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Comment

Comment on: "Reduction of static field equation of Faddeev model to first order PDE" [Phys. Lett. B 652 (2007) 384]

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Abstract

The authors of the article [M. Hirayama, C.-G. Shi, Phys. Lett. B 652 (2007) 384, arXiv: 0707.2207] propose an interesting method to solve the Faddeev model by reducing it to a set of first order PDEs. They first construct a vectorial quantity $\boldsymbol{\alpha}$, depending on the original field and its first derivatives, in terms of which the field equations reduce to a linear first order equation. Then they find vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ which identically obey this linear first order equation. The last step consists in the identification of the $\boldsymbol{\alpha}_i$ with the original $\boldsymbol{\alpha}$ as a function of the original field. Unfortunately, the authors overlook a constraint implied by their construction, which invalidates most of their subsequent results.

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The Faddeev model [1,2] (also known as the Skyrme–Faddeev model or the Faddeev–Niemi model) is a nonlinear field theory in $3+1$ dimensions which is known to support knotted solitons, both from an analysis of its topology and stability [3], and from numerical calculations [4–7]. Apart from their existence, however, the analytic information on these solitons is rather sparse.

In the Letter [8], the authors proposed a method to partially solve the static field equations by effectively reducing them to a set of first order equations. Unfortunately, that Letter contains an error which invalidates most of its results. In the sequel we briefly review the construction of [8], point out the error and demonstrate that from their (incorrect) results, incorrect conclusions may be drawn (i.e., one may construct "solutions" which are well known *not* to be solutions of the Faddeev model).

The target space of the Faddeev model is the two-sphere and may be described either by a three-component unit vector field $\vec{n}$ or by a complex field u via stereographic projection. The energy functional for static configurations of the Faddeev model (in terms of the complex field u) is

$$E[u,\bar{u}] = \int d^3\mathbf{x}\,(c_2\epsilon_2 + c_4\epsilon_4), \tag{1}$$

with

$$\epsilon_2 = \frac{4}{(1+|u|^2)^2}(\nabla u \cdot \nabla u^*), \tag{2}$$

$$\epsilon_4 = -8\frac{(\nabla u \times \nabla u^*)^2}{(1+|u|^2)^4}. \tag{3}$$

Following the conventions of [8], we now assume a choice of length units such that $c_2 = 4c_4$ and re-express u by its modulus and phase,

$$u = Re^{i\Phi}, \tag{4}$$

with real functions R and Φ. Then the static field equations can be written like

$$\nabla \cdot \boldsymbol{\alpha} + i\boldsymbol{\beta} \cdot \boldsymbol{\alpha} = 0 \tag{5}$$

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and its complex conjugate, where

$$\boldsymbol{\alpha} \equiv \frac{\nabla u^*}{1+R^2} - \frac{\nabla u^* \times (\nabla u \times \nabla u^*)}{(1+R^2)^3} \tag{6}$$

and

$$\boldsymbol{\beta} \equiv -i\frac{u^*\nabla u - u\nabla u^*}{1+R^2} = \frac{2R^2}{1+R^2}\nabla\Phi. \tag{7}$$

Eq. (5) is the starting point for the analysis in Ref. [8]. Next, the authors observe that the vectors

$$\boldsymbol{\alpha}_1 = (\nabla R \times \nabla\rho)\exp\left(-2i\Phi\frac{R^2}{1+R^2}\right),$$
$$\boldsymbol{\alpha}_2 = (\nabla\Phi \times \nabla\mu) \tag{8}$$

identically obey Eq. (5) for arbitrary complex functions ρ and μ. Due to the linearity of Eq. (5), also the sum $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ obeys this equation.

For a further analysis, the authors then regard ρ and μ as functions of R, Φ and a third function ζ which is unknown at this moment but should obey $\frac{\partial(R,\Phi,\zeta)}{\partial(x_1,x_2,x_3)} \neq 0$ such that the three functions R, Φ, ζ may be used as a new system of curvilinear coordinates. The idea is then to expand the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_i$ into the basis

$$\nabla R, \qquad R\nabla\Phi, \qquad R\nabla R \times \nabla\Phi \tag{9}$$

and to compare coefficients. For the gradient of ζ the authors assume

$$\nabla\zeta = \gamma\nabla R \times R\nabla\Phi + \xi\nabla R + R\eta\nabla\Phi, \tag{10}$$

