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Barycentric systems and stretchability☆

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Abstract

Using a general resolution of barycentric systems we give a generalization of Tutte's theorem on convex drawing of planar graphs. We deduce a characterization of the edge coverings into pairwise non-crossing paths which are stretchable: such a system is stretchable if and only if each subsystem of at least two paths has at least three free vertices (vertices of the outer face of the induced subgraph which are internal to none of the paths of the subsystem). We also deduce that a contact system of pseudo-segments is stretchable if and only if it is extendible.

MSC: 05C10; 57M15

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1. Introduction

It is a classical result established independently by Wagner [11], Fáry [2] and Stein [7] (which is also a consequence of the Steinitz theorem on convex polytopes [8]) that a simple planar graph has a straight line representation in the plane.

In the early 60s, Tutte [9,10] proved that a planar graph admits a convex embedding (in which each interior vertex is the barycenter of its neighbors) if and only if it is a subdivision of a 3-connected graph. The proof given by Tutte may be extended to the case where the barycentric coefficients giving the position of a vertex are arbitrary strictly positive values summing up to 1.

Both results may be viewed in a more general framework where we ask that vertices in a given subset are barycenters of a given subset of their neighbors with prescribed (possibly null) coefficients summing up to 1:

Problem 1. *Assume we are given a connected plane graph G, a subset S of vertices (which are "free") and a weight function $\alpha : (V(G)\backslash S) \times V(G) \to [0, 1]$ such that, for any $(x, y) \in (V(G)\backslash S) \times V(G)$:*

$$\alpha(x, y) > 0 \quad \Rightarrow \quad \{x, y\} \in E(G), \tag{1}$$

$$\sum_{y \in V(G)} \alpha(x, y) = 1. \tag{2}$$

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Does there exist a function $\Psi : V(G) \to \mathbb{R}^2$, *such that*

$$\forall x \in V(G)\backslash S, \quad \Psi(x) = \sum_{y \in V(G)} \alpha(x, y)\Psi(y) \tag{3}$$

and such that Ψ *induces a planar straight line drawing of G homeomorphic to the original one*?

In such a formalism, Fáry's theorem means that the problem has a positive answer whenever $S = V(G)$, and Tutte's theorem asserts that the problem also has a positive answer when G is a subdivision of a 3-connected planar graph, S is the vertex set of the outer face of G and $\alpha(x, y) > 0$ for any couple (x, y) of adjacent vertices (where $x \notin S$).

As a special instance of the problem, one may also ask when a decomposition of the edge set of G into disjoint pairwise non-crossing paths can be "stretched":

Problem 2. *Let G be a connected plane graph, and let* $\mathscr{P}$ *be a covering of* $E(G)$ *with edge-disjoints and pairwise non-crossing paths. Does there exists a planar straight line embedding of G homeomorphic to the original one, such that all the vertices belonging to a path in* $\mathscr{P}$ *are collinear*?

The latter problem may in turn be viewed as a stretching problem for contact systems of pseudo-segments:

Problem 3. *Let* $\mathscr{A}$ *be a contact system of pseudo-segments. Is it possible to "stretch"* $\mathscr{A}$ *into a homeomorphic contact system of straight line segments*?

We shall give a precise answer to the first problem by proving the necessity and sufficiency of some simple combinatorial properties. We will derive a simplified statement for the second problem and an answer to the third problem in terms of extendibility of the contact system.

Notice that if crossings are allowed in Problems 2 and 3, these problem become NP-hard, as proved by Mnëv [3,4] (see also [5,6]).

The paper is organized as follows: Section 2 will be concerned with general resolutions of barycentric systems defined on graphs; in Section 3, we shall give a generalization of Tutte's theorem that will answer Problem 1; Problem 2 will be given a solution in Section 4, while we shall give a topological solution of Problem 3 in Section 5.

2. Barycentric systems

2.1. Definitions and notation

Let G be a graph and let $A \subseteq V(G)$. We use the notation G_A for the subgraph of G induced by A.

Definition 4. A *barycentric system* Σ is a triple (G, S, α), where the *ground graph* G of Σ is a simple connected finite graph, the *source set* S of Σ is a subset of vertices of G, the *weight function* α of Σ is a function from $(V(G)\backslash S) \times V(G)$ to $[0, 1]$, and where, for $x \in V(G)\backslash S$ and $y \in V(G)$:

$$\alpha(x, y) \neq 0 \;\Rightarrow\; \{x, y\} \in E(G), \tag{4}$$

$$\sum_{v \in V(G)} \alpha(x, v) = 1. \tag{5}$$

Given a function $f : S \to \mathbb{R}^k$, a *solution* of Σ for f is a function $\Psi : V(G) \to \mathbb{R}^k$, such that:

$$\Psi(x) = \begin{cases} f(x) & \text{if } x \in S, \\ \sum_{v \in V(G)} \alpha(x, v)\Psi(v) & \text{otherwise.} \end{cases}$$

Σ is *non-singular* if any function $f : S \to \mathbb{R}^k$ gives rise to a unique solution $\Sigma(f)$; it is *singular*, otherwise.

Lemma 5. *A barycentric system is non-singular if and only if it has a unique solution for some function.*

If a barycentric system Σ is non-singular, the mapping $f \mapsto \Sigma(f)$ is a linear function.

Proof. Let $\Sigma = (G, S, \alpha)$ and let n be the order of G. Consider a fixed numbering of the vertices of G. Define the $n \times n$ square matrix A (with coefficients in [0, 1]) by

$$A_{i,j} = \begin{cases} \alpha(v_i, v_j) & \text{if } v_i \notin S, \\ 0 & \text{otherwise,} \end{cases}$$

and, given any function $f : S \to \mathbb{R}^k$, define the n column matrix F (with coefficients in $\mathbb{R}^k$) by

$$F_i = \begin{cases} f(v_i) & \text{if } v_i \in S, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then the solutions of Σ for f correspond to the solutions Ψ of the equation $\Psi = A\Psi + F$. Hence, a barycentric system Σ has a unique solution for a function f if and only if $(I - A)$ is non-singular. Hence, Σ has a unique solution for any function if and only if it has a unique solution for some function.

Moreover, if Σ is non-singular, $\Psi = (I - A)^{-1}F$ and thus $f \mapsto \Sigma(f)$ is a linear function. □

Definition 6. Let $\Sigma = (G, S, \alpha)$ be a barycentric system and let A be a subset of $V(G)$. The *relative source set* $\mathrm{Source}(\Sigma, A)$ of A is defined by

$$\mathrm{Source}(\Sigma, A) = (S \cap A) \cup \{x \in A \backslash S : \exists v \notin A,\ \ \alpha(x, v) \neq 0\}.$$

That is, $\forall x \in A$:

$$x \notin \mathrm{Source}(\Sigma, A) \iff x \notin S \quad \text{and} \quad \sum_{v \in A} \alpha(x, v) = 1.$$

Notice that, according to this definition, $\mathrm{Source}(\Sigma, V(G)) = S$.

Lemma 7. *Let G be a simple graph, let $\Sigma = (G, S, \alpha)$ be a barycentric system, let A be a subset of $V(G)$, let $S_A =$* $\mathrm{Source}(\Sigma, A)$ *and let α_A be the restriction of α to $(A \backslash S_A) \times A$.*

If Ψ is a solution of Σ for some function f, then $\Sigma_A = (G_A, S_A, \alpha_A)$ is a barycentric system and the restriction of Ψ to A is a solution of Σ_A for $\Psi_{|S_A}$.

Proof. According to the definition of $\mathrm{Source}(\Sigma, A)$, the function α_A is such that

$$\forall (x, y) \in (A \backslash S_A) \times A, \quad \alpha(x, y) \neq 0 \ \Rightarrow \ \{x, y\} \in E(G_A),$$

$$\forall x \in A \backslash S_A, \quad \sum\nolimits_{v \in A} \alpha(x, v) = 1.$$

Thus, Σ_A is a barycentric system, obviously having $\Psi_{|A}$ as a solution for $\Psi_{|S_A}$. □

Definition 8. Let $\Sigma = (G, S, \alpha)$ be a barycentric system.

The *flow graph* $\vec{D}_\Sigma$ of Σ is the directed graph with vertex set $V(G)$, having S as its set of sources and whose arcs are the $(x, y) \in V(G) \times (V(G) \backslash S)$ such that $\alpha(y, x) \neq 0$ (see Fig. 1). The *distance function* $\mathrm{dist}_\Sigma : V(G) \to \mathbb{N} \cup \{\infty\}$ associated with Σ is defined as the minimum length of a directed path in $\vec{D}_\Sigma$ from S to a vertex, or ∞ is such a path does not exist.

