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Positive Values of Non-homogeneous Indefinite Quadratic Forms of Type (2, 4)

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1. INTRODUCTION

Let $Q(x_1, ..., x_n)$ be a real indefinite quadratic form in n variables of type $(r, n-r)$ and determinant $D \neq 0$. Blaney [9] has shown that there exist constants Γ, independent of Q and depending only on n and r, such that given any real numbers $c_1, ..., c_n$ there exist $(x_1, ..., x_n) \equiv (c_1, ..., c_n) \pmod 1$ such that

$$0 < Q(x_1, ..., x_n) \leqslant (\Gamma |D|)^{1/n}.$$

Let $\Gamma_{r, n-r}$ denote the infimum of all such numbers Γ. In this notation the following results are known:

$\Gamma_{1,1} = 4$, Davenport and Heilbronn [11].

$\Gamma_{2,1} = 4$, Blaney [10] and Barnes [7].

$\Gamma_{1,2} = 8$, $\Gamma_{3,1} = 16/3$, $\Gamma_{2,2} = 16$, Dumir [12–14].

$\Gamma_{1,3} = 16$, Dumir and Hans-Gill [15].

$\Gamma_{3,2} = 16$, $\Gamma_{4,1} = 8$, Hans-Gill and Madhu Raka [19, 20].

$\Gamma_{r, n-r}$ for $s = 2r - n = 0, \pm 1, 2, 3$, Bambah *et al.* [4–6].

$\Gamma_{r, r+2}$ and $\Gamma_{r, r+3}$ for $r \geqslant 3$, Aggarwal and Gupta [1, 2].

$\Gamma_{r+4, r}$ for $r \geqslant 1$, Aggarwal and Gupta [3].

$\Gamma_{2,5} = 32$, Dumir and Sehmi [17].

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Dumir *et al.* [16] have proved that $\Gamma_{r,n-r}$ depends only on signature $s = 2r - n \pmod 8$ for $n \geqslant 6$. Thus $\Gamma_{r,n-r}$ is known except for $\Gamma_{2,4}$ and $\Gamma_{1,4}$. It is easy to see that $\Gamma_{1,4} \geqslant 8$. Dumir and Sehmi [18] have shown that $\Gamma_{1,4} \leqslant 16$. The expected value is 8. It may be remarked here that for larger values of n the evaluation of $\Gamma_{r,n-r}$ is relatively easy. (For $n \geqslant 21$, see M. Flahive, *Indian J. Pure Appl. Math.* **19** (1988), 931–959.) For small values of n, detailed analysis and careful investigation is needed. In this paper we shall prove that $\Gamma_{2,4} = 64/3$, thereby proving the conjecture of Bambah *et al.* [4] in this case. More precisely we prove:

THEOREM. *Let $Q(x_1, ..., x_6)$ be a real indefinite quadratic form of type $(2, 4)$ and determinant $D \neq 0$. Then given any real numbers $c_1, ..., c_6$ there exist $(x_1, ..., x_6) \equiv (c_1, ..., c_6) \pmod 1$ such that*

$$0 < Q(x_1, ..., x_6) \leqslant \left(\frac{64}{3}\,|D|\right)^{1/6}. \tag{1.1}$$

Moreover, equality in (1.1) is needed if and only if Q is equivalent to pQ_1 or pQ_2 and $(c_1, ..., c_6)$ is equivalent to P_1 or P_2 respectively, where $p > 0$ and

$$Q_1 = x_1x_2 + x_3x_4 - x_5^2 - x_5x_6 - x_6^2, \qquad P_1 = (0, ..., 0)$$

and

$$Q_2 = x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_5x_6 - x_6^2, \qquad P_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0).$$

2. SOME LEMMAS

In the course of the proof we shall use the following lemmas:

LEMMA 1 [Lagrange]. *If $Q(x, y)$ is a positive definite form of determinant Δ, then $Q \sim ax^2 + bxy + cy^2$, where $0 \leqslant b \leqslant a \leqslant c$ and*

$$0 < a \leqslant (\tfrac{4}{3}\Delta)^{1/2}.$$

LEMMA 2 [Markoff]. *If $Q(x, y)$ is an indefinite non-zero quadratic form of determinant Δ, and if Q is not equivalent to $x^2 + xy - y^2$, then there exist integers u, v such that*

$$0 < |Q(u, v)| \leqslant (\Delta/2)^{1/2}.$$

LEMMA 3 [Gauss and Seeber]. *Any positive definite ternary form of determinant Δ represents a number b with $0 < b \leqslant (2\Delta)^{1/3}$.*

LEMMA 4. (Watson [27]). *Any non-zero ternary form of type* (1, 2) *and determinant* Δ *represents a number* b *with* $0 < b \leqslant (4\Delta)^{1/3}$.

LEMMA 5 (Venkov [25]). *Any non-zero ternary form of type* (1, 2) *and determinant* Δ *represents a number* b *with* $|b| \leqslant (2\Delta/3)^{1/3}$.

LEMMA 6 (Oppenheim [24]). *Any non-zero form* $Q_{1,3}$ *of determinant* Δ *represents a number* b *with* $|b| \leqslant (2\,|\Delta|/9)^{1/4}$ *except when* $Q_{1,3} \sim pG_i$, $i = 1, 2, 3$, *where* $G_1 = -[x^2 + y^2 + z^2 - u^2 - xu - yu - zu]$, $G_2 = -[x^2 + xy - y^2 + 2(z^2 + zu + u^2)]$, *and* $G_3 = -[2(x^2 + xy - y^2) + z^2 + zu + u^2]$.

LEMMA 7 (Dumir [13, 14]). *Let* α, β, γ *be real numbers with* $\gamma > 1$. *Suppose that* m *is the integer defined by* $m < \gamma \leqslant m + 1$. *Let* x_0 *be any real number.*

(a) *There exists* $x \equiv x_0 \pmod 1$ *satisfying*

$$0 < -(x + \alpha)^2 + \beta < \gamma,$$

provided

$$\frac{1}{4} < \beta < \frac{m^2}{4} + \gamma.$$

(b) *There exists* $x \equiv x_0 \pmod 1$ *satisfying*

$$0 < (x + \alpha)^2 + \beta < \gamma,$$

provided

$$-\frac{m^2}{4} < \beta < \gamma - \frac{1}{4}.$$

It is convenient to use the following convention: For a polynomial $P(x_1, ..., x_n)$ and real numbers α, β we say that the inequality

$$\alpha < P(x_1, ..., x_n) < \beta$$

is soluble if for any real numbers $c_1, ..., c_n$ there exist $(x_1, ..., x_n) \equiv (c_1, ..., c_n) \pmod 1$ satisfying this inequality.

LEMMA 8 (Dumir and Hans-Gill [15]). *If* $Q(x_1, ..., x_4)$ *is a quadratic form of type* (1, 3) *with determinant* D, *then*

$$0 < Q(x_1, ..., x_4) \leqslant (16\,|D|)^{1/4}$$

is soluble.

LEMMA 9 (Jackson [21]). *Let* $Q(x_1, ..., x_5)$ *be a zero form of type* (1, 4) *or* (2, 3) *and determinant* D. *Then*

$$\alpha_1 < Q(x_1, ..., x_5) < \alpha_2$$

is soluble provided $\alpha_2 - \alpha_1 > 2\,|D|^{1/5}$.

LEMMA 10 (Macbeath [22]). *Let α and β be given real numbers with $\alpha \neq 0$. Then for any real number v, there exist integers x, y satisfying*

$$0 < x + \beta y - \alpha y^2 + v \leqslant (2|\alpha|)^{1/3}.$$

LEMMA 11 (Macbeath [22]). *Let α, β, A be real numbers with $\alpha \neq 0$. Let $2h$, k be positive integers such that*

$$|h - k^2|\alpha|| + \tfrac{1}{2} < A. \tag{2.1}$$

Further, suppose that either $|\alpha| \neq h/k^2$ or $\beta \not\equiv h/k(\text{mod } 1/k, 2\alpha)$, i.e., $\beta - h/k$ is not an integral linear combination of $1/k$ and 2α. Then for any real number v, there exist integers x, y satisfying

$$0 < x + \beta y - \alpha y^2 + v < A. \tag{2.2}$$

This result follows from Lemma 6 of Macbeath [22]. The special case $h = 1/2$, $k = 1$ in this lemma will be used several times. So we state it separately.