where γ, ξ and η are, at this moment, *unconstrained* real functions. This assumption is the error we announced at the beginning. The expansion into the basis (9) with unconstrained coefficient functions is only true for a general vector field. However, the l.h.s. of Eq. (10) is a gradient and, therefore, obeys $\nabla \times \nabla\zeta = 0$. Applying this condition to the r.h.s. of the same equation produces constraints which the coefficient functions γ, ξ and η have to obey. Concretely, in an index notation the constraints are

$$(\gamma R R_k \Phi_j)_j - (\gamma R R_j \Phi_k)_j$$
$$+ \epsilon_{kjl}(\xi_j R_l + R\eta_j\Phi_l + \eta R_j\Phi_l) = 0, \tag{11}$$

where the subindices mean partial derivatives. Obviously, the constraints contain first derivatives of the functions γ, ξ, η, as well as second derivatives of R and Φ, and it is not known how to expand these expressions into the basis (9). In fact, coefficient functions γ, ξ, η which obey the constraints (11) cannot be found unless the basis vector fields (9) are known. As this point is crucial for our criticism, we demonstrate it in some more detail in Appendix A. This problem invalidates all the subsequent analysis of Ref. [8], where the comparison of $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ essentially leads to a system of linear equations. All conditions derived in Ref. [8] stem from this linear system of equations (Eqs. (32)–(34) of Ref. [8]) and from the integrability conditions on the arbitrary functions μ and ρ (Eqs. (46), (47) of Ref. [8]).

Let us illustrate how a strict application of the results of Ref. [8] leads to wrong conclusions, demonstrating thereby the incorrectness of these results by a reductio ad absurdum. Concretely, we will show that using their results one may derive easily "solutions" of the Faddeev model which are well known not to be solutions at all. For this purpose, we first summarize the (incorrect) final result of Ref. [8]. As said, they expand the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$ into the basis (9) and compare coefficients. This leads to three complex (i.e., six real) equations for the coefficients. Four of these six equations provide a linear system of equations for the three quantities

$$p = (\nabla R)^2, \tag{12}$$
$$q = \nabla R \cdot R\nabla\Phi, \tag{13}$$
$$r = (R\nabla\Phi)^2, \tag{14}$$

see Eqs. (33), (34) of Ref. [8]. The solution of this linear system allows to express the three quantities p, q and r in terms of ζ derivatives of μ and ρ (multiplied by some given coefficients depending on R, Φ). As μ and ρ are arbitrary and complex, this allows to express the solution by four arbitrary real functions, which are conveniently abbreviated by the letters a, b, c, d (their precise form is given in Eqs. (35)–(38) of Ref. [8]). As the system consists of four equations for only three unknowns, there must exist a linear dependency in the linear system, i.e., a relation among the four real functions a, b, c, d. Concretely, the relation is $a = d$. Further, the functions a, b, c, d obey some inequalities which are related to the definition of the quantities p, q and r. See Eqs. (42) and (43) of Ref. [8].

There remains a third complex equation (two more real equations), see Eq. (32) of Ref. [8]. This equation may be re-expressed as a system of two real linear first order partial differential equations with (R, Φ, ζ) as independent variables, and $(\gamma, \xi, \eta, a, b, c)$ as dependent variables (see Eq. (54) of Ref. [8] for the precise form of these PDEs).

There are no more conditions in Ref. [8], as is explicitly stated in that Letter ("We have thus found the relation that γ, ξ, η, a, b and c should satisfy. It consists of two partial differential equations of first order".). Therefore, any choice of the six functions $(\gamma, \xi, \eta, a, b, c)$ obeying the two linear first order PDEs (Eq. (54) of Ref. [8]) should provide a static solution for the Faddeev model. More precisely, it directly provides a solution for the three quantities

$$p = (\nabla R)^2 = S(R, \Phi, \zeta), \tag{15}$$
$$q = \nabla R \cdot R\nabla\Phi = T(R, \Phi, \zeta), \tag{16}$$
$$r = (R\nabla\Phi)^2 = U(R, \Phi, \zeta), \tag{17}$$

(i.e., it provides the r.h.s. of these equations), from which R and Φ still have to be calculated.