2.2. General resolution

In the next lemmas, $\Sigma = (G, S, \alpha)$ is a barycentric system. Elements of $\mathbb{R}^k$ are compared with respect to lexicographic order.

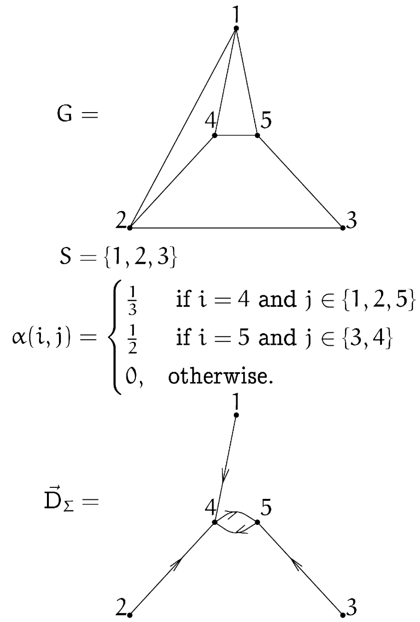

Fig. 1. Example of a flow graph defined by a barycentric system $\Sigma = (G, S, \alpha)$.

Lemma 9. *If* $\sup_{v \in V(G)} \mathrm{dist}_\Sigma(v) < \infty$, *any solution* Ψ *of* Σ *for some function f reaches its extremal values on S and, for any vertex x, there exists in* $\vec{D}_\Sigma$:

- *a directed path* P_x^+ *from S to x such that* $(\Psi, \mathrm{dist}_\Sigma)$ *is strictly increasing on* P_x^+ *with respect to lexicographic order*;
- *a directed path* P_x^- *from S to x such that* $(\Psi, -\mathrm{dist}_\Sigma)$ *is strictly decreasing on* P_x^- *with respect to lexicographic order*.

Proof. We only have to prove the existence of P_x^+ since the existence of the directed path P_x^- is then obtained by considering $-\Psi$ instead of Ψ, and the monotony of these paths implies that the extrema of Ψ are reached on S.

For a vertex v, note P_v^+ some directed path from S to v such that $(\Psi, \mathrm{dist}_\Sigma)$ is strictly increasing on P_v^+ with respect to lexicographic order, when such a directed path exists. Assume there is at least one vertex v such that such a path P_v^+ does not exist, and choose the one with the smallest possible value of $(\Psi(v), \mathrm{dist}_\Sigma(v))$. Obviously, v does not belong to S. So,

$$\Psi(v) = \sum_{(w,v) \in E(\vec{D}_\Sigma)} \alpha(v, w)\Psi(w) \geqslant \min_{(w,v) \in E(\vec{D}_\Sigma)} \Psi(w).$$

- If there exists an arc $(x, v) \in E(\vec{D}_\Sigma)$ such that $\Psi(x) < \Psi(v)$, then a directed path P_x^+ exists (by minimality assumption on v) and the directed path $P_v^+ = P_x^+ + (x, v)$ contradicts the hypothesis.
- Otherwise, any edge $(x, v) \in E(\vec{D}_\Sigma)$ is such that $\Psi(x) = \Psi(v)$. As there exists an edge $(x, v) \in E(\vec{D}_\Sigma)$, such that $\mathrm{dist}_\Sigma(x) = \mathrm{dist}_\Sigma(v) - 1$ the directed path P_x^+ exists, and $P_v^+ = P_x^+ + (x, v)$ contradicts the hypothesis.

In both cases, we are led to a contradiction. □

Lemma 10. *If* $\sup_{v \in V(G)} \mathrm{dist}_\Sigma(v) < \infty$, *then* Σ *is non-singular*.

Proof. According to Lemma 5, Σ is non-singular if and only if Σ has a unique solution for the constant function f_0 mapping S to $\mathbf{0}$. But this is an implication of the previous result, which asserts that the extrema of a solution of a barycentric system are reached on its source set, that is that a solution has the same extremal values as f_0. □

Lemma 11. *If* $\sup_{v\in V(G)} \mathrm{dist}_\Sigma(v) = \infty$, *the barycentric system has an infinite set of solutions for any function f.*

Proof. Let V_S be the subset of $V(G)\backslash S$ reachable from S by means of a directed path. For any value $c \in \mathbb{R}^k$, define the function $f' : V(G) - V_S \to \mathbb{R}^k$ by $f'(v) = f(v)$ if $v \in S$, and $f'(v) = c$, otherwise. According to Lemma 10, the barycentric system $(G - V_S, V(G) - V_S, \alpha)$ has a solution Ψ for f', which is also a solution of Σ. This solution is such that $\Psi|_{V(G)-V_S} \equiv c$, and thus we get an infinite set of solutions of Σ. $\square$

Theorem 12. *Let* $\Sigma = (G, S, \alpha)$ *be a barycentric system and let f be a function from S to* $\mathbb{R}^k$.
Then Σ *has at least a solution for f and the following conditions are equivalent*:

(1) *the solution of* Σ *for f is unique*,
(2) Σ *is non-singular* (*that is*: *for any function* $f' : S \to \mathbb{R}^k$, *the solution of* Σ *for* f' *is unique*),
(3) *any vertex of the flow graph* $\vec{D}_\Sigma$ *may be reached from S by a directed path*,
(4) $\mathrm{Source}(\Sigma, A)$ *is non-empty for any non-empty subset A of vertices of G*.

If Σ *is non-singular*, *the solution* $\Psi = \Sigma(f)$ of Σ *for f reaches its extremal values on S and, for any vertex x*, *there exists in* $\vec{D}_\Sigma$:

- *a directed path* P_x^+ *from S to x such that* $(\Psi, \mathrm{dist}_\Sigma)$ *is strictly increasing on* P_x^+ (*with respect to lexicographic order*),
- *a directed path* P_x^- *from S to x such that* $(\Psi, -\mathrm{dist}_\Sigma)$ *is strictly decreasing on* P_x^- (*with respect to lexicographic order*).

Proof. The existence of at least a solution for any function and the equivalence of (2) and (3) is ensured by Lemmas 10 and 11, while the equivalence of (1) and (2) is given by Lemma 5.

If Σ is non-singular, the properties of $\Sigma(f)$ stated in the theorem are those stated in Lemma 9.

On the one hand, these properties obviously imply that $\mathrm{Source}(\Sigma, A)$ is non-empty for any non-empty subset A of vertices of G: if $x \in A$, the first intersection of P_x^+ and A belongs to $\mathrm{Source}(\Sigma, A)$. Hence (3) implies (4).

On the other hand, assume (4) holds and (3) does not. Then, there exists a vertex x of $\vec{D}_\Sigma$ that cannot be reached from S by a directed path. Let A be the set of vertices of $\vec{D}_\Sigma$ that may reach x through a (possibly empty) directed path. Then $\mathrm{Source}(\Sigma, A) = \emptyset$, which contradicts (4). Thus (4) implies (3). $\square$

3. Planarity of a barycentric representation

3.1. Topological convexity

Let A be a subset of vertices of a connected plane graph G.

Definition 13. The *topological closure* $\mathcal{TC}(A)$ of A is the subset of the vertices of G that are either on G_A or inside G_A.

In other words, $\mathcal{TC}(A)$ is the maximal subset of $V(G)$, such that the outer face of $G_{\mathcal{TC}(A)}$ is the same as the outer face of G_A.

A subset $A \subseteq V(G)$ is *topologically convex*/if $\mathcal{TC}(A) = A$. A vertex $x \in A$ is an *extremal vertex* of A if x belongs to the outer face of G_A. We denote $\mathrm{Extr}(A)$ the set of the extremal vertices of A.

Remark 1. Notice that the topological closure operator is

- extensive: $A \subseteq \mathcal{TC}(A)$,
- isotone: $A \subseteq B \;\Rightarrow\; \mathcal{TC}(A) \subseteq \mathcal{TC}(B)$,
- idempotent: $\mathcal{TC}(\mathcal{TC}(A)) = \mathcal{TC}(A)$,
- convex: $\mathcal{TC}(A) = \mathcal{TC}(B)$ and $A \subseteq X \subseteq B \;\Rightarrow\; \mathcal{TC}(X) = \mathcal{TC}(A)$.