LEMMA 11′. *Let a, β, A be real numbers with $a \neq 0$. Suppose that* (i) $1/2 < |a| < A$ *or* (ii) $1 - A < |a| < 1/2$ *or* (iii) $|a| = 1/2 < A$ *and* $\beta \not\equiv 1/2$ (mod 1). *Then for any real number v, there exist integers x, y satisfying*

$$0 < x + \beta y - ay^2 + v < A. \tag{2.3}$$

3. PROOF OF THE THEOREM

If Q is an incommensurable quadratic form, then the result follows by well known results of Margulis [23] and Watson [27]. So we can suppose that Q is a rational form of determinant $D \neq 0$. By Meyer's Theorem it is a zero form. Following the proof of Lemma 12 of Birch [8] and using homogeneity we can suppose that either

$$\begin{aligned} Q = (x_1 + a_2 x_2 + a_4 x_4 + a_5 x_5 + a_6 x_6) x_2 \\ + m(x_3 + b_4 x_4 + b_5 x_5 + b_6 x_6) x_4 - Q_{2,0}(x_5, x_6), \end{aligned}$$

or

$$Q = (x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_6 x_6) x_2 + Q_{1,3}(x_3, x_4, x_5, x_6),$$

where m is a positive integer, $Q_{2,0}$ is a positive definite quadratic form and $Q_{1,3}$ is a non-zero rational form of type (1, 3). We can suppose that $-1/2 < a_i \leqslant 1/2$ and $-1/2 < b_j \leqslant 1/2$ for each i and j. Further, Theorem 13 of Watson [28] gives that if $a_2 = 0$ then $a_i = 0$ for each i and if $b_4 = 0$ then

$h_5 = h_6 = a_4 = 0$. We can also suppose that $-1/2 < c_i \leqslant 1/2$ for each i. Let $d = (64\,|D|/3)^{1/6}$. We shall show that

$$0 < Q(x_1, ..., x_6) < d \tag{3.1}$$

is soluble except when Q is equivalent to ρQ_1 or ρQ_2, $\rho > 0$ and c_i are as stated in the theorem.

LEMMA 12. *If Q represents a number a such that $0 < |a| < d/3$ or $d/2.48 \leqslant |a| < d/2$, then* (3.1) *is soluble.*

Proof. We can suppose that Q represents a primitively. Replacing Q by an equivalent form we can suppose that

$$Q = a(x_1 + h_2 x_2 + \cdots + h_6 x_6)^2 + \phi(x_2, ..., x_6).$$

By homogeneity we can suppose that $a = \pm 1$, so that $d > 2$. Let m be the integer satisfying $m < d \leqslant m+1$. Then $m \geqslant 2$.

Case (i) $a = 1$.

Here (3.1) becomes

$$0 < (x_1 + h_2 x_2 + \cdots + h_6 x_6)^2 + \phi(x_2, ..., x_6) < d.$$

By Lemma 7(b), it is enough to show that

$$-\frac{m^2}{4} < \phi(x_2, ..., x_6) < d - \frac{1}{4} \tag{3.2}$$

is soluble.

Since Q is a rational form, so is ϕ. Also ϕ is indefinite being of type (1, 4). Hence by Meyer's Theorem, ϕ is a zero form. By Lemma 9, (3.2) is soluble if

$$\frac{m^2-1}{4} + d > 2\,|D|^{1/5} = \left(\frac{3}{2}\,d^6\right)^{1/5},$$

i.e. if

$$f(d) = \left(\frac{m^2-1}{4} + d\right) d^{-6/5} > \left(\frac{3}{2}\right)^{1/5}. \tag{3.3}$$

Now $f(d)$ is a decreasing function of d and $d \leqslant m+1$. Therefore (3.3) is satisfied if

$$f(m+1) = \tfrac{1}{4}(m+3)(m+1)^{-1/5} > (\tfrac{3}{2})^{1/5}.$$

Since $f(m+1)$ is an increasing function of m, for $m \geqslant 3$, we have

$$f(m+1) \geqslant f(4) = \tfrac{3}{2}(4)^{-1/5} > (\tfrac{3}{2})^{1/5}.$$

For $m=2$, $f(d) = (d+3/4)d^{-6/5} > (3/2)^{1/5}$ if $2 < d \leqslant 2.48$.

Case (ii) $a = -1$.

This case is dealt in an analogous manner using Lemma 7(a) and Lemma 9.

$$4. \quad Q = (x_1 + a_2 x_2 + a_4 x_4 + a_5 x_5 + a_6 x_6) x_2 + m(x_3 + b_4 x_4 + b_5 x_5 + b_6 x_6) x_4 - Q_{2,0}(x_5, x_6)$$

4.1. Let Δ = determinant of $Q_{2,0}$. Then $m^2\Delta/16 = D = 3d^6/64$ and so $\Delta = 3d^6/4m^2$. Let $a = \min\{Q_{2,0}(X) : 0 \neq X \in \mathbb{Z}^2\}$. By Lemma 1, $Q_{2,0}$ represents a primitively with

$$0 < a \leqslant \left(\frac{4\Delta}{3}\right)^{1/2} = \frac{d^3}{m}. \tag{4.1}$$

Since Q represents $-a$, by Lemma 12, (3.1) is soluble except when

$$\frac{d}{2} \leqslant a \leqslant \frac{d^3}{m} \quad \text{or} \quad \frac{d}{3} \leqslant a \leqslant \frac{d}{2.48}, \tag{4.2}$$

so that

$$d^2 \geqslant \frac{m}{3} \geqslant \frac{1}{3} \quad \text{and hence} \quad d > \frac{1}{2}. \tag{4.3}$$

LEMMA 13. *Inequality* (3.1) *is soluble if* (i) $c_2 \neq 0$, *or* (ii) $c_2 = 0$ *and* $d > 1$. *In particular this is so if* $c_2 = 0$ *and* $m \geqslant 4$.

Proof. Choose $x_2 = c_2$ or 1 according as $c_2 \neq 0$ or $c_2 = 0$. Take $(x_3, ..., x_6) = (c_3, ..., c_6)$ and then choose $x_1 \equiv c_1 \pmod 1$ such that

$$0 < Q = (x_1 + \cdots)x_2 + m(x_3 + \cdots)x_4 - Q_{2,0}(x_5, x_6) \leqslant |x_2| < d.$$

Since $m \geqslant 4$ implies that $d > 1$, the lemma is proved.

Remark 1. Now we suppose that $c_2 = 0$, $m \leqslant 3$ and $d \leqslant 1$. Moreover, we can suppose that

$$Q_{2,0} = a(x_5 + \lambda x_6)^2 + \left(\frac{\Delta}{a}\right) x_6^2,$$

where $0 \leqslant \lambda \leqslant \frac{1}{2}$. We notice that if we write $x_1 = x + c_1$, $x_5 = y + c_5$, $x_2 = \pm 1$ and choose $(x_3, x_4, x_6) \equiv (c_3, c_4, c_6) \pmod 1$ arbitrarily, then (3.1) reduces to an inequality of the type

$$0 < x + \beta y - a y^2 + v < d, \tag{4.4}$$

where $\beta = \pm a_5 + m b_5 x_4 - 2ac_5 - 2a\lambda x_6$ and v is some constant. Solubility of (3.1) follows if we can find integers x and y satisfying (4.4). This inequality is of the type (2.2) with $d = \Delta$. We shall make repeated use of Macbeath's result (Lemmas 11 and 11′).

LEMMA 14. *If $m = 2$ or 3, then (3.1) is soluble.*

Proof. Here (4.3) along with Remark 1, implies that $d = 1$ if $m = 3$ and $d > 3/4$ if $m = 2$ and so $a \geqslant d/3 > 1 - d$. Also

$$a \leqslant \frac{d^3}{m} \leqslant \frac{1}{m} \leqslant \frac{1}{2}.$$

Therefore by Lemma 11′, there exist integers x, y satisfying (4.4) unless $m = 2$, $d = 1$, $a = 1/2$ and $\beta = \pm a_5 + 2b_5 x_4 - c_5 - \lambda x_6 \equiv 1/2 \pmod 1$. Taking $x_6 = c_6$ and $1 + c_6$ we get $\lambda \equiv 0 \pmod 1$, i.e., $\lambda = 0$. Since $a = 1/2$, $d = 1$ therefore

$$Q = (x_1 + \cdots)x_2 + 2(x_3 + \cdots)x_4 - (1/2)x_5^2 - (3/8)x_6^2.$$

So 3/8 is a value of $Q_{2,0}$, which is not possible since $a = 1/2$ is the minimum value of $Q_{2,0}$.

