-1}(D(1, \delta)) \supset \pi^{-1}(D(1, \tau))$ and this completes the proof.

1.7. Now in the following diagram

$$\begin{array}{ccc}
\Gamma_1 & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
D \times \mathcal{F} & \longrightarrow & \mathcal{F}
\end{array}$$
the maps $\Gamma_1 \to \Gamma$ and $\mathcal{F} \to D \times \mathcal{F}$ are homotopic equivalences and his leads us to the cohomology sequence (from now on cohomology means $\mathbb{Q}$-cohomology: $H^*(\cdot) = H^*(\cdot, \mathbb{Q})$): $H^*(\mathcal{F}) \to H^*(\mathcal{F}_o) \to H^*(\mathcal{F}_{o(t)})$. Like $\mathcal{F}_o$, the variety $\mathcal{F}_{o(t)}$ is paved by affine cells being a disjoint union of products of flag varieties. Since the cells of $\mathcal{F}_o$ and $\mathcal{F}_{o(t)}$, are all even dimensional we have: $H^i(\mathcal{F}_o) = H^i(\mathcal{F}_{o(t)}) = 0$ for $i$ odd; so the map $H^*(\mathcal{F}) \to H^*(\mathcal{F}_o)$ induced by the inclusion is surjective (and $S_n$-equivariant [4]).

2. Relation between the Cohomologies of the Fibres

2.1. The cohomology algebra $H^*(\mathcal{F})$ can be described as follows: let $\mathcal{V}_j$ be the subbundle of the trivial vector bundle $\mathcal{F} \times V$ over $\mathcal{F}$ whose fibre at $(V_i)_{i=1}^n \in \mathcal{F}$ is just $V_i$. If $x'_i$ denotes the first Chern class of the line bundle $\mathcal{V}_i/\mathcal{V}_{i-1}$ we have [6]:

1. the cohomology $H^*(\mathcal{F})$ is generated by $x'_1, \ldots, x'_n$ as an algebra,
2. the kernel of the mapping $\alpha: \mathbb{Q}[x_1, \ldots, x_n] \to H^*(\mathcal{F})$, $\alpha(x_i) = x'_i$ is the ideal $(e_1, \ldots, e_n)$ generated by the elementary symmetric functions.

$S_n$ acts on $\mathcal{F}$ as follows: for any $(V_i)_{i=1}^n \in \mathcal{F}$ there exists $g \in U(n)$ so that $V_i = \bigoplus_{j=1}^n C g(v_j)$ where $\{v_j | j = 1, \ldots, n\}$ is the canonical basis of $V = \mathbb{C}^n$. Then the action of $\omega \in S_n$ on $\mathcal{F}$ can be defined by:

$$(V_i)_{i=1}^n \omega = (V'_j)_{j=1}^n \text{ with } V'_j = \bigoplus_{j=1}^n C g(v_{\omega^{-1}(j)}) .$$

Thus $S_n$ acts on $H^*(\mathcal{F})$ which becomes isomorphic, as $S_n$-algebra, to $\mathbb{Q}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$, that is, the regular representation of $S_n$.

2.2. Proposition. $H^*(\mathcal{F}_{o(t)})$ as an $S_n$-module is isomorphic to the representation induced from the regular representation of the subgroup $S_{\mu}$ (that is, the regular representation of $S_n$).

Proof: $S_n/S_{\mu}$ acts transitively on the components of $\mathcal{F}_{o(t)}$ and so the result follows from Proposition 1.4 and from the picture of the cohomology of a flag variety.

2.3. We set $A = H^*(\mathcal{F})$, $B = H^*(\mathcal{F}_o)$, $C = H^*(\mathcal{F}_{o(t)})$; we obtain $C = \bigoplus_{\eta} C_\eta$, $\eta$ running through a set of representatives of $S_n/S_{\mu}$, $C_\eta$ isomorphic to $H^*(\mathcal{F}(V_\eta)) \otimes \cdots \otimes H^*(\mathcal{F}(V_1))$. The map $d: A \to C$ as in 1.7 is the map $d: x_i \to (x_i^\eta)_\eta$, $x_i^\eta \in C_\eta$, and so clearly $S_n$-equivariant. Furthermore $d$ factors through $l: B \to C$. In the next proposition we prove the injectivity of the map $l$. Note that this will provide a new proof of the existence of a Springer representation for $SL(n, \mathbb{C})$. 