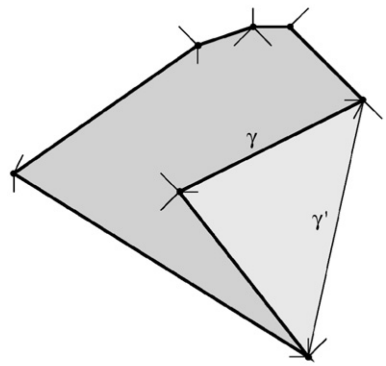

Fig. 2. The topological closure of the cycle γ may be greater than the interior of γ (because of external chords), but has a cycle for external face (γ'): the interior of γ is dark shaded, the topological closure of γ is the union of the two shaded regions, that is, the interior of γ'.

The topological closure operator actually defines a convex geometry on $V(G)$ (hence the term of "topologically convex") and we have

- $\mathscr{T}\mathscr{C}(A) = \mathscr{T}\mathscr{C}(B) \;\Rightarrow\; \mathrm{Extr}(A) = \mathrm{Extr}(B)$.
- $\mathscr{T}\mathscr{C}(A) = \mathscr{T}\mathscr{C}(\mathrm{Extr}(A))$.
- For any topological convex A and any $x \in A$, $A - x$ is a topological convex if and only if $x \in \mathrm{Extr}(A)$.

Notice that a topological convex set needs not to induce a connected subgraph of G.

Definition 14. A *disk* of G is the topological closure of a cycle of G.

Remark 2. If A is a disk of G, $\mathrm{Extr}(A)$ is the vertex set of a cycle of G: by definition, there exists a cycle γ of G so that $A = \mathscr{T}\mathscr{C}(V(\gamma))$. Hence, the outer face of G_A is 2-connected and thus is a cycle γ', with $V(\gamma') \subseteq V(\gamma)$ (see Fig. 2).

Definition 15. A plane graph G is *internally maximal* if every inner face of G is a triangle.

3.2. Admissibility of a barycentric system

Definition 16. Let G be a plane graph and let $\Sigma = (G, S, \alpha)$ be a barycentric system.
The *interface* of a subset $A \subseteq V(G)$ in Σ is the set

$$\mathrm{Interface}(\Sigma, A) = \mathrm{Source}(\Sigma, A) \cap \mathrm{Extr}(A).$$

Σ is *admissible* if, for any subset $A \neq \emptyset$ of vertices of G inducing a connected subgraph:

$$|\mathrm{Interface}(\Sigma, A)| > 0,$$

$$|\mathrm{Interface}(\Sigma, A)| = 1 \;\Rightarrow\; |A| = 1,$$

$$|\mathrm{Interface}(\Sigma, A)| = 2 \;\Rightarrow\; G_A \text{ is a path}.$$

Remark 3. According to Theorem 12, an admissible barycentric system Σ is non-singular, as $\mathrm{Interface}(\Sigma, A) \subseteq \mathrm{Source}(\Sigma, A)$ is non-empty for any non-empty subset A of vertices.

Lemma 17. *For any subset A of vertices of G*:

$$\mathrm{Interface}(\Sigma, \mathscr{T}\mathscr{C}(A)) \subseteq \mathrm{Interface}(\Sigma, A). \tag{6}$$

Proof. First notice that $(\mathscr{TC}(A) \cap S) \cap \mathrm{Extr}(A) = S \cap \mathrm{Extr}(A) = (A \cap S) \cap \mathrm{Extr}(A)$. Thus, $\mathrm{Interface}(\Sigma, \mathscr{TC}(A)) \cap S = \mathrm{Interface}(\Sigma, A) \cap S$.

Let $x \in \mathrm{Interface}(\Sigma, \mathscr{TC}(A))\setminus S$. Then there exists $y \notin \mathscr{TC}(A)$, such that $\alpha(x, y) > 0$. As $y \notin \mathscr{TC}(A) \supseteq A$ and $x \in \mathrm{Extr}(A) \subseteq A$, $x \in \mathrm{Source}(\Sigma, A)$ and thus $x \in \mathrm{Interface}(\Sigma, A)$. □

Lemma 18. *Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be an admissible barycentric system.*
Then any vertex in $V(G)\setminus S$ has indegree at least 2 in $\vec{D}_\Sigma$.

Proof. Assume there exists $x \in V(G)\setminus S$ having indegree at most 1 in $\vec{D}_\Sigma$. Then, either x has indegree 1 and x has a neighbor y, so that $|\mathrm{Source}(\Sigma, \{x, y\})| = 1$, or x has indegree 0 and $|\mathrm{Source}(\Sigma, \{x\})| = 0$. Both cases contradict the admissibility of Σ. □

Lemma 19. *Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be an non-singular barycentric system. Assume that all vertices in $V(G)\setminus S$ have indegree at least 2 in $\vec{D}_\Sigma$.*
Then, for any subset $A \subseteq V(G)$ inducing a tree, $|\mathrm{Interface}(\Sigma, A)|$ is at least equal to the number of leaves of A.

Proof. A leaf of G_A belongs to $\mathrm{Interface}(\Sigma, A)$, as it either belongs to S or has at least one incoming arc in $\vec{D}_\Sigma$ from a vertex outside A. □

Lemma 20. *Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be a non-singular barycentric system. Assume that all vertices in $V(G)\setminus S$ have indegree at least 2 in $\vec{D}_\Sigma$, and let A be a connected topological convex of G different from a tree.*
If $|\mathrm{Interface}(\Sigma, D)| \geqslant 3$ for any disk $D \subseteq A$, then $|\mathrm{Interface}(\Sigma, A)| \geqslant 3$ and there exists $v_1, v_2, v_3 \in \mathrm{Interface}(\Sigma, A)$, such that G_A is contractible to the triangle $\{v_1, v_2, v_3\}$.

Proof. The proof is by induction on $|A|$.

Assume there exists $x \in A$ having degree 1 in G_A. Then $x \in \mathrm{Extr}(A)$ and either x belongs to S or x has at least one incoming arc in $\vec{D}_\Sigma$ from a vertex outside A. Thus, $x \in \mathrm{Interface}(\Sigma, A)$. Moreover, $A - x$ is a topological convex and the neighbor of x is the only vertex that may belong to $\mathrm{Interface}(\Sigma, A - x)\setminus\mathrm{Interface}(\Sigma, A)$. By induction, $|\mathrm{Interface}(\Sigma, A - x)| \geqslant 3$ and there exists $v_1, v_2, v_3 \in \mathrm{Interface}(\Sigma, A - x)$ such that G_{A-x} is contractible to the triangle $\{v_1, v_2, v_3\}$. Hence, if x is adjacent to v_i (we may assume $i = 1$) G_A is contractible to the triangle $\{v_2, v_3, x\} \subseteq \mathrm{Interface}(\Sigma, A)$ and, otherwise, G_A is contractible to the triangle $\{v_1, v_2, v_3\}\mathrm{Interface}(\Sigma, A)$.

Otherwise, if G_A is 2-connected, then A is a disk and thus $|\mathrm{Interface}(\Sigma, A)| \geqslant 3$. Given any three vertices $v_1, v_2, v_3 \in \mathrm{Interface}(\Sigma, A)$, G_A is contractible to the triangle $\{v_1, v_2, v_3\}$.

Otherwise, let x be a cut-vertex of G_A and let $A_1, \ldots, A_p$ $(p \geqslant 2)$ be the vertex sets of the blocs of G_A having x as common intersection. Obviously, $A_1, \ldots, A_p$ are connected topological convexes of G and each of the G_{A_i} has at least one cycle, for otherwise there would exist a vertex of degree 1 in G_A. Let $B = A\setminus A_1 \cup \{x\}$. By induction, $|\mathrm{Interface}(\Sigma, B)| \geqslant 3$ and there exists $v_1, v_2, v_3 \in B$ such that G_B is contractible to the triangle $\{v_1, v_2, v_3\}$. Moreover, it is easily checked that $\mathrm{Interface}(\Sigma, A)\setminus\{x\} = \mathrm{Interface}(\Sigma, A_1)\setminus\{x\} \cup \mathrm{Interface}(\Sigma, B)\setminus\{x\}$. Thus, as $|\mathrm{Interface}(\Sigma, A_1)| \geqslant 3$, $|\mathrm{Interface}(\Sigma, A)| \geqslant 3$. If x is different from v_1, v_2, v_3, then G_A is contractible to the triangle $\{v_1, v_2, v_3\} \subseteq \mathrm{Interface}(\Sigma, A)$. Otherwise, if $x = v_1$, let $y \in \mathrm{Interface}(\Sigma, A)\setminus\{x\}$. By connectivity of G_{A_1}, there exists a path in G_{A_1} joining x to y. Thus, G_A is contractible to the triangle $\{y, v_2, v_3\} \subseteq \mathrm{Interface}(\Sigma, A)$. □

Lemma 21. *Let G be a connected plane graph and $\Sigma = (G, S, \alpha)$ a non-singular barycentric system.*
Then Σ is admissible if and only if the two following conditions hold:

- *any vertex in $V(G)\setminus S$ has indegree at least 2 in $\vec{D}_\Sigma$*,
- *for any disk A of G*, $|\mathrm{Interface}(\Sigma, A)| \geqslant 3$.