Remark 2. We are now left with $m = 1$.

4.2. $m = 1$.

Here $Q = (x_1 + a_2 x_2 + \cdots)x_2 + (x_3 + b_4 x_4 + \cdots)x_4 - a(x_5 + \lambda x_6)^2 - (\Delta/a)x_6^2$. Arguing as in Lemma 13, we see that (3.1) is soluble if $c_4 \neq 0$. So we can now suppose that

$$c_2 = c_4 = 0, \qquad \frac{d}{3} \leqslant a \leqslant d^3 \leqslant d, \qquad \frac{1}{\sqrt{3}} \leqslant d \leqslant 1. \tag{4.5}$$

By Lemma 11′, there exist integers x, y satisfying (4.4) if (i) $1/2 < a < d$ or (ii) $a < 1/2$ and $a + d > 1$ or (iii) $a = 1/2$ and $\beta \not\equiv 1/2 \pmod 1$. Therefore we are through by Lemma 11′ except when (i) $a = d = 1$ or (ii) $a < 1/2$, $a + d \leqslant 1$ or (iii) $a = 1/2$ and $\beta \equiv 1/2 \pmod 1$.

LEMMA 15. *If $a=d=1$, then (3.1) is soluble except when Q is equivalent to pQ_1 or pQ_2 and $(c_1, ..., c_6)$ is equivalent to P_1 or P_2 respectively, where $p>0$ and Q_1, Q_2, P_1, P_2 are as in the Theorem. In these cases (1.1) is soluble with the sign of equality being necessary.*

Proof. Here

$$Q=(x_1+a_2x_2+\cdots)x_2+(x_3+b_4x_4+\cdots)x_4-(x_5+\lambda x_6)^2-3/4x_6^2.$$

Choosing $(x_1, x_2, x_5)=(x+c_1, \pm 1, y+c_5)$, $(x_3, x_4, x_6)\equiv(c_3, c_4, c_6)$ (mod 1), (3.1) reduces to an inequality of the type (2.2) with $a=A=1$ and $\beta=\pm a_5+b_5x_4-2c_5-2\lambda x_6$. Applying Lemma 11 with $h=k=1$ it is easy to see that (3.1) is soluble unless

$$\pm a_5+b_5x_4-2c_5-2\lambda x_6\equiv 0 \quad (\text{mod } 1). \tag{4.6}$$

Taking $x_6=c_6$ and $1+c_6$ we get $\lambda\equiv 0(\text{mod } \frac{1}{2})$ and thus

$$\lambda=0 \text{ or } 1/2. \tag{4.7}$$

If $\lambda=0$, then 3/4 is a value of $Q_{2,0}$, which is not possible since $a=1$ is the minimum value. Let $\lambda=1/2$. Then (4.6) becomes

$$\pm a_5+b_5x_4-2c_5-x_6\equiv 0 \quad (\text{mod } 1) \tag{4.8}$$

Taking $x_4=c_4$ and $1+c_4$, we get $b_5=0$. Interchanging the roles of x_2 and x_4 in the above argument we get $a_5=0$. Thus (4.8) reduces to

$$2c_5+c_6\equiv 0 \quad (\text{mod } 1).$$

Symmetry w.r.t. x_5 and x_6 gives $a_6=b_6=0$ and

$$2c_6+c_5\equiv 0 \quad (\text{mod } 1).$$

so that $c_5=c_6=0$, $1/3$ or $-1/3$. Thus

$$Q=(x_1+a_2x_2+a_4x_4)x_2+(x_3+b_4x_4)x_4-x_5^2-x_5x_6-x_6^2.$$

Now b_4 is a value of Q therefore (3.1) is soluble except when $b_4=0$ or $|b_4|\geqslant d/3$. For $b_4\neq 0$, $|1/2-|b_4||+1/2<1$, so that choosing $x_1=x+c_1$, $x_2=\pm 1$, $x_3=c_3$, $x_5=c_5$, $x_6=c_6$ and $x_4=y+c_4$ and applying Lemma 11′, (3.1) is soluble unless $b_4=1/2$ and $\pm a_4+c_3\equiv 1/2(\text{mod } 1)$, i.e., $(a_4, c_3)=(0, 1/2)$ or $(1/2, 0)$. Thus we are left with (i) $b_4=0$ in which case by symmetry we can suppose that $c_3=0$ or (ii) $b_4=1/2$ and $(a_4, c_3)=(0, 1/2)$ or $(1/2, 0)$.

Similarly we can show that either (i) $a_2=1/2$ and $(a_4, c_1)=(0, 1/2)$ or $(1/2, 0)$ or (ii) $a_2=0$ in which case $a_4=0$ by a result of Watson [26] and hence $c_1=0$.

Case (i) $b_4 = 0 = c_3$.

Choose $(x_1, x_2, x_3, x_4) = (c_1, 0, 1, 1)$, $|x_6| \leqslant 1/2$, $|x_5 + (1/2)x_6| \leqslant 1/2$ then $0 < Q < 1$ unless $x_6 = 0$ and $x_5 + (1/2)x_6 = 0$, i.e., $c_5 = c_6 = 0$. Now if $a_2 = a_4 = c_1 = 0$ then

$$Q = x_1 x_2 + x_3 x_4 - (x_5 + (1/2)x_6)^2 - (3/4)x_6^2$$
$$= x_1 x_2 + x_3 x_4 - x_5^2 - x_5 x_6 - x_6^2 \equiv 0 \pmod 1$$

for integers x_i and $Q(1, 1, 0, 0, 0, 0) = 1$ so that (1.1) is soluble with equality where as (3.1) is not soluble.

If $a_2 = 1/2$ and $(a_4, c_1) = (1/2, 0)$ then $Q(0, 1, 0, 0, 0, 0) = 1/2$ so that (3.1) is soluble in this case.

If $a_2 = 1/2$ and $(a_4, c_1) = (0, 1/2)$ then

$$Q = (x_1 + (1/2)x_2)x_2 + x_3 x_4 - x_5^2 - x_5 x_6 - x_6^2$$

and

$$(c_1, ..., c_6) = (1/2, 0, ..., 0)$$

so that $Q(x_1, ..., x_6) \equiv 0 \pmod 1$ for $(x_1, ..., x_6) \equiv (1/2, 0, ..., 0) \pmod 1$ and $Q(1/2, 0, 1, 1, 0, 0) = 1$, i.e., (3.1) is not soluble whereas (1.1) is soluble with equality. Moreover Q is equivalent to ρQ_2 and $(c_1, ..., c_6)$ goes to P_2 under the corresponding transformations.

Case (ii). $b_4 = 1/2$ and $(a_4, c_3) = (1/2, 0)$ or $(0, 1/2)$.

If $(a_4, c_3) = (1/2, 0)$ then $a_2 = 1/2$ and $c_1 = 0$. (Since $a_2 = 0$ implies $a_4 = 0$ and since $a_4 = 1/2$ we have $c_1 = 0$). Therefore

$$Q = (x_1 + (1/2)x_2 + (1/2)x_4)x_2 + (x_3 + (1/2)x_4)x_4$$
$$- (x_5 + (1/2)x_6)^2 - (3/4)x_6^2.$$

Choosing $(x_1, ..., x_4) = (0, 0, 0, 1)$ and $|x_6| \leqslant 1/2$, $|x_5 + (1/2)x_6| \leqslant 1/2$, we have $0 < Q < 1$ so that (3.1) is soluble.

If $(a_4, c_3) = (0, 1/2)$ and $a_2 = c_1 = 0$, as before it is easy to see that (3.1) is soluble unless

$$Q = x_1 x_2 + (x_3 + (1/2)x_4)x_4 - x_5^2 - x_5 x_6 - x_6^2$$

and

$$(c_1, ..., c_6) = (0, 0, 1/2, 0, 0, 0),$$

in which case (1.1) is soluble with equality. Moreover in this case Q is equivalent to ρQ_2 and $(c_1, \ldots, c_6)$ goes to P_2 under the corresponding transformations.

If $(a_4, c_3) = (0, 1/2)$ and $a_2 = 1/2$ we must have $c_1 = 1/2$ (since $a_4 = 0$). Now

$$Q = (x_1 + (1/2)x_2)x_2 + (x_3 + (1/2)x_4)x_4 - (x_5 + (1/2)x_6)^2 - (3/4)x_6^2.$$

Again we can show that (3.1) is soluble unless $c_5 = c_6 = 0$ in which case $Q(1/2, 1, 1/2, 0, 0, 0) = 1$ and $Q(x_1, \ldots, x_6) \equiv 0 \pmod 1$ for $(x_1, \ldots, x_6) \equiv (1/2, 0, 1/2, 0, 0, 0) \pmod 1$ so that (1.1) is soluble with equality being necessary. Again Q is equivalent to ρQ_2 and $(c_1, \ldots, c_6)$ goes to P_2 under the corresponding transformations. This proves the lemma.