2.4. Proposition. The map \( l: H^*(\mathcal{F}_n) \rightarrow H^*(\mathcal{F}_{n+1}) \) is injective.

Proof. We proceed by induction on the number \( n \) of variables and so we may assume the assertion is true for \( n-1 \) variables. Let \( I = (x_1) \) and \( G = (l(x_1)) \) be the ideals generated, in \( B \) and in \( C \) respectively, by \( x_1 \) and \( l(x_1) \) and consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & I^s/I^{s+1} & \rightarrow & B/I^{s+1} & \rightarrow & B/I^s & \rightarrow & 0 \\
& & \downarrow l_i & & \downarrow l_i & & & \\
0 & \rightarrow & G^s/G^{s+1} & \rightarrow & C/G^{s+1} & \rightarrow & C/G^s & \rightarrow & 0
\end{array}
\]

where \( l_i \) and \( l_i \) are induced maps. By induction, we may assume \( l_i \) injective and so we have only to prove the injectivity of \( l^s \).

For every \( 1 \leq s \leq \mu_1 \), in the commutative diagram of cohomology exact sequences

\[
\begin{array}{cccccc}
0 & \rightarrow & H^*(V^{s+1}_j, V^s_j) & \rightarrow & H^*(V^s_j) & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \\
0 & \rightarrow & H^*(\mathcal{P}(W_{s+1}), \mathcal{P}(W_s)) & \rightarrow & H^*(\mathcal{P}(W_{s+1})) & \rightarrow & H^*(\mathcal{P}(W_s)) & \rightarrow & 0
\end{array}
\]

we have \( H^*(\mathcal{P}(W_j)) = \mathbb{Q}[x_1]/(x_1^i), j = s, s + 1 \), and \( H^*(\mathcal{P}(W_{s+1}), \mathcal{P}(W_s)) = H^{2s}(\mathcal{P}(W_{s+1}), \mathcal{P}(W_s)) = u_s \), where \( u_s \) maps to \( x_1^i \).

Furthermore [2], \( H^*(V^{s+1}_j, V^s_j) \) is isomorphic to \( I^s/I^{s+1} \) and as \( H^*(\mathcal{F}_{n+1}) \)-module is free over \( u_s \). Since between the direct factors of \( G^s/G^{s+1} \) there are the \( (n-1)!/\lambda_1! \cdots \lambda_r! \), \( \lambda = \mu^{s+1} \), that make it possible to apply induction, we get the injectivity of \( q \circ l \mid H^*(\mathcal{F}_{n+1}) \), where \( q \) is the projection over these factors, whence the assertion.

3. Harmonic Polynomials

3.1. Let us recall the results of Hulsurkar [5]. Let \( W \) be a Weyl group acting faithfully in a real space \( E \) and let \( \Delta \) be its root system; denote by \( \delta_1, \ldots, \delta_l \) the fundamental weights determined by \( \Delta \) and by \( X \) the weight-lattice. For \( \omega \in W \) define \( \delta_\omega = \sum a_\omega \delta_i \) where \( I_\omega = \{ i | 1 \leq i \leq l, l(\cos_i) < l(\omega) \} \); here \( s_1, \ldots, s_l \) are the Coxeter generators of \( W \), being the reflections corresponding to the simple roots \( \alpha_1, \ldots, \alpha_l \), and \( l(\cdot) \) denotes the length function on \( W \) (see, for example, [11]). For linear and affine transformations of \( E \) we shall denote the operation on the right and shall compose them accordingly. Put

\[
P_\delta(x) = \prod_{x \in \Delta_+} (x, x') / \prod_{x \in \Delta_+} (\delta, x'), \text{ for } x \in V;
\]

\[
P_{\Delta, \omega}(x) = P_\delta(x \omega + \delta_\omega), \text{ for } x \in V \text{ and } \omega \in W.
\]
Here $(\cdot, \cdot)$ is a $W$-invariant symmetric bilinear form on $E = X \otimes \mathbb{R}$ and $\alpha' = 2\alpha/\alpha, \alpha)$. The polynomials $P_\alpha(x)$ and $P_{\alpha, \omega}(x)$ belong to the space $\mathcal{H}$ of all polynomials annihilated by all homogeneous differential operators (with constant coefficients) of positive degree that are invariant under $W$. In fact $P_{\alpha, \omega}(x) = \text{sgn } \omega P_\alpha(x + \epsilon_\omega)$, where $\epsilon_\omega = (\delta_\omega) \omega^{-1}$ and $\text{sgn } \omega$ is defined as in [1], and we know that the translate $H(x + \lambda)$ of an harmonic polynomial $H(x)$ is again harmonic (use Taylor's theorem and the fact that the derivatives of $H(x)$ are harmonic).