Proof. Assume Σ is admissible. According to Lemma 18, condition (1) holds. Moreover, since any disk includes a cycle, (2) also holds.

Assume (1) and (2) hold. Assume $A \subseteq V(G)$ induces a connected subgraph of G. According to Lemma 17, $\text{Interface}(\Sigma, \mathscr{TC}(A)) \subseteq \text{Interface}(\Sigma, A)$. According to Lemmas 19 and 20, $|\text{Interface}(\Sigma, \mathscr{TC}(A))| = 1$ if $|\mathscr{TC}(A)| = 1$ (hence $|A| = 1$), $|\text{Interface}(\Sigma, \mathscr{TC}(A))| \geqslant 2$ if $G_{\mathscr{TC}(A)}$ is a path (hence G_A is a path) and $|\text{Interface}(\Sigma, \mathscr{TC}(A))| \geqslant 3$ otherwise. Thus Σ is admissible. $\square$

3.3. Augmentation

In this section, we shall prove that a connected plane graph G with an admissible barycentric system $\Sigma = (G, S, \alpha)$ may be augmented into a maximal plane graph G^+ with an admissible barycentric system $\Sigma^+ = (G^+, T, \alpha^+)$ such that

- G is a topological induced subgraph of G^+,
- T is the vertex set of the outer face of G^+,
- α is the restriction of α^+ to $(V(G)\backslash S) \times V(G)$.

Here, *topological induced subgraph* means that the embedding of the graph G corresponds to the restriction of the embedding of the graph G^+ to its subgraph induced by $V(G)$.

This augmentation will be performed in two steps: first, we augment the graph to a 2-connected plane graph, then we augment it to a maximal plane graph.

Definition 22. Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be a barycentric system. Let $x \in V(G)$.

The vertex x is *neutral* for Σ if the union of the faces of G including x is a wheel, and if either $x \in S$ or $\alpha(x, y) > 0$ for any neighbor y of x.

The vertex x is *strongly neutral* for Σ if x and all its neighbors are neutral for Σ.

Remark 4. Let $A \subseteq V(G)$. A vertex of $\text{Extr}(A)$ which is neutral for Σ belongs to $\text{Interface}(\Sigma, A)$.

Lemma 23. *A connected plane graph G with an admissible barycentric system* $\Sigma = (G, S, \alpha)$ *may be augmented into a* 2*-connected plane graph* G^+ *with an admissible barycentric system* $\Sigma = (G^+, T, \alpha^+)$, *so that*

- *G is a topological induced subgraph of* G^+,
- $S \subseteq T$,
- α *is the restriction of* α^+ *to* $(V(G)\backslash S) \times V(G)$.

Proof. Assume x is a cut-vertex of G, let u and v be two consecutive neighbors of x belonging to the same face of G and to different blocs (i.e. 2-connected components) of G. We add a vertex s and the edges $\{s, u\}$ and $\{s, v\}$ to G, while preserving the planarity (and the embedding, see Fig. 3).

Let G^+ be the so-obtained graph, $T = S \cup \{s\}$, let α^+ be the extension of α to $(V(G^+)\backslash T) \times V(G^+)$ defined by $\alpha^+(x, s) = 0, \forall x \in V(G^+)\backslash T$ and let $\Sigma^+ = (G^+, T, \alpha^+)$. As Σ is non-singular, any vertex of $V(G^+)\backslash T = V(G)\backslash S$ may be reached from S by a directed path in $\vec{D}_\Sigma$, and thus it may be reached from $T \supseteq S$ by a directed path in $\vec{D}_{\Sigma^+} \supseteq \vec{D}_\Sigma$. Hence Σ^+ is non-singular. Moreover, any vertex in $V(G^+)\backslash T = V(G)\backslash S$ has indegree at least 2 in $\vec{D}_{\Sigma^+}$.

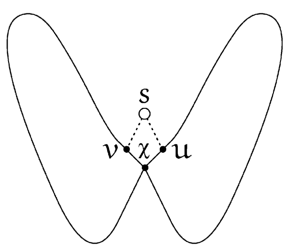

Fig. 3. Augmentation at a cut-vertex.

Let A be a disk of G^+. If $s \notin \mathrm{Extr}(A)$, then

$$|\mathrm{Interface}(\Sigma^+, A)| = |\mathrm{Interface}(\Sigma, A \backslash \{s\})| \geqslant 3.$$

Otherwise, $A - s$ has x as cut-vertex and

$$|\mathrm{Interface}(\Sigma, A \backslash \{s\}) \backslash \{x\}| \geqslant 2.$$

Hence $|\mathrm{Interface}(\Sigma^+, A)| \geqslant 3$ and Σ^+ is admissible, according to Lemma 21.

By iterating this process, we eventually obtain a topological supergraph G^+ of G and a barycentric system Σ^+ having the required properties. □

Lemma 24. *Let G be a 2-connected plane graph and let $\Sigma = (G, S, \alpha)$ be an admissible barycentric system.*

Then there exists a maximal plane graph G^+ with outer face vertex set $T = \{r_1, r_2, r_3\}$ and an admissible barycentric system $\Sigma^T = (G^+, T, \alpha^+)$, such that

- *G is a topological induced subgraph of G^+,*
- *$V(G) \backslash S \subseteq V(G^+) \backslash T$ and α is the restriction of α^+ to $(V(G) \backslash S) \times V(G)$.*

Proof. The graph G^+ and a partition of $V(G^+) \backslash V(G)$ into three sets A, B, C is obtained as follows (see Figs. 4 and 5): starting with $A = B = C = \emptyset$, for each face $F = (v_1, \ldots, v_k)$ of G,

- planarly add vertices $c_1, \ldots, c_k$ so that c_i is adjacent to v_i and v_{i+1} (or v_1 if $i = k$) and put them in C;
- planarly add vertices $b_1, \ldots, b_k$ so that b_i is adjacent to c_i, v_i and c_{i-1} (or c_k if $i = 1$) and put them in B;

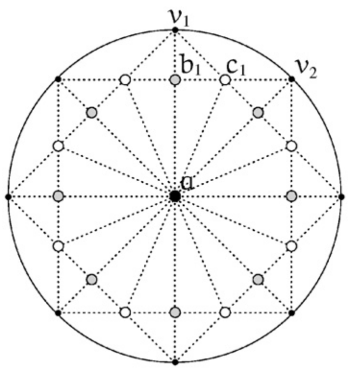

Fig. 4. Augmentation of an internal face.

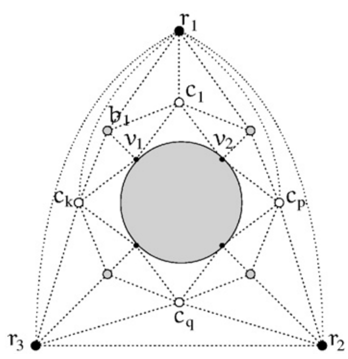

Fig. 5. Augmentation of the external face.

- if F is an inner face of G, planarly add a vertex a adjacent to $c_1, \dots, c_k, b_1, \dots, b_k$ and put it in A; otherwise, planarly add three mutually adjacent vertices r_1, r_2, r_3 adjacent to the vertices in $\{c_k, b_1, c_1, \dots, b_p, c_p\}$, $\{c_p, b_{p+1}, c_{p+1}, \dots, b_q, c_q\}$ and $\{c_q, b_{q+1}, c_{q+1}, \dots, b_k, c_k\}$, respectively, where $1 \leqslant p < q < k$, so that $\{r_1, r_2, r_3\}$ will be the new outer face of the graph, and put r_1, r_2, r_3 in A.