LEMMA 16. *If $a < 1/2$, $a + d \leqslant 1$ and $d \leqslant 3/4$, then (4.4) is soluble for $d > 0.7$ unless $a = 1/4$, $\lambda = c_5 = 0$, $a_5 = 0$ or $1/2$ and $b_5 = 0$ or $1/2$.*

Proof. Taking $h = 1$, $k = 2$ and $A = d$ in Lemma 11, it is easy to see that

$$|1 - 4a| + \tfrac{1}{2} < d,$$

is satisfied for $d > 0.7$. Hence (4.4) is soluble unless $a_4 = 1/4$, and $\beta \equiv 0 \pmod{1/2}$, i.e., $\pm a_5 + b_5 x_4 - (1/2)c_5 - (1/2)\lambda x_6 \equiv 0 \pmod{1/2}$. Taking $x_6 \equiv c_6$ and $1 + c_6$ we get $\lambda \equiv 0 \pmod 1$ and so $\lambda = 0$. Taking $x_4 = 0$ and 1, we get $b_5 \equiv 0 \pmod{1/2}$. Since $m = 1$, by symmetry $a_5 \equiv 0 \pmod{1/2}$. For $a_5 \equiv 0 \pmod{1/2}$, $b_5 \equiv 0 \pmod{1/2}$ and $\lambda = 0$ we get $c_5 = 0$.

Remark 3. If $a \geqslant d/2$, then $d^3 \geqslant a \geqslant d/2$ gives $d^2 \geqslant 1/2$, i.e., $d > 0.7$, and $a \geqslant d/2 \geqslant 1/2\sqrt{2} > 1/4$. Thus, in this case result follows by Lemma 16, so we can now suppose by (4.2) that

$$\frac{d}{3} \leqslant a \leqslant \frac{d}{2.48}. \tag{4.9}$$

LEMMA 17. *If $d/3 \leqslant a \leqslant d/2.48$, then (3.1) is soluble for $a < 1/2$, $a + d \leqslant 1$ and $d \leqslant 3/4$.*

Proof. Taking $(x_2, x_3, x_4) = (1, c_3, 0)$, the inequality (3.1) becomes

$$0 < x_1 + a_2 + a_5 x_5 + a_6 x_6 - a(x_5 + \lambda x_6)^2 - (\Delta/a)\, x_6^2 < d.$$

This can be written as

$$0 < a^{-1}(x_1 + a_6' x_6 + v_6' - (\Delta/a)x_6^2) - (x_5 + \lambda x_6 + a_5/2a)^2 < d/a, \tag{4.10}$$

where a'_6 and v' are suitable real numbers. By Lemma 7(a), the inequality (4.10) is soluble if we can solve

$$\frac{1}{4} < a^{-1}\left(x_1 + a'_6 x_6 + v' - \frac{\Delta}{a} x_6^2\right) < 1 + \frac{d}{a}.$$

Write $x_1 = x + c_1$ and $x_6 = y + c_6$. Then this inequality becomes

$$0 < x + a''_6 y + v'' - \frac{\Delta}{a} y^2 < d + \frac{3a}{4}, \tag{4.11}$$

for some real numbers a''_6 and v''.

Case (i). $d^2 < 125/288$.

By Lemma 10, (4.11) is soluble in integers x and y if

$$\left(\frac{2\Delta}{a}\right)^{1/3} < d + \frac{3a}{4},$$

which is satisfied because $\Delta = 3d^6/4$, $a \geqslant d/3$, and $d^2 < 125/288$.

Case (ii). $d^2 \geqslant 125/288$.

Here we shall use Lemma 11′ with $A = d + 3a/4$ and Δ/a instead of a. Since $d/3 \leqslant a \leqslant d^3$ and $\Delta/a + 3a/4 + d > 1$, therefore by Lemma 11′, it remains to consider the case $\Delta/a = 1/2$ or $a = 3d^6/2$. Since $a \geqslant d/3$, this gives $d > 0.7$. By Lemma 16, it follows that (4.4) is soluble unless $a = 1/4$ (so that $d^6 = 1/6$) and $a_5 = 0$ or $1/2$, $b_5 = 0$ or $1/2$ and $\lambda = c_5 = 0$. In this case

$$Q = (x_1 + a_2 x_2 + \cdots)x_2 + (x_3 + b_4 x_4 + b_5 x_5 + b_6 x_6)x_4 - \tfrac{1}{4}x_5^2 - \tfrac{1}{2}x_6^2.$$

If $b_5 = 1/2$, we have

$$\begin{aligned} Q &= (x_1 + a_2 x_2 + \cdots)x_2 + (x_3 + b'_4 x_4 + b_6 x_6)x_4 - \tfrac{1}{4}(x_5 - x_4)^2 - \tfrac{1}{2}x_6^2 \\ &\sim (x_1 + a_2 x_2 + a'_4 x_4 + a'_5 x_5 + a_6 x_6)x_2 + (x_3 + b'_4 x_4 + b_6 x_6)x_4 \\ &\quad - \tfrac{1}{4}x_5^2 - \tfrac{1}{2}x_6^2. \end{aligned}$$

So we can suppose that $b_5 = 0$. Similarly we can suppose that $a_5 = 0$. Therefore

$$Q = (x_1 + a_2 x_2 + a_4 x_4 + a_6 x_6)x_2 + (x_3 + b_4 x_4 + b_6 x_6)x_4 - \tfrac{1}{4}x_5^2 - \tfrac{1}{2}x_6^2.$$

Take $(x_1, ..., x_6) = (x + c_1,\ 1,\ c_3, 0, 0, y + c_6)$ or $(c_1, 0, x + c_3, 1, 0, y + c_6)$ or $(x + c_1, 1, c_3, 1, 0, y + c_6)$. By Lemma 11′, it is easy to see that (3.1) is soluble unless

$$a_6 - c_6 \equiv \tfrac{1}{2} \pmod 1$$

$$b_6 - c_6 \equiv \tfrac{1}{2} \pmod 1$$

and

$$a_6 + b_6 - c_6 \equiv \tfrac{1}{2} \pmod 1.$$

Therefore $a_6 = b_6 = 0$ and $c_6 = 1/2$. Now we have

$$Q = (x_1 + a_2 x_2 + a_4 x_4) x_2 + (x_3 + b_4 x_4) x_4 - \tfrac{1}{4} x_5^2 - \tfrac{1}{2} x_6^2$$

and

$$(c_2, c_4, c_5, c_6) = (0, 0, 0, \tfrac{1}{2}).$$

We shall now show that (3.1) is soluble unless $b_4 = 0,\ \pm 1/4,\ 1/2$. Let $b_4 \neq 0$. Take $(x_1, ..., x_6) = (x + c_1,\ 1,\ c_3,\ y,\ 0,\ 1/2)$. Then (3.1) will be soluble if there exist integers x, y satisfying

$$0 < x + (a_4 + c_3)\, y + b_4 y^2 + v < 6^{-1/6} = d = 0.7418. \tag{4.12}$$

If $1 - d < |b_4| < 1/2$, then this follows by Lemma 11′. So let $0 < |b_4| \leqslant 1 - d$, $|b_4| \neq 1/4$. By Lemma 12, (3.1) is soluble except when $|b_4| \geqslant d/3$. Now using Lemma 11, with $h = 1$, $k = 2$, the condition $|1 - 4\,|b_4|| + 1/2 < d$, is easily seen to be satisfied. Therefore by Lemma 11 and Lemma 11′, the inequality (4.12) is soluble unless $b_4 = 0,\ \pm 1/4,\ 1/2$. Now we discuss the special cases depending on b_4.

Case (i). $b_4 = 0$.

In this case $Q = (x_1 + \cdots) x_2 + x_3 x_4 - (1/4) x_5^2 - (1/2) x_6^2$. If $c_3 \neq 0$, choose $(x_1, x_2, x_5, x_6) = (c_1, c_2, c_5, c_6)$ and x_4 such that

$$0 < Q \leqslant |c_3| \leqslant \tfrac{1}{2} < d.$$

If $c_3 = 0$, then $0 < Q(c_1, 0, 1, 1, 1, 1/2) = 5/8 < d$.