Now, in order to continue with our demonstration of the incorrectness of the procedure just described, let us make some simplifying assumptions for the functions a, b, c. Concretely, we assume $a = 0$ and $b = c$, which immediately leads to

$$p = r = \left[b\left(1 + R^2\right)\right]^{-1}, \tag{18}$$
$$q = 0, \tag{19}$$

see Eqs. (39)–(41) of Ref. [8]. We emphasize that our simplifying assumptions are fully compatible with the conditions derived in Ref. [8], i.e., both with the inequalities of Eq. (43) of [8] and with the PDEs of Eq. (54) of that reference. In fact, the system of two linear first order PDEs (Eq. (54) of Ref. [8]) decouples under these assumptions. Next, we make the further (fully compatible!) simplifying assumption that $\gamma = \text{const.}$, then the l.h.s. of Eq. (54) of Ref. [8] is zero. The resulting two first order differential equations are now ordinary ones and are just the defining equations for the (up to now, arbitrary) functions ξ and η, respectively, for a given but completely arbitrary function b. This implies that *any* solution to the equations

$$(\nabla R)^2 = (R\nabla \Phi)^2, \tag{20}$$

$$\nabla R \cdot R\nabla\Phi = 0 \tag{21}$$

(the so-called complex eikonal equation) should be a solution to the field equations of the Faddeev model (due to the arbitrariness of the function b). But this conclusion is certainly wrong. It is, for instance, well known that the ansatz in toroidal coordinates

$$u = f(\tilde\eta)e^{in\tilde\xi + im\tilde\varphi} \tag{22}$$

provides solutions to the complex eikonal equation for arbitrary integers m and n, see [9], [10] (we use tildes for the torus coordinates in order not to confuse them with the functions introduced above; for the conventions used for the torus coordinates, we refer, e.g., to [9]).

On the other hand, it is well known that the ansatz (22) in toroidal coordinates is *incompatible* with the field equations of the Faddeev model, see, e.g., [11]. We emphasize again that in our reductio ad absurdum we strictly followed the prescription provided in Ref. [8] in that we obeyed all conditions derived there and only used further simplifying assumptions which are completely compatible with the conditions of Ref. [8].

In short, we have demonstrated that the analysis of Ref. [8] contains an error, and that the use of the (incorrect) results of that Letter may lead to wrong conclusions about solutions of the Faddeev model, which was the purpose of this comment.

We think, nevertheless, that the starting point of the Letter [8], i.e., the linear equation (5) and the observation that it is identically obeyed by the family of vectors of Eq. (8), is interesting and deserves further investigation.

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Appendix A

Here we demonstrate that there does not exist a choice for the coefficient functions γ, ξ and η which obeys the constraints (11) unless the basis vectors (9) (and, therefore, the sought after soliton solution of the original field equations) are known explicitly. Following Ref. [8], we assume that γ, ξ and η are functions of the variables R, Φ and ζ, and we re-express the partial derivatives accordingly, i.e.,

$$\gamma_j = \gamma_R R_j + \gamma_\Phi \Phi_j + \gamma_\zeta \zeta_j, \tag{A.1}$$

etc., and we use expression (10) for ζ_j. Then we find after some calculation that the constraints (11) can be written like follows,

$$\begin{aligned} & R\left(\gamma_R + \gamma_\zeta \xi - \xi_\zeta \gamma + \frac{\gamma}{R}\right)(R_k R_j \Phi_j - \Phi_k R_j R_j) \\ & \quad + R^2(\gamma_\Phi + \gamma_\zeta \eta - \eta_\zeta \gamma)(R_k \Phi_j \Phi_j - \Phi_k R_j \Phi_j) \\ & \quad + \epsilon_{kjl} R_j \Phi_l (\eta + R\eta_R + R\eta_\zeta \xi - \xi_\Phi - R\xi_\zeta \eta) \\ & \quad + \gamma R \partial_j (R_k \Phi_j - R_j \Phi_k) = 0. \end{aligned} \tag{A.2}$$

The problem is with the last term $\partial_j(R_k\Phi_j - R_j\Phi_k)$. As long as R and Φ are not known, the expansion of this term into the basis (9) is completely arbitrary, and therefore the constraints on the coefficient functions γ, ξ and η cannot be imposed. The choice $\gamma = 0$, which would make this last term disappear, is forbidden because then $\nabla\zeta$ would be a linear combination of ∇R and $\nabla\Phi$, and the condition $\frac{\partial(R,\Phi,\zeta)}{\partial(x_1,x_2,x_3)} \neq 0$ would be violated.

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