Let $T = \{r_1, r_2, r_3\}$. Clearly, G^+ is a maximal plane graph having T as the vertex set of its outer face. Define $\alpha^+ : V(G^+)\backslash T$ to $[0, 1]$ by

$$\alpha^+(x, y) = \begin{cases} \alpha(x, y) & \text{if } (x, y) \in (V(G)\backslash S) \times V(G), \\ \dfrac{1}{d(x)} & \text{if } x \in (V(G^+)\backslash T)\backslash (V(G)\backslash S) \text{ and } \{x, y\} \in E(G^+), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Sigma^+ = (G^+, T, \alpha^+)$. By construction, each vertex in S is neutral for Σ^+ and the sets A, B, C have the following properties:

- vertices in A are strongly neutral for Σ^+ and $T \subseteq A$;
- each vertex in B is neutral for Σ^+ and has, in circular order, a neighbor in C, a neighbor in $V(G)$, a neighbor in C and a neighbor in A;
- each vertex in C is neutral for Σ^+ and has two consecutive neighbors in $V(G)$ and its other neighbors in $A \cup B$.

We shall first prove that Σ^+ is non-singular. Let $A \subseteq V(G^+)\backslash T$:

- If the outer face of G^+_A includes a vertex x in $A \cup B \cup C \cup S$, this vertex is neutral for Σ^+ and thus has a neighbor y outside A so that $\alpha(x, y) > 0$. Hence, $|\text{Interface}(\Sigma^+, A)| > 0$.
- Otherwise, $G_{A \cap V(G)}$ has the same outer face as G^+_A and, as Σ is admissible, $|\text{Interface}(\Sigma, A \cap V(G))| > 0$. As $\text{Interface}(\Sigma, A \cap V(G)) \cap S = \emptyset$, there exists, in the outer face of $G_{A \cap V(G)}$ (that is: the outer face of G^+_A) a vertex x having a neighbor y in $V(G)\backslash A$, so that $\alpha(x, y) > 0$. Thus $|\text{Interface}(\Sigma^+, A)| > 0$.

According to Theorem 12, Σ^+ is non-singular.

Moreover, any vertex x in $V(G^+)$ has indegree at least 2 in $\vec{D}_{\Sigma^+}$: either $x \in V(G)\backslash S$ and then x has indegree at least 2 in $\vec{D}_\Sigma$ according to Lemma 18, or $x \in A \cup B \cup C \cup S$ and x has indegree at least 2 in $\vec{D}_{\Sigma^+}$ as it is neutral for Σ^+ and has degree at least 3 in G^+.

Assume there exists a disk A of G^+ such that $|\text{Interface}(\Sigma^+, A)| \leqslant 2$, and assume A is minimal for the inclusion.

- Assume the outer face of G^+_A includes a vertex $v \in A$, the two neighbors of v on the outer face of A are neutral for Σ^+. Thus $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction;
- otherwise, assume the outer face of G^+_A includes a vertex $v \in B$ and let v_1, v_2 be the two neighbors of v on the outer face of G^+_A. Then, either v_1 and v_2 are non-consecutive neighbors of v and thus belong to C, in which case they are neutral and $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction, or $\{v, v_1, v_2\}$ is a face of G^+. In the latter case, either $A - v$ is a disk and thus $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction, or $A - v$ is reduced to $\{v_1, v_2\}$ and $|\text{Interface}(\Sigma^+, A)| = 3$, a contradiction;
- otherwise, assume the outer face of G^+_A includes a vertex $v \in C$ and let v_1, v_2 be the neighbors of v on the outer face of G^+_A. Then either v_1 and v_2 are non-consecutive neighbors of v and thus belong to $A \cup B$, in which case they are neutral and $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction, or $\{v, v_1, v_2\}$ is a face of G^+. In the latter case, either $A - v$ is a disk and thus $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction, or $A - v$ is reduced to $\{v_1, v_2\}$ and $|\text{Interface}(\Sigma^+, A)| = 3$, a contradiction;
- otherwise, the outer face of G^+_A is included in $V(G)$. Hence $A \cap V(G)$ is a disk of G and $G_{A \cap V(G)}$ and G^+_A have the same outer face. Let $x \in \text{Interface}(\Sigma, A \cap V(G))$. Then either x has a neighbor y in G (thus in G^+) such that $y \notin A$ and $\alpha(y, x) > 0$, or $x \in S$ and thus x is neutral for Σ^+. Thus $x \in \text{Interface}(\Sigma^+, A)$. As Σ is admissible and $A \cap V(G)$ is a disk of G, $|\text{Interface}(\Sigma, A \cap V(G))| \geqslant 3$. Hence $|\text{Interface}(\Sigma^+, A)| \geqslant 3$, a contradiction.

Thus, any disk A of G^+ is such that $|\text{Interface}(\Sigma^+, A)| \geqslant 3$ and, according to Lemma 21, Σ^+ is admissible. $\square$

3.4. Good barycentric systems

Definition 25. Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be a non-singular barycentric system.

Σ is *good* if, for some function f from S to the plane, $\Sigma(f)$ induces a planar straight line drawing of G homeomorphic to G.

The aim of this section is to prove that a barycentric system is good if and only if it is admissible.

Let G be a 2-connected planar graph. A straight line drawing of G is a *convex drawing* if each face of the graph is drawn as a convex polygon. The drawing is *strictly convex* if the vertices of a face are mapped to the extremal points of the polygon corresponding to the face. Convex drawings are deeply related to good barycentric systems:

Lemma 26. *Let G be a 2-connected plane graph. A straight line representation of G (with the same embedding) is convex if and only it is induced by a solution f of a good barycentric system $\Sigma = (G, S, \alpha)$, where S is the vertex set of a face of G and f embeds S in a convex polygon (in a compatible order).*

Proof. Consider a convex drawing of G and let S be the vertex set of the external face. Let v be an internal vertex. As the faces including v are convex, we may augment the neighborhood of v to a wheel by adding straight line segments between consecutive non-adjacent neighbors while preserving planarity. Thus, v belongs to the convex hull of its neighbors. Hence, the drawing is induced by the solution of a good barycentric system $\Sigma = (G, S, \alpha)$ mapping S to a convex polygon.

Conversely, assume S is the vertex set of a face of G embedded in a convex polygon of the plane. Assume the representation includes a non-convex internal face, and let v_1, v_2, v_3 be three consecutive vertices of such a face defining a concave angle at v_2. Then v_2 does not belong to the convex hull of its neighbors, a contradiction. □

Lemma 27. *Let G be a maximal planar graph and let $\mu : [0, 1] \times V(G) \to \mathbb{R}^2$ be a mapping such that*

- *$\mu(0, \cdot)$ induces a plane straight line representation of G,*
- *$\mu(\cdot, x)$ is continuous for any vertex x of G,*
- *$\forall t \in]0, 1]$ and for all triangle (x_1, x_2, x_3) of G, $\mu(t, x_1)$, $\mu(t, x_2)$ and $\mu(t, x_3)$ are not collinear.*

Then, $\mu(t, \cdot)$ induces a plane straight line representation of G for every t in $[0, 1]$.

Proof. First notice that $\mu(0, \cdot)$ induces a plane straight line representation of G by hypothesis and that we only have to consider the case where $t \in]0, 1]$.

In order to check if a straight line representation of a maximal planar graph is plane, it is sufficient to check that no edge has null length, that no two adjacent edges are overlapping and that the circular orders of the edges around a vertex define a planar map. The first two cases cannot occur, as they imply the existence of a flat triangle. As $\mu(\cdot, x)$ is continuous for any vertex x, the circular order around a vertex may only change if two consecutive edges overlap for some $t > 0$, which would give rise to a contradiction. □

Theorem 28. *Let G be a connected plane graph and let $\Sigma = (G, S, \alpha)$ be a barycentric system. Then Σ is good if and only if it is admissible.*

Proof. Assume Σ is good. Consider a function f from S to the plane inducing a straight line drawing homeomorphic to G and let $\Psi = \Sigma(f)$. Then, for any subset A of $V(G)$, any point in $\Psi(A)$ belongs to the convex hull of $\Psi(\mathrm{Interface}(\Sigma, A))$. Thus, if G_A is connected, the three conditions of admissibility are obviously satisfied.

Now, assume Σ is admissible. According to Lemmas 23 and 24, we may assume that G is a maximal plane graph with outer face $S = \{s_1, s_2, s_3\}$. It is a classical result [2,7,11] that G has a straight line embedding Γ with external face $\{s_1, s_2, s_3\}$. Let $f : S \to \mathbb{R}^2$ be the function which maps the vertices in S to the corresponding points of the straight line embedding Γ. According to Lemma 26, this embedding is induced by the solution for f of some barycentric system $\Sigma' = (G, S, \alpha')$. Let $\alpha_t = (1 - t)\alpha' + \alpha$ and let $\Sigma^t = (G, S, \alpha_t)$, for $t \in [0, 1]$.