Case (ii) $b_4 = 1/2$ or $1/4$.

Choosing $(x_1, ..., x_4, x_6) = (c_1, 0, c_3, \pm 1, 1/2)$ so that $x_3 x_4 = |c_3|$ and $x_5 = 0$ or 1 according as $b_4 = 1/4$ or $1/2$, it can be seen that

$$0 < Q = |c_3| + b_4 - \tfrac{1}{4} x_5^2 - \tfrac{1}{8} \leqslant \tfrac{5}{8} < d.$$

Case (iii). $b_4 = -1/4$.

Choosing $(x_1, \ldots, x_6) = (c_1, 0, \pm 1 + c_3, \pm 1, 0, 1/2)$ in such a way that $x_3 x_4 = 1 - |c_3|$ we have $0 < Q = 1 - |c_3| - 1/4 - 1/8 \leqslant 5/8 < d$.

LEMMA 18. *The inequality* (3.1) *is soluble when* $a = 1/2$.

Proof. By Lemma 1, we can take

$$Q = (x_1 + a_2 x_2 + \cdots)x_2 + (x_3 + b_4 x_4 + \cdots)x_4 - (\tfrac{1}{2}x_5^2 + b x_5 x_6 + c x_6^2),$$

where $0 \leqslant b \leqslant 1/2 \leqslant c$.

As before we convert the inequality $0 < Q < d$ to an inequality of the type (2.2) by making different substitutions given below:

$$\begin{aligned}(x_1, x_2, x_3, x_4, x_5, x_6) = {} & (x + c_1, 1, c_3, 0, y + c_5, c_6)\\ \text{or } & (x + c_1, 1, c_3, 0, y + c_5, 1 + c_6)\\ \text{or } & (x + c_1, 1, c_3, 1, y + c_5, c_6)\\ \text{or } & (c_1, 0, x + c_3, 1, y + c_5, c_6).\end{aligned}$$

By Lemma 11′, the inequality (3.1) is soluble except when

$$\begin{aligned}-bc_6 - c_5 + a_5 &\equiv \tfrac{1}{2} \pmod 1\\ -b(c_6 + 1) - c_5 + a_5 &\equiv \tfrac{1}{2} \pmod 1\\ -bc_6 - c_5 + a_5 + b_5 &\equiv \tfrac{1}{2} \pmod 1\\ -bc_6 - c_5 + b_5 &\equiv \tfrac{1}{2} \pmod 1.\end{aligned}$$

From these congruences we get

$$b = a_5 = b_5 = 0 \qquad \text{and} \qquad c_5 = 1/2. \tag{4.13}$$

In this case

$$Q = (x_1 + a_2 x_2 + a_4 x_4 + a_6 x_6)x_2 + (x_3 + b_4 x_4 + b_6 x_6)x_4 - \tfrac{1}{2}x_5^2 - c x_6^2,$$

$d^6 = (64/3)\ |D| = 2c/3$ and so $c < 3/2$ because $d < 1$. Since $c \geqslant 1/2$ we get $d^6 \geqslant 1/3$. Since b_4 is a value of Q, therefore (3.1) is soluble except when $|b_4| \geqslant d/3$ or $b_4 = 0$. If $b_4 \neq 0$ then $|b_4| + d \geqslant d/3 + d = 4d/3 > 1$, therefore by Lemma 11′, (3.1) is soluble unless $b_4 = 0$ or $1/2$.

Again we convert the inequality (3.1) to that of type (2.2) with $\alpha = c$, $A = d$ by the substitution $(x_1, \ldots, x_6) = (c_1, 0, x + c_3, 1, c_5, y + c_6)$. For $1/2 < c < d$, the result follows by Lemma 11′. So let us suppose that $c \geqslant d$. Then $d^5 \geqslant 2/3$ and so $d > 0.9$. Since

$$|1 - c| + \tfrac{1}{2} < d,$$

applying Lemma 11 with $h=k=1$, it follows that (3.1) is soluble unless $c=1$. Thus, we are left with $c=\frac{1}{2}$ and 1.

Case (i). $c=\frac{1}{2}$.

Interchange of x_5 and x_6 shows that $c_6=\frac{1}{2}$, $a_6=b_6=0$. Thus $Q=(x_1+a_2x_2+a_4x_4)x_2+(x_3+b_4x_4)x_4-\frac{1}{2}x_5^2-\frac{1}{2}x_6^2$, $c_5=c_6=\frac{1}{2}$, $b_4=0$ or $\frac{1}{2}$.

Choosing $(x_1, x_2, x_3, x_5, x_6)=(c_1, 0, c_3, \frac{1}{2}, \frac{1}{2})$ and $x_4=\pm 1$ so that $x_3x_4=|c_3|$, it can be easily seen that (3.1) is satisfied for $b_4=\frac{1}{2}$. If $b_4=0$, then interchanging x_3 and x_4 we see that $c_3=0$. Here take $x_2=0$, $x_3=x_4=1$, $x_5=x_6=\frac{1}{2}$.

Case (ii). $c=1$.

Here $d^6=2/3$; i.e., $d=0.93, \ldots$. We convert (3.1) into an inequality of the type (2.2) by making different substitutions given below

$$(x_1, \ldots, x_6)=(x+c_1, 1, c_3, 0, \tfrac{1}{2}, y+c_6) \quad \text{or} \quad (x+c_1, 1, c_3, 1, \tfrac{1}{2}, y+c_6),$$

or

$$(c_1, 0, x+c_3, 1, \tfrac{1}{2}, y+c_6).$$

Applying Lemma 11 with $\alpha=h=k=1$, $A=d$, it follows that (3.1) is soluble unless

$$a_6-2c_6\equiv 0 \pmod 1$$

$$a_6+b_6-2c_6\equiv 0 \pmod 1$$

$$b_6-2c_6\equiv 0 \pmod 1.$$

These congruences imply $a_6=b_6=0$ and $c_6=0$ or $\frac{1}{2}$. In this case

$$Q=(x_1+a_2x_2+a_4x_4)x_2+(x_3+b_4x_4)x_4-\tfrac{1}{2}x_5^2-x_6^2,$$

where $b_4=0$ or $\frac{1}{2}$. These special cases can be dealt with easily as done in Case (i).

5. $Q=(x_1+a_2x_2+\cdots+a_6x_6)x_2+Q_{1,3}(x_3, \ldots, x_6)$

Here $Q_{1,3}$ is a non-zero rational form of type (1, 3) and determinant $\Delta=4|D|=3d^6/16$.

LEMMA 19. *The inequality* (3.1) *is soluble if* (i) $c_2\neq 0$ *and* $d>1/2$ *or* (ii) $c_2=0$ *and* $d>1$ *or* (ii) $c_2=0$ *and* $d<1/\sqrt{3}$.

Proof. Proof of (i) and (ii) is similar to that of Lemma 13. For the proof of (iii) we note that by Lemma 8, the inequality $0 < Q_{1,3} \leqslant (16\,|\Delta|)^{1/4} = (3d^6)^{1/4}$ is soluble. Therefore taking $x_2 = 0$, it follows that for $d < 1/\sqrt{3}$ and $c_2 = 0$, (3.1) is soluble. This completes the proof of the lemma.

Suppose first that $Q_{1,3}$ is not equivalent to ρG_i, $i = 1, 2, 3$. Since $Q_{1,3}$ is a rational form, $Q_{1,3}$ represents a, where

$$|a| = \min\{\,|Q_{1,3}(X)| : 0 \neq X \in \mathbb{Z}^4\}.$$

By Lemma 6, we have

$$0 < |a| \leqslant \left(\frac{2\,|\Delta|}{9}\right)^{1/4} = \left(\frac{d^6}{24}\right)^{1/4}. \tag{5.1}$$

Since Q represents a, the inequality (3.1) is soluble by Lemma 12, if $|a| < d/3$. So let us suppose that $|a| \geqslant d/3$ and hence $d^2 \geqslant 8/27 > 1/4$.