Denote by $W'$ the semidirect extension of $W$ by the normal subgroup of translation with respect to the elements of $X$: then $W'$ acts on $\mathcal{H}$ (the action is defined to be $P(x) \varphi = P(x \varphi^{-1})$, $x \in E$, $\varphi \in W'$, $P \in \mathcal{H}$). Let $T_\delta_{\omega}$ be the translation corresponding to $\delta_{\omega}$ and let $w_\omega$ be the map $w_{\omega} = \varphi(x + \epsilon_\omega)$, which lies in $W'$; we have $P_{\alpha, \omega}(x) = P_\alpha(xw_\omega)$. The following theorem tells us that $\mathcal{H}$ is the cyclic $W'$-module generated by $P_\alpha$.

3.2. THEOREM [5]. The set of polynomials $\{P_{\alpha, \omega} | \omega \in W\}$ is a basis of the space of the $W$-harmonic polynomials, moreover $\{P_{\alpha, \omega} | \omega \in W\}$ is a $\mathbb{Z}$-basis of the lattice $\mathcal{H}_\mathbb{Z}$ consisting of those harmonic polynomials $\Phi \in \mathcal{H}$ for which $\Phi(x) \in \mathbb{Z}$ for all $x \in X$.

3.3. We can now prove the theorem stated in the introduction.

THEOREM. Under the identification of $H_* (\mathcal{H}, \mathbb{Q})$ with the $\mathbb{Q}$-space of $S_\mu$-harmonic polynomials, the homology $H_* (\mathcal{H}, \mathbb{Q})$ becomes the cyclic $S_\mu$-submodule generated by $P_{\alpha}$, where $\Delta$ is the root system of the Weyl group $S_\mu$.

Proof. We have the following sequence of maps (homology is $\mathbb{Q}$-homology):

$$H_* (\mathcal{H} (V_1) \times \cdots \times \mathcal{H} (V_r)) \xrightarrow{\alpha} H_* (\mathcal{H} (\mathcal{F}_*) \times \mathcal{H} (\mathcal{F}_*) \times \cdots \times \mathcal{H} (\mathcal{F} (V_r))) \xrightarrow{\beta} H_* (\mathcal{H} (\mathcal{F}_*)) \xrightarrow{\gamma} H_* (\mathcal{H}).$$

where $\beta \gamma$ is the dual of the map $\alpha$ in (2.3) and so is $S_\mu$-equivariant; $\gamma$ is the map induced by inclusion and so is injective (1.7) as well as $\alpha \beta \gamma$.

Now let $F$ be an element of $H_* (\mathcal{F} *)$; there exists (Prop. 2.4) $G \in H_* (\mathcal{H} (\mathcal{F}_*))$ with $G \beta = F$ and (Prop. 2.2) $G = \sum G_\eta \eta$, $\{\eta\}$ being a set of representatives of $S_\mu/S_\mu$, $G_\eta \in H_* (\mathcal{H} (V_1) \times \cdots \times \mathcal{H} (V_r))$. By Theorem 3.2 we have $G_\eta = \sum c_{\eta, \omega}^\eta P_{\alpha, \omega}$ where $c_{\eta, \omega}^\eta \in \mathbb{Q}$ and $\Delta$ is the root system of $S_\mu$. By abuse of notations, denote by $P_{\alpha, \omega}$ the image of $P_{\alpha, \omega}$ under $\alpha \beta \gamma$; then we get

$$F = \sum_{\eta, \omega} c_{\eta, \omega}^\eta (P_{\alpha, \omega} \eta)$$

as an element of $H_* (\mathcal{F})$. 

In this way we obtain both an expression of an element of $H_\ast(\mathcal{F}_\sigma)$ in terms of $S_n$-harmonic polynomials (since a $S_\mu$-harmonic polynomial is clearly $S_n$-harmonic) and a group action of $S'_n$ on $H_\ast(\mathcal{F}_\sigma)$ extending the action of $S_n$ which yields the claimed result. In fact let $\mathcal{O}$ be a weight which belongs to the weight-lattice of $S_n$, then we have:

$$FT_\mathcal{O} = \left( \sum_{\omega, \omega} c_n^\mathcal{O}(P_{A, \omega} \eta) \right) T_\mathcal{O} = \sum_{\omega, \omega} c_n^\mathcal{O}(P_{A, \omega} T_{S_n} \eta) \eta.$$

By definition of $P_A$ the only significant translation are those relative to the elements of the weight-lattice of $S_\mu$, so $P_{A, \omega} T_{S_n}$ lies in $H_\ast(\mathcal{F}(V_1) \times \cdots \times \mathcal{F}(V_r))$, the $S'_n$-action on $H_\ast(\mathcal{F}_\sigma)$ is well defined, and we are done.

3.4. Note that we can even work over the integers; in fact by Theorem 3.2 $H_\ast(\mathcal{F}, \mathbb{Z})$ is isomorphic to $\mathbb{H}_\mathcal{Z}$ while $H_\ast(\mathcal{F}_\sigma, \mathbb{Z})$ is without torsion.

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