For $0<t<1$ and $x, y \in V(G)$, $\alpha_t(x, y)>0$ if and only if $\alpha(x, y)>0$ or $\alpha'(x, y)>0$. Thus, for any subset $A \subseteq V(G)$ inducing a connected subgraph, $\text{Interface}(\Sigma^t, A) \subseteq \text{Interface}(\Sigma, A)$ and hence the barycentric system Σ^t is admissible for any $0<t\leqslant 1$.

Let $\mu : [0, 1] \times V(G) \to \mathbb{R}^2$ be defined by $\mu(t, \cdot) = \Sigma^t(f)$. This function continuously depends on t.

Assume there exists $t_0 \in]0, 1]$ and a triangle $\{x_1, x_2, x_3\}$ of G, such that $\mu(t_0, x_1)$, $\mu(t_0, x_2)$ and $\mu(t_0, x_3)$ are collinear. Let $ax+by+c=0$ be an equation of the straight line Δ including these points. Denote f_x and f_y the x and y coordinates of f. Let $f_\Delta = af_x + bf_y + c$ and let $\Psi_\Delta = \Sigma^{t_0}(f_\Delta)$ be the solution of the barycentric system Σ^{t_0} for f_Δ. According to Lemma 5, the mapping $g \to \Sigma^{t_0}(g)$ is linear and thus $\Psi_\Delta(v) = a\mu_x(t_0, v) + b\mu_y(t_0, v) + c$, where $\mu_x(t_0, v)$ and $\mu_y(t_0, v)$ denote the x and y coordinates of $\mu(t_0, v)$. Hence $\Psi_\Delta(x_1) = \Psi_\Delta(x_2) = \Psi_\Delta(x_3) = 0$, as $\mu(t_0, x_1)$, $\mu(t_0, x_2)$ and $\mu(t_0, x_3)$ belong to Δ. Let A be the vertex set of the connected component of $G_{\Psi_\Delta^{-1}(0)}$ including the vertices x_1, x_2, x_3.

Let G^+ be the planar graph obtained from G by adding two vertices ω^- and ω^+ and the edges $\{\omega^-, s\}$ (for $s \in S$ and $\Psi_\Delta(s)\leqslant 0$), the edges $\{\omega^+, s\}$ (for $s \in S$ and $\Psi_\Delta(s)\geqslant 0$) and the edge $\{\omega^-, \omega^+\}$.

For each vertex $a_i \in \text{Interface}(\Sigma^{t_0}, A)\backslash S$, there exists a vertex v such that (v, a_i) is an arc of $\vec{D}_{\Sigma^{t_0}}$ and $\Psi_\Delta(v) \neq 0$. From the equation

$$\sum_{(x,a_i)\in E(\vec{D}_{\Sigma^{t_0}})} \Psi_\Delta(x) = \Psi_\Delta(a_i) = 0,$$

we deduce that, for each such a_i, there exist two vertices x_i and y_i, such that (x_i, a_i) and (y_i, a_i) are arcs of $\vec{D}_{\Sigma^{t_0}}$ and such that $\Psi_\Delta(x_i) < 0 < \Psi_\Delta(y_i)$. According to Theorem 12, for each such a_i, there exist in $\vec{D}_{\Sigma^{t_0}}$ directed paths $P^+_{a_i}$ and $P^-_{a_i}$, whose vertices (except a_i) have a negative (resp., positive) Ψ_Δ-value.

Thus, for each $a_i \in \text{Interface}(\Sigma, A)$, there exist in G^+ paths $P_1(a_i)$ and $P_2(a_i)$ from ω^- (resp., ω^+) to a_i, which vertices (except a_i) have a negative (resp., positive) Ψ_Δ-value. Hence, for any a, a' in $\text{Interface}(\Sigma, A)$:

$$P_1(a) \cap P_2(a') = \emptyset,$$

$$P_1(a) \cap P_2(a) = P_1(a) \cap A = P_2(a) \cap A = \{a\},$$

$$\omega^- \in P_1(a) \quad \text{and} \quad \omega^+ \in P_2(a).$$

According to Lemma 20, there exist $a_1, a_2, a_3 \in \text{Interface}(\Sigma^{t_0}, A)$, such that G_A is contractible to the triangle $\{a_1, a_2, a_3\}$. Thus, G_A, the paths $P_i(a_j)$ ($i \in \{1, 2\}$, $j \in \{1, 2, 3\}$), and the edge $\{\omega^-, \omega^+\}$ may be contracted to K_5, contradicting the planarity of G^+ (Fig. 6).

Thus, there exists no $t \in]0, 1]$, such that some triangle (x_1, x_2, x_3) of G is that $\mu(t, x_1)$, $\mu(t, x_2)$ and $\mu(t, x_3)$ are collinear. According to Lemma 27, $\Sigma(f) = \mu(1, \cdot)$ defines a barycentric representation of G (obviously homeomorphic to Γ) and thus Σ is good. $\square$

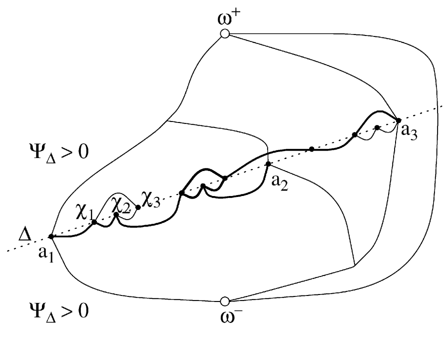

Fig. 6. When $\mu(t_0, x_1)$, $\mu(t_0, x_2)$ and $\mu(t_0, x_3)$ are collinear, a K_5 minor of G^+ is exhibited (thick edges may be contracted to a triangle).

4. Stretching paths

In this section, we consider a connected plane graph G, a covering $\mathscr{P} = (P_1, \ldots, P_k)$ of G by edge disjoint and pairwise non-crossing paths and a function $\alpha : V(G) \times V(G) \to [0, 1]$, such that

- $\alpha(x, y) > 0$ if and only if x is internal to a path in $\mathscr{P}$ in which y is adjacent to x,
- $\sum_{y\in V(G)}\alpha(x, y) = 1$ if x is internal to a path in $\mathscr{P}$.

We note by S the set of vertices of G which are internal to no path in $\mathscr{P}$ and $\Sigma = (G, S, \alpha)$ the barycentric system defined by G, S and α.

Definition 29. Let $\mathscr{P}'$ be a subset of $\mathscr{P}$. A vertex $x \in V(G)$ is a *free vertex* of $\mathscr{P}'$ if it belongs to the outer face of $\bigcup_{P\in\mathscr{P}'} P$ and if it is interior to no path in $\mathscr{P}'$.

Remark 5. According to the definition of α, the set of the free vertices of $\mathscr{P}' \subseteq \mathscr{P}$ is $\mathrm{Interface}(\Sigma, \bigcup_{P\in\mathscr{P}'} V(P))$.

Definition 30. We define the function $l : \mathscr{P} \to \mathbb{N}$ and, for $P \in \mathscr{P}$, the function $\pi_P : \{0, 1, \ldots, l(P)\} \to V(G)$ by

- $l(P)$ is the length of the path $P \in \mathscr{P}$,
- $\pi_P(i)$ is the ith vertex of the path P (vertices are numbered from one end to the other).

Definition 31. The *closure* $\mathrm{clos}_{\mathscr{P}}(A)$ of a subset $A \subseteq V(G)$ of at least two vertices inducing a connected subgraph of G is the smallest subset of $\mathscr{P}$, such that $A \subseteq \bigcup_{P\in\mathrm{clos}_{\mathscr{P}}(A)} V(P)$ and such that any path in $\mathscr{P}\backslash\mathrm{clos}_{\mathscr{P}}(A)$ share at most one vertex with $\bigcup_{P\in\mathrm{clos}_{\mathscr{P}}(A)} V(P)$.

Notice that $\mathrm{clos}_{\mathscr{P}}(A)$ may easily be computed as follows:

- let $X = \emptyset$;
- while there exists $P \in \mathscr{P}\backslash X$, such that P shares at least two vertices with $A \cup \bigcup_{P'\in X} V(P')$, let $X = X \cup \{P\}$;
- let $\mathrm{clos}_{\mathscr{P}}(A) = X$.