Remark 4. In view of Lemma 19, we can suppose that $c_2 = 0$ and $1/\sqrt{3} \leqslant d \leqslant 1$. Moreover we have

$$\frac{d}{3} \leqslant |a| \leqslant \left(\frac{d^6}{24}\right)^{1/4} < \frac{d}{2} \leqslant \frac{1}{2}. \tag{5.2}$$

Since $Q_{1,3}$ represents a, we can write

$$Q = (x_1 + a_2\, x_2 + \cdots)x_2 + a(x_3 + b_4 x_4 + \cdots)^2 + \phi(x_4, x_5, x_6).$$

Putting $(x_1, ..., x_6) = (x + c_1,\ 1,\ y + c_3,\ c_4,\ c_5,\ c_6)$, the inequality (3.1) is converted into an inequality of the type (2.2). By Lemma 11′, this inequality is soluble in integers x, y if $|a| + d > 1$. So we can suppose that

$$|a| + d \leqslant 1 \qquad \text{and} \qquad d \leqslant \tfrac{3}{4}. \tag{5.3}$$

LEMMA 20. *The inequality* (3.1) *is soluble for* $|a| > 2/9$ *unless* $|a| = 1/4$.

Proof. Proceeding as in the above Remark and applying Lemma 11 with $h = 1$ and $k = 2$, the inequality (3.1) is soluble if

$$|1 - 4\,|a|\,| + \tfrac{1}{2} < d. \tag{5.4}$$

If $|a| > \frac{1}{4}$, then using (5.3) for $d > 0.7$ and using (5.2) for $d \leqslant 0.7$, it is easy to see that (5.4) is satisfied. If $|a| < 1/4$, then (5.4) is satisfied if $3/2 < d + 4\,|a|$. Otherwise $2/9 < |a| \leqslant (1/4)(3/2 - d)$. Again apply Lemma 11 with $h = 2$ and $k = 3$. Then it is easy to see that $|2 - 9\,|a|\,| + 1/2 < d$, so that (2.2) and hence (3.1) is soluble in this case.

LEMMA 21. *The inequality* (3.1) *is soluble if* $d/3 \leqslant a \leqslant 2/9$ *or if* $a = 1/4$.

Proof. Here

$$Q = (x_1 + a_2 x_2 + \cdots)x_2 + a(x_3 + \cdots)^2 - Q_{3,0},$$

where the positive definite form $Q_{3,0}$ has determinant $\delta = 4D/a = 3d^6/16a$. Let b be the minimum value of $Q_{3,0}$. By Lemma 3, $Q_{3,0}$ represents b with

$$0 < b \leqslant (2\delta)^{1/3} = \left(\frac{3d^6}{8a}\right)^{1/3} \leqslant \left(\frac{9d^5}{8}\right)^{1/3}. \tag{5.5}$$

Now we can suppose that

$$Q = (x_1 + a_2 x_2 + \cdots)x_2 + a(x_3 + b_4 x_4 + \cdots)^2 - b(x_4 + \cdots)^2 - Q_{2,0}.$$

Take $x_2 = 1$. Then (3.1) is soluble if we can solve

$$0 < \left(x_3 + b_4 x_4 + b_5 x_5 + b_6 x_6 + \frac{1}{2} a_3 a^{-1}\right)^2$$
$$+ a^{-1}[x_1 + a'_4 x_4 + \cdots + a'_6 x_6 - b(x_4 + \cdots)^2 + \cdots + v'] < \frac{d}{a} \tag{5.6}$$

Here v' is a suitable real number and $d/3 \leqslant a \leqslant 1/4 < d/2$ and so $2 < d/a \leqslant 3$. Therefore by Lemma 7(a) with $m = 2$, (5.6) is soluble if we can solve

$$-1 < a^{-1}[x_1 + a'_4 x_4 + \cdots + a'_6 x_6 - b(x_4 + \cdots)^2 - Q_{2,0} + v'] < \frac{d}{a} - \frac{1}{4},$$

or

$$0 < x_1 + a'_4 x_4 + \cdots + a'_6 x_6 - b(x_4 + \cdots)^2 - Q_{2,0} + v < d + \frac{3a}{4}, \tag{5.7}$$

where v is a constant.

Putting $x_1 = x + c_1$, $x_4 = y + c_4$, $x_5 = c_5$ and $x_6 = c_6$, we get an inequality of the type (2.2). Using (5.5) it is easy to see that $b < d + 3a/4$ so that applying Lemma 11′ with b and $d + 3a/4$ in place of a and A respectively if follows that (5.7) is soluble for $b > 1/2$. Now suppose that $b < 1/2$. Since a is a value of section $a(x_3 + b_4 x_4)^2 - bx_4^2 = f(x_3, x_4)$ of $Q_{1,3}$ and $a = \min\{|Q_{1,3}(X)| : 0 \neq X \in \mathbb{Z}^4\}$, therefore $a = \min\{|f(x_3, x_4)| : x_3, x_4$ integers not both zero$\}$ and hence by Lemma 2, either $f \sim a(x_3^2 + x_3 x_4 - x_4^2)$ or $0 < a \leqslant (ab/2)^{1/2}$, i.e., $a \leqslant b/2$. Consequently for $a \leqslant b/2$ we have $b + d + 3a/4 \geqslant d + 11a/4 \geqslant d + 11d/12 > 1$, so that the condition of Lemma 11′ is satisfied for $b < 1/2$. Thus (3.1) is soluble unless $b = 1/2$ or $f \sim a(x_3^2 + x_3 x_4 - x_4^2)$.

Case (i) $b=1/2$.

$$Q=(x_1+a_2x_2+\cdots)x_2+a(x_3+\cdots)^2-\tfrac{1}{2}(x_4\cdots)^2-Q_{2,0},$$

where $Q_{2,0}$ represents α such that

$$0<\alpha\leqslant\left(\frac{d^6}{2a}\right)^{1/2}. \tag{5.8}$$

Without loss of generality, we can suppose that

$$Q_{2,0}=\alpha(x_5+\lambda_6x_6)^2+\alpha'x_6^2.$$

Now (5.7) becomes

$$0<-(x_4+\cdots)^2-2[\alpha(x_5+\cdots)^2+\alpha'x_6^2-(x_1+\cdots)+\nu_1]$$
$$<2d+\frac{3a}{2}. \tag{5.9}$$

Since $1<2d+3a/2<2$, by Lemma 7(a), (5.9) is soluble if we can solve

$$\frac{1}{4}<-2[\alpha(x_5+\cdots)^2+\alpha'x_6^2-(x_1+\cdots)+\nu_1]<2d+\frac{3a}{2}+\frac{1}{4},$$

or

$$0<-\alpha[x_5+\cdots]^2-\alpha_6'x_6^2+x_1+\cdots+\nu_2<d+\frac{3a}{4}. \tag{5.10}$$

Consider the section $-(1/2)(x_4+\lambda_5x_5)^2-\alpha x_5^2$ of $-Q_{3,0}$. It represents $-k$, where $0<k\leqslant(2\alpha/3)^{1/2}$. Also $k\geqslant b=1/2$. Therefore $\alpha\geqslant 3/8$. This gives $\alpha+d+(3a/4)>1$. If $d/3\leqslant a\leqslant 2/9$, then (5.8) gives $\alpha<1/2$, so that applying Lemma 11′, it is easy to see that (5.10) is soluble. If $a=1/4$, then (5.8) gives $\alpha\leqslant\sqrt{2}d^3$. It can be easily verified that if $\alpha\neq 1/2$, then the conditions of Lemma 11′ are satisfied and so (5.10) is soluble for $\alpha\neq 1/2$.

If $\alpha=1/2$, then $Q_{1,3}(x_3,x_4,x_5,0)=(1/4)(x_3+\cdots)^2-(1/2)(x_4+\cdots)^2-(1/2)x_5^2$ is rationally equivalent to a zero form which is a contradiction.

Case (ii) $a(x_3+b_4x_4)^2-bx_4^2\sim a(x_3^2+x_3x_4-x_4^2)$.

Here $b=5a/4\leqslant 5/16<1/2$. If $b+d+3a/4=d+2a>1$, then (5.7) is soluble by Lemma 11′. Let us now suppose that $d+2a\leqslant 1$. Since $a\geqslant d/3$, we have $d\leqslant 3/5$. Using Lemma 11 with $h=1$, $k=2$, the inequality (5.7) is soluble for $b\neq 1/4$ if

$$|1-4b|+\frac{1}{2}<d+\frac{3a}{4}. \tag{5.11}$$

Since $b=5a/4$, $1/4 \geqslant a \geqslant d/3$, and $d \geqslant 1/\sqrt{3}$, it is easy to see that (5.11) is satisfied and hence (5.7) is soluble unless $b=1/4$.