Lemma 32. *Let A be a subset of at least two vertices of G inducing a connected subgraph. Let A^+ be the vertex set of the union of the paths in* $\mathrm{clos}_{\mathscr{P}}(A)$. *Then* $|\mathrm{Interface}(\Sigma, A)| \geqslant |\mathrm{Interface}(\Sigma, A^+)|$, *that is*: $|\mathrm{Interface}(\Sigma, A)|$ *is at least equal to the number of free vertices of* $\mathrm{clos}_{\mathscr{P}}(A)$.

Proof. As G_A is not reduced to an isolated vertex, A is a subset of A^+. Thus, it is sufficient to prove the property for the addition to A of the vertex set of a path $P \in \mathscr{P}$ sharing at least one edge with A.

If there exists $x, y \in A \cap V(P)$ so that the subpath $P_{x,y}$ of P from x to y does not meet A except at x and y, then $\mathrm{Interface}(\Sigma, A \cup V(P_{x,y})) \subseteq \mathrm{Interface}(\Sigma, A)$, as the added vertices have all their incoming edges (in $\vec{D}_\Sigma$) in $V(P_{x,y})$.

Otherwise, if $P \nsubseteq G_A$, there exists $x \in A$ such that the subpath $P_{a,x}$ from an extremity of P to x does not meet A except at x. Notice that x has in P one neighbor outside A and one neighbor in A (for otherwise, the previous case would apply). Then $x \in \mathrm{Interface}(\Sigma, A)$ (as $(a, x) \in \vec{D}_\Sigma$). Moreover, as in the previous case $\mathrm{Interface}(\Sigma, A \cup V(P_{a,x}))$ does not contain any vertex internal to $P \cap G_{A\cup V(P_{a,x})}$, so it does not contain any vertex in $V(P_{a,x})\backslash\{a\}$. Hence $\mathrm{Interface}(\Sigma, A \cup V(P_{a,x})) \subseteq \mathrm{Interface}(\Sigma, A)\backslash\{x\} \cup \{a\}$ and $|\mathrm{Interface}(\Sigma, A \cup V(P_{a,x}))| \leqslant |\mathrm{Interface}(\Sigma, A)|$. $\square$

Theorem 33. *Let G be a connected plane graph and let $\mathscr{P} = (P_1, \ldots, P_k)$ be a covering of G by edge disjoint and pairwise non-crossing paths. Then there exists an embedding of G, such that each path in $\mathscr{P}$ is stretched if and only if any subset of $\mathscr{P}$ of at least two paths has at least three free vertices.*

Moreover, in this case, for any family of strictly increasing functions $\rho_P : \{0, 1, \ldots, l(P)\} \to [0, 1]$ indexed by $\mathscr{P}$ such that $\rho_P(0) = 0$ and $\rho_P(l(P)) = 1$ (for any $P \in \mathscr{P}$), there exists an embedding of G, such that each path in $\mathscr{P}$ is *stretched and such that, for* $P \in \mathscr{P}$ *and* $i \in \{0, 1, \ldots, l(P)\}$:

$$\rho_P(i) = \frac{\|\pi_P(0)\pi_P(i)\|}{\|\pi_P(0)\pi_P(l(P))\|}. \tag{7}$$

Proof. Let S be the set of vertices of G which are internal to no path in $\mathscr{P}$. Let $(x, y) \in (V(G)\backslash S) \times V(G)$. As x is internal to exactly one path $P \in \mathscr{P}$, it is equal to exactly one $\pi_P(i)$ (with $0 < i < l(P)$). Define $\alpha(x, y)$ as follows:

$$\alpha(x, y) = \begin{cases} \dfrac{\rho_P(i) - \rho_P(i-1)}{\rho_P(i+1) - \rho_P(i-1)} & \text{if } y = \pi_P(i+1), \\ \dfrac{\rho_P(i+1) - \rho_P(i)}{\rho_P(i+1) - \rho_P(i-1)} & \text{if } y = \pi_P(i-1), \\ 0 & \text{otherwise} \end{cases}$$

and let $\Sigma = (G, S, \alpha)$.

Assume that any subset of $\mathscr{P}$ of at least two paths has at least three free vertices. Let A be a subset of $V(G)$ inducing a connected subgraph. If $|A| = 1$, $|\text{Interface}(\Sigma, A)| = 1$. Otherwise, according to Lemma 32, $|\text{Interface}(\Sigma, A)|$ is at least equal to the number of free vertices of $\text{clos}_{\mathscr{P}}(A)$, thus at least 2, and at least 3 if $|\text{clos}_{\mathscr{P}}(A)| > 1$ (hence at least 3 if G_A is not a path). Thus Σ is admissible and, according to Theorem 28, Σ is good. The embedding induced by Σ is then such that each path in $\mathscr{P}$ is stretched and such that (7) holds for $P \in \mathscr{P}$ and $0 \leqslant i \leqslant l(P)$ (according to the definition of α).

Conversely, assume G has an embedding such that each path in $\mathscr{P}$ is stretched, and consider such an embedding. Let $\mathscr{P}' \subseteq P$ be a subset of at least 2 paths. Then all the vertices of the paths in $\mathscr{P}'$ belong to the convex hull of the set of the free vertices of $\mathscr{P}'$. If a path in $\mathscr{P}'$ has an extremity which is not a free vertex or if $\mathscr{P}'$ contains a cycle, the vertices in $\mathscr{P}'$ are all not collinear, so that $\mathscr{P}'$ has at least three free vertices. Otherwise, $\mathscr{P}'$ has $|\mathscr{P}'| + 1 \geqslant 3$ free vertices. □

5. Pseudo-segments contact systems

5.1. Introduction

A *Jordan arc* is an arc of the plane homeomorphic to a straight line segment. A set of Jordan arcs is called a family of *pseudo-segments* if every pair of arcs in the set intersect in at most one point. A *contact system* is a set of pseudo-segments such that two arcs intersect at most once at a point which is internal to at most one arc. A contact system is *stretchable* if there exists a homeomorphism which transforms it into a contact system whose arcs are straight line segments. A *pseudo-line* is an arc of the plane homeomorphic to a straight line; an *arrangement of pseudo-lines* is a family of pseudo-lines, such that any two pseudo-lines in the family intersect at most once. A contact system is *extendible* if there exists an arrangement of pseudo-lines such that each pseudo-segment of the contact system is included in a corresponding pseudo-line of the family. See [1] for a discussion about extendible pseudo-segments.

5.2. Extendibility and extremal points

An *extremal point* of a contact system is a point of the union of the arcs which is interior to no arc. Notice that the notion of an extremal point is clearly equivalent to the one of a free vertex, when considering the plane graph induced by the union of the arcs (where points are the points and the edges are the arc portions between two consecutive vertices) and the partition of the edge set into paths corresponding to the arcs in $\mathscr{A}$. A *maximal contact system* is a contact system whose extremal points belong to the unbounded region. A contact system $\mathscr{A} = (A_1, \ldots, A_k)$ is *expandable* if there exists a maximal contact system $\mathscr{A}' = (A'_1, \ldots, A'_k)$, such that $A_i \subseteq A'_i$ for each $i \in \{1, \ldots, k\}$.

Remark 6. If a contact system is extendible, it is expandable to a maximal contact system, which in turn is extendible.

Let $\mathscr{A}$ be a maximal contact system extendible to an arrangement of pseudo-lines $\mathscr{L}$.

- Let p be an interior point of an arc A (but of no other arc) on the unbounded region of $\mathscr{A}$.
- Let L be the pseudo-line extending A in $\mathscr{L}$, and let L^- and L^+ be the two halves of L delimited by p.
- Let H be a half pseudo-line having its endpoint at p and having no other intersection in $\mathscr{A}$.

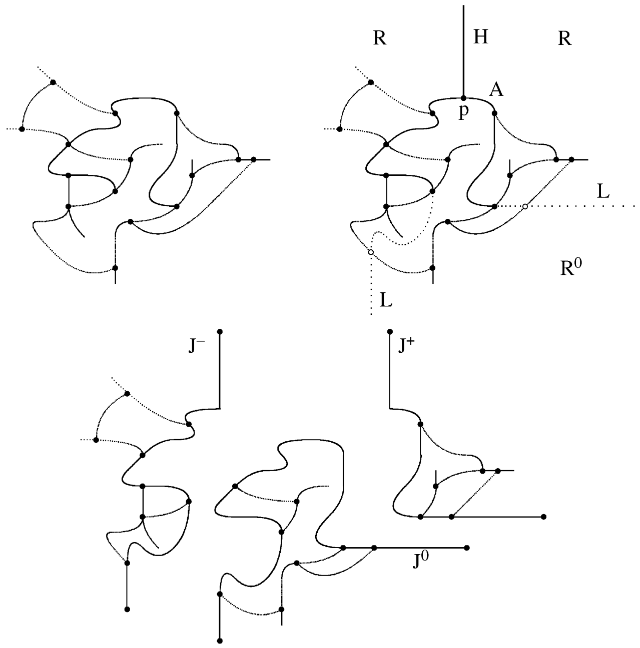

Fig. 7. Partitioning a contact system.