If $b=1/4$, then $a=1/5$. Again we proceed as in Case (i). Here

$$Q=(x_1+a_2x_2+\cdots)x_2+\tfrac{1}{5}(x_3+\cdots)^2-\tfrac{1}{4}(x_4+\cdots)^2-Q_{2,0},$$

where $Q_{2,0}$ represents α with $0<\alpha \leqslant (5d^6)^{1/2}$. Since $d/3 \leqslant a=1/5$ we have $d \leqslant 3/5$.Therefore $\alpha<1/2$. In this case (5.7) can be written as

$$0< -(x_4+\cdots)^2+4[x_1+\cdots-\alpha(x_5+\cdots)^2-\alpha' x_6^2+v'']<4d+\tfrac{3}{5}. \quad (5.12)$$

Since $2<4d+3/5 \leqslant 3$, by Lemma 7(a), (5.12) is soluble if we can solve

$$\tfrac{1}{4}<4[x_1+\cdots-\alpha(x_5+\cdots)^2-\alpha' x_6^2+v'']<1+4d+\tfrac{3}{5},$$

i.e.,

$$0<x_1+\cdots-\alpha(x_5+\cdots)^2-\alpha' x_6^2+v''<d+\frac{27}{80}. \quad (5.13)$$

Consider the section

$$Q_{1,3}(x_3,x_4,x_5,0)=\tfrac{1}{5}(x_3+\cdots)^2-\tfrac{1}{4}(x_4+\cdots)^2-\alpha x_5^2.$$

By Lemma 5, it represents a number k with $|k| \leqslant (\alpha/30)^{1/3}$. Also $|k| \geqslant a=1/5$ and so $\alpha \geqslant 6/25$. Therefore $\alpha+d+27/80>1$ and hence (5.13) is soluble by Lemma 11'.

LEMMA 22. *The inequality* (1.1) *is soluble if* $a<0$, $d/3 \leqslant |a| \leqslant 2/9$ *or* $a=-1/4$.

Proof. For convenience, writing $-a$ instead of a, we have $d/3 \leqslant a \leqslant 2/9$ or $a=1|/4$ and

$$Q=(x_1+\cdots)x_2-a(x_3+b_4x_4+b_5x_5+b_6x_6)^2+Q_{1,2},$$

where $Q_{1,2}$ is a non-zero form of determinant $-\Delta/a=3d^6/16a$. By Lemma 4, $Q_{1,2}$ represents b with $0<b \leqslant (3d^6/4a)^{1/3}$. Let b be the smallest such number and write $Q_{1,2}=b(x_4+\lambda_5x_5+\lambda_6x_6)^2-Q_{2,0}$, where $0 \leqslant \lambda_5 \leqslant \frac{1}{2}$, $0 \leqslant \lambda_6 \leqslant \frac{1}{2}$.

Now proceeding as in the proof of Lemma 21, using Lemma 7(b), one can easily see that it is enough to prove that

$$0<(x_1+a_4'x_4+\cdots)+b(x_4+\lambda_5x_5+\lambda_6x_6)^2-Q_{2,0}+v<d+\frac{3a}{4}, \quad (5.14)$$

is soluble.

Proceeding as in Lemma 21, it is easy to see that either $-a(x_3+b_4x_4)^2+bx_4^2$ is equivalent to $-a(x_3^2+x_3x_4-x_4^2)$ or $2a \leqslant b$,

$b+d+3a/4>1$ and $b<d+3a/4$. Taking $x_1=x+c_1$, $x_2=1$ $x_4=y+c_4$ and $(x_5, x_6)\equiv(c_5, c_6) \pmod 1$ arbitrarily and applying Lemma 11′, it follows that (5.14) is soluble unless

(i) $b=1/2$ and $a_4'+c_4+\lambda_5 x_5+\lambda_6 x_6\equiv 1/2 \pmod 1$, or

(ii) $b=5a/4$ and $-a(x_3+b_4x_4)^2+bx_4^2\sim -a(x_3^2+x_3x_4-x_4^2)$.

If $b=5a/4$ and $-a(x_3+b_4x_4)^2+bx_4^2\sim -a(x_3^2+x_3x_4-x_4^2)\sim a(x_3^2+x_3x_4-x_4^2)$ then a binary section of $Q_{1,3}$ represents a and so the result follows as in case (ii) of Lemma 21.

Now we are left with $b=1/2$ and $a_4'+c_4+\lambda_5x_5+\lambda_6x_6\equiv 1/2 \pmod 1$. Taking $x_5=c_5$ and $1+c_5$, this congruence implies that $\lambda_5\equiv 0 \pmod 1$. Since $0\leqslant\lambda_5\leqslant 1/2$, we get $\lambda_5=0$. Similarly $\lambda_6=0$. Therefore

$$Q=(x_1+a_2x_2+\cdots)x_2-a(x_3+b_4x_4+\cdots)^2+\tfrac{1}{2}x_4^2-Q_{2,0}(x_5, x_6).$$

By Lemma 2, $Q_{2,0}$ represents c such that

$$0<c\leqslant\left(\frac{4}{3}\cdot\frac{3d^6}{8a}\right)^{1/2}\leqslant\left(\frac{3d^5}{2}\right)^{1/2}<d, \tag{5.15}$$

because $a\geqslant d/3$ and $d\leqslant 3/4$. Without loss of generality we can suppose that

$$Q_{2,0}=c(x_5+\cdots)^2+\cdots.$$

If $c<1/2$, then $(1/2)-c>0$ is a value of $Q_{1,2}$ and is less than $1/2=b$, which is not possible by definition of b. Therefore $c\geqslant 1/2$. If $c=1/2$, then $Q_{1,2}=(1/2)x_4^2-(1/2)(x_5+\cdots)^2+\cdots$ is rationally equivalent to a zero form, which is not the case. If $c>1/2$, then choose $x_1=x+c_1$, $x_2=1$, $(x_3, x_4, x_6)=(c_3, c_4, c_6)$ and $x_5=y+c_5$ and apply Lemma 11′. Since $1/2<c<d$, by (5.15), it follows by Lemma 11′ that (3.1) is soluble in this case.

6. Exceptional Cases: $Q_{1,3}\sim\rho G_i$, $i=1, 2, 3$, $\rho>0$

Case (i) $Q_{1,3}=-\rho[x_3^2+x_4^2+x_5^2-x_6^2-x_6(x_3+x_4+x_5)]$, $\rho>0$.

Here $Q=(x_1+\cdots)x_2+Q_{1,3}$ and $(7/16)\rho^4=D=(3d^6/64)$, so that $\rho=(3d^6/28)^{1/4}$. Since ρ is a value of Q, by Lemma 12, we can suppose

$$\frac{d}{3}\leqslant\rho=\left(\frac{3d^6}{28}\right)^{1/4}\leqslant\frac{d}{2.48} \quad\text{or}\quad \rho\geqslant\frac{d}{2} \tag{6.1}$$

which gives

$$d^2>\tfrac{1}{9}. \tag{6.2}$$

By Lemma 19, it remains to discuss the following cases

(i) $c_2=0,\ 1/\sqrt{3}\leqslant d\leqslant 1,$

(ii) $c_2=0,\ d\leqslant |c_2|.$

First suppose that $c_2=0$ and $1/\sqrt{3}\leqslant d\leqslant 1$. Then $\rho=(3d^6/28)^{1/4}\geqslant d(1/28)^{1/4}>d/2.48$, so that (6.1) gives $\rho\geqslant d/2$ and hence $d>3/4$. Take $x_1=x+c_1,\ \ x_2=1,\ \ x_3=y+c_3,\ \ (x_4,x_5,x_6)=(c_4,c_5,c_6)$. By Lemma 11′, it is easy to see that (3.1) is soluble unless $\rho=1/2$. If $\rho=1/2$, then $d=(7/12)^{1/6}=0.914,\ldots$. Taking $x_2=1$, (3.1) can be written as

$$0<(x_1+a_2+a_3x_3+\cdots)$$
$$-\tfrac{1}{2}(x_3^2+x_4^2+x_5^2-x_6^2-x_3x_6-x_4x_6-x_5x_6)<d.$$

By Lemma 7(a), it is soluble if we can solve

$$0<x_1+a_4x_4+a_5x_5+a_6'x_6+v-\tfrac{1}{2}(x_4^2+x_5^2-\tfrac{5}{4}x_6^2-x_4x_6-x_5x_6)<d. \tag{6.3}$$

Taking $x_1=x+c_1,\ x_6=y+c_6$ and $(x_4,x_5)=(c_4,c_5)$ it reduces to an inequality of the type (2.2). Since $d>5/8$, taking $a=5/8$ and $A=d$ in Lemma 11′, it follows that the inequality is soluble.

Now suppose that $c_2\neq 0$ and $d\leqslant |c_2|\leqslant 1/2$. Let $d'=d/|c_2|$ and $\rho'=\rho/|c_2|$. Then $\rho'\leqslant\rho/d=(3d^2/28)^{1/4}<1/2$. Taking $x_2=c_2$ it is enough to solve

$$0<\pm(x_1+\cdots)-\rho'[(x_3-x_6/2)^2-\tfrac{5}{4}x_6^2-\cdots]<d'. \tag{6.4}$$

Taking $x_1=x+c_1,\ x_3=y+c_3,\ (x_4,x_5)=(c_4,c_5)$ it reduces to an inequality of the type (2.2) with a and A replaced by ρ' and d' respectively. By Lemma 11′, it is soluble if $\rho'+d'>1$ or $\rho+d>|c_2|$, which is satisfied if $d>3/8$ and $\rho\geqslant d/3$. Otherwise suppose that

$$\rho+d\leqslant |c_2|\leqslant \tfrac{1}{2} \qquad \text{and} \qquad d\leqslant \tfrac{3}{8}. \tag{6.5}$$

(6.4) can be rewritten as

$$0<-\left[x_3-\frac{1}{2}x_6-\frac{1}{2\rho'}a_3\right]^2+\frac{5}{4}x_6^2$$
$$+(x_1+a_2x_2+a_4x_4+a_5x_5+a_6'x_6)/\rho+v$$
$$-[x_4^2+x_5^2+\cdots]<\frac{d'}{\rho'}=\frac{d}{\rho}.$$

Since $2 < d/\rho \leqslant 3$, by Lemma 7(a) it is soluble if we can solve

$$\frac{1}{4} < \frac{5}{4} x_6^2 + (x_1 + a_2 x_2 + a_4 x_4 + a_5 x_5 + a_6' x_6)/\rho + v$$
$$- [x_4^2 + x_5^2 - x_4 x_6 - x_5 x_6] < \frac{d'}{\rho'} + 1,$$

i.e.,

$$0 < x_1 + \cdots + \frac{5\rho'}{4} x_6^2 + v' - \rho'[x_4^2 + x_5^2 - x_4 x_6 - x_5 x_6] < d' + \frac{3}{4}\rho'. \quad (6.6)$$

Now

$$\frac{5\rho'}{4} = 5\rho/(4\,|x_2|) \leqslant \frac{5\rho}{4d} \leqslant \frac{5}{4}\left(\frac{3d^2}{28}\right)^{1/4} \leqslant \frac{5}{4}\left[\frac{3}{28}\left(\frac{3}{8}\right)^2\right]^{1/4} < \frac{1}{2}.$$

Since

$$\frac{5\rho'}{4} + d' + \frac{3\rho}{4} = d' + 2\rho' = (d + 2\rho)/|x_2| \geqslant 2\left(d + \frac{2d}{3}\right)$$
$$= \frac{10d}{3} > 1, \qquad \text{by (6.2)}.$$

Therefore taking $x_1 = x + c_1$, $x_6 = y + c_6$, $(x_4, x_5) = (c_4, c_5)$ and $5\rho'/4$ and $d' + 3\rho'/4$ in place of a and A in Lemma 11′, it follows that (6.6) is soluble.

Case (ii)

$$Q_{1,3} = -\rho[x_3^2 + x_3 x_4 - x_4^2 + 2(x_5^2 + x_5 x_6 + x_6^2)] = \rho G_2 \text{ or}$$
$$Q_{1,3} = -\rho[2(x_3^2 + x_3 x_4 - x_4^2) + x_5^2 + x_5 x_6 + x_6^2] = \rho G_3.$$

Here

$$Q = (x_1 + a_2 x_2 + \cdots + a_6 x_6) x_2 + Q_{1,3}.$$

In this case $(15/16)\rho^4 = D = (3/64) d^6$ so that $\rho = (d^6/20)^{1/4}$. Since ρ is a value of Q, therefore $d/3 \leqslant \rho = (d^6/20)^{1/4}$ and hence $d^2 \geqslant 20/81$. By Lemma 19, (3.1) is soluble if either $c_2 \neq 0$ and $d > |c_2|$, or $c_2 = 0$ and $d > 1$, or $c_2 = 0$ and $d < 1/\sqrt{3}$.

Suppose first that $c_2 \neq 0$ and $d \leqslant |c_2| \leqslant 1/2$. We want to solve

$$0 < (x_1 + \cdots) x_2 - \rho[x_3^2 + x_3 x_4 - x_4^2 + 2(x_5^2 + x_5 x_6 + x_6^2)] < d \quad (6.7)$$

and

$$0<(x_1+\cdots)x_2-\rho[2(x_3^2+x_3x_4-x_4^2)+x_5^2+x_5x_6+x_6^2]<d. \quad (6.8)$$

Take $x_1=x+c_1$, $x_2=c_2$, $(x_4, x_6)=(c_4, c_6)$ and $(x_3, x_5)=(y+c_3, c_5)$ or $(c_3, y+c_5)$ according as inequality is (6.7) or (6.8), respectively. Then these inequalities reduce to an inequality of the type (2.2) with $a=\rho/|c_2|$ and $A=d/|c_2|$. Since $(\rho+d)/|c_2|\geqslant 4d/|c_2|\geqslant 8d/3>1$ and $\rho/|c_2|\leqslant\rho/d=(d^2/20)^{1/4}\leqslant(1/80)^{1/4}<1/2$, therefore the inequality is soluble by Lemma 11′.

Now suppose that $c_2=0$ and $1/\sqrt{3}\leqslant d\leqslant 1$ and hence $\rho<1/2$. Take $(x_1, ..., x_6)=(x+c_1, 1, y+c_3, c_4, c_5, c_6)$ or $(x+c_1, 1, c_3, c_4, y+c_5, c_6)$ according as inequality considered is (6.7) or (6.8) respectively. By Lemma 11′ with $a=\rho$ and $A=d$, these inequalities are soluble if $\rho+d>1$ which is satisfied if $d>3/4$. Otherwise suppose that $\rho+d\leqslant 1$ and $d\leqslant 3/4$, then $2<d/\rho\leqslant 3$. Taking $x_2=1$, (6.7) and (6.8) can be written as

$$0<(x_1+a_4'x_4+a_5x_5+a_6x_6+v)/\rho-\left[\left(x_3+\frac{1}{2}x_4-\frac{1}{2\rho}a_3\right)^2-\frac{5}{4}x_4^2\right]$$
$$-2x_5^2+\cdots<\frac{d}{\rho},$$

and

$$0<(x_1+a_3x_3+a_4x_4+a_6'x_6+v)/\rho-\left[\left(x_5+\frac{1}{2}x_6-\frac{1}{2\rho}a_5\right)^2+\frac{3}{4}x_6^2\right]$$
$$-2x_3^2+\cdots<\frac{d}{\rho}.$$

By Lemma 7(a) these are soluble if we can solve

$$0<x_1+\cdots+v'+\frac{5\rho}{4}x_4^2-2\rho x_5^2+\cdots<d+\frac{3\rho}{4}, \quad (6.9)$$

and

$$0<x_1+\cdots+v'+\frac{3\rho}{4}x_6^2-2\rho x_3^2+\cdots<d+\frac{3\rho}{4}. \quad (6.10)$$

Take $(x_1, x_4, x_5, x_6)=(x+c_1, y+c_4, c_5, c_6)$ in (6.9) and $(x_1, x_3, x_4, x_6)=(x+c_1, c_3, c_4, y+c_6)$ in (6.10). They reduce to an inequality of the type (2.2). By Lemma 11′, (6.9) and (6.10) are soluble if $d+3\rho/4+5\rho/4>1$ and $d+3\rho/4+3\rho/4>1$, respectively. Otherwise suppose that

$$d+2\rho\leqslant 1 \quad \text{and} \quad d+\frac{3\rho}{2}\leqslant 1, \text{ respectively.}$$

It is easy to see that $2p < 1/2$ in each case. Then taking $(x_1, x_4, x_5, x_6) = (x + c_1, c_4, y + c_5, c_6)$ in (6.9) and $(x_1, x_3, x_4, x_6) = (x + c_1, y + c_3, c_4, c_6)$ in (6.10) and applying Lemma 11′, these inequalities are soluble since $d + 3p/4 + 2p = d + 11p/4 > d + 11d/12 > 1$. This completes the proof of case (ii).

Lemmas 1–22 along with Section 6 complete the proof of the theorem.

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