Then, L and H induce a "partition" of A into three new contact systems A^-, A^+ and A^0 (see Fig. 7):

- H, L^- and L^+ define three unbounded regions $\mathscr{R}^-$, $\mathscr{R}^+$ and $\mathscr{R}^0$ having $H \cup L^-$, $H \cup L^+$ and L as respective frontiers.
- We define J^-, J^+ and J^0 as the sub-arcs of $H \cup L^-$, $H \cup L^+$ and L strictly including all the intersections of these later pseudo-lines with $\mathscr{A}$.

These regions define three contact systems of pseudo-segments:

$$\mathscr{A}^- = \{J^-\} \cup \{A \cap \mathscr{R}^- : A \in \mathscr{A} \text{ and } A \cap \mathscr{R}^- \neq \emptyset\},$$
$$\mathscr{A}^+ = \{J^+\} \cup \{A \cap \mathscr{R}^+ : A \in \mathscr{A} \text{ and } A \cap \mathscr{R}^+ \neq \emptyset\},$$
$$\mathscr{A}^0 = \{J^0\} \cup \{A \cap \mathscr{R}^0 : A \in \mathscr{A} \text{ and } A \cap \mathscr{R}^0 \neq \emptyset\}.$$

These three contact systems have some nice properties:

Lemma 34. *The contact systems $\mathscr{A}^-$, $\mathscr{A}^+$ and $\mathscr{A}^0$ are maximal and extendible.*

Proof. The maximality is straightforward, as the extremal points of these contact systems are either extremal points of $\mathscr{A}$ or the extremities of J^-, J^+ or J^0.

The contact systems $\mathscr{A}^-$ and $\mathscr{A}^+$ are homeomorph to the contact systems where J^- (resp., J^+) is replaced by J^0. Then, the arrangement of pseudo-lines which extends $\mathscr{A}$ is an extension of all of the contact systems. □

Lemma 35. *If the arc A includes no extremal point of $\mathscr{A}$, we have*

$$2 \leqslant |\mathscr{A}^-| < |\mathscr{A}|, \quad 2 \leqslant |\mathscr{A}^+| < |\mathscr{A}|, \quad 3 \leqslant |\mathscr{A}^0| \leqslant |\mathscr{A}|,$$

and each of $\mathscr{A}^-$, $\mathscr{A}^+$ and $\mathscr{A}^0$ has an arc including two extremal points.

Proof. As A includes no extremal point of $\mathscr{A}$ the extremity of A in A^- (resp., A^+) is interior to some arc B^- (resp., B^+). As A may not intersect an arc twice, we have $B^- \neq B^+$. Moreover, each of these arcs intersects at most once the extension L of A, and hence the arc B^- (resp., B^+) does not meet the region $\mathscr{R}^+$ (resp., $\mathscr{R}^-$). Thus, $|\mathscr{A}^+| \leqslant |\mathscr{A}| - 1$ and the same holds for $|\mathscr{A}^-|$. Notice also that $\mathscr{A}^-$ and $\mathscr{A}^+$ include at least two arcs: J^- (resp., J^+) with two extremal points on it and the non-empty portion of B^- (resp., B^+) in $\mathscr{R}^-$ (resp., $\mathscr{R}^+$). Moreover, $\mathscr{A}^0$ includes at least three arcs: J^0 (with two extremal points on it) and the portions of B^- and B^+ in $\mathscr{R}^0$. □

Lemma 36. *Any extendible contact system has at least three extremal points, unless it has cardinality at most one.*

Proof. First notice that, according Remark 6 and the fact that the expanded contact system cannot have more extremal points than the original one, we can only prove the lemma for maximal contact systems.

The lemma is straightforward for contact systems of cardinality at most two. So, assume that for $2 \leqslant i < k$, the lemma holds for maximal contact systems of cardinality i and assume there exists a maximal contact system $\mathscr{A}$ of cardinality k having at most two extremal points.

Notice that at least three arcs of $\mathscr{A}$ meet the unbounded region in a sub-arc. Otherwise, the contact system would be bounded by a single arc or by two arcs intersecting each other twice, and the system would not be extendible.

- Assume $\mathscr{A}$ has two extremal points belonging to a same arc.
  From the previous construction and according to Lemma 34, $\mathscr{A}^-$ and $\mathscr{A}^+$ are extendible and, according to Lemma 35, have a smaller cardinality than $\mathscr{A}$ but include at least two arcs. By induction, they have at least three extremal points. Among these, there is one which is an extremal point of $\mathscr{A}$ belonging to the interior of $\mathscr{R}^-$ (resp., $\mathscr{R}^+$). Notice that these extremal points do not belong to a same arc: otherwise, this arc would intersect A twice. Hence, as the contact system $\mathscr{A}$ has an arc with two extremal points on it, it has at least three extremal points, what leads to a contradiction.
  Hence, the lemma holds for maximal contact system with at most k arcs having two extremal points belonging to the same arc.
- Assume $\mathscr{A}$ does not include two extremal points belonging to the same arc.
  From the same construction and according to Lemma 34, the contact system $\mathscr{A}^0$ is maximal and, according to Lemma 35, it has cardinality at most k and has two extremal points belonging to the same arc. According to the previous case, it has at least three extremal points. Among these, one is an extremal point of $\mathscr{A}$ belonging to the interior of $\mathscr{R}^0$. Together with the two extremal points of $\mathscr{A}$ belonging to the interior of $\mathscr{R}^-$ and $\mathscr{R}^+$ (as in the previous case), we get three extremal points, a contradiction. □

Lemma 37. *Any extendible contact system $\mathscr{A}$ is expandable to a maximal contact system, each subsystem of which has at least three extremal points unless it has cardinality at most one.*

Proof. As any subsystem $\mathscr{A}'$ of the expansion of $\mathscr{A}$ into a maximal contact system is extendible to a weak arrangement of pseudo-lines, the result follows from Lemma 36. □

Theorem 38. *Let $\mathscr{A}$ be a contact system of pseudo-segments. Then, the following conditions are equivalent:*

1. *$\mathscr{A}$ is stretchable,*
2. *each subsystem of $\mathscr{A}$ has at least three extremal points, unless it has cardinality at most one,*
3. *$\mathscr{A}$ is extendible.*

Proof. It is straightforward that (1) implies (3). The equivalence of (1) and (2) results from Theorem 33. According to Lemma 37, (3) implies that $\mathscr{A}$ is expendable to a maximal contact system, each subsystem of which has at least three extremal points or has cardinality at most one, thus implying (2) on the expanded contact system and thus the stretchability of the expanded contact system (and thus (1)). □

References

[1] T. Chan, On levels in arrangements of curves, Discrete Comput. Geom. 29 (2003) 375–393.
[2] I. Fáry, On straight line representation of planar graphs, Acta Sci. Math. (Szeged) II (1948) 229–233.
[3] N. Mnëv, On manifolds of combinatorial types of projective configurations and convex polyhedra, Soviet Math. Deokl. (32) (1985) 335–337.

[4] N. Mnëv, The universality theorems on the classification problem of configuration varieties and convex polytopes varieties, in: O.Y. Viro (Ed.), Topology and Geometry—Rohlin Seminar, Lecture Notes in Mathematics, vol. 1346, Springer, Berlin, 1988, pp. 527–544.
[5] J. Richter-Gebert, Realization spaces of polytopes, Lecture Notes in Mathematics, vol. 1643, Springer, Berlin, 1996.
[6] P.W. Shor, Stretchability of pseudolines is NP-hard, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 4 (1991) 531–554.
[7] S. Stein, Convex maps, in: Proceedings of the American Mathematical Society, vol. 2, 1951, pp. 464–466.
[8] E. Steinitz, H. Rademacher, Vorlesung über die Theorie der Polyeder, Springer, Berlin, 1934.
[9] W. Tutte, Convex representations of graphs, in: Proceedings of the London Mathematical Society, vol. 10, 1960, pp. 304–320.
[10] W. Tutte, How to draw a graph, in: Proceedings of the London Mathematical Society, vol. 13, 1963, pp. 743–768.
[11] K. Wagner, Über eine eigenschaft der Ebenen Komplexe, Math. Ann. 114 (1937) 570–590.