Double vertex digraphs of digraphs

Yubin Gao *, Yanling Shao
Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, PR China

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Let D be a digraph of order n. The double vertex digraph $S_2(D)$ of D is the digraph whose vertex set consists of all ordered pairs of distinct vertices of $V(D)$ such that there is an arc in $S_2(D)$ from $(x, y)$ to $(u, v)$ if and only if $x = u$ and there is an arc in D from y to v, or $y = v$ and there is an arc in D from x to u. In this paper, we establish some relationships between a digraph and its double vertex digraph.

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1. Introduction

For a simple graph G (with no loops and multiple edges), in [1] the authors gave the following definition. The double vertex graph $U_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two distinct vertices $(x, y)$ and $(u, v)$ are adjacent if and only if $|(x, y) \cap (u, v)| = 1$ and if $x = u$, then $y$ and $v$ are adjacent in G. The order and the size of $U_2(G)$ are $n(n - 1)/2$ and $q(n - 2)$, respectively, where n is the order and q is the size of G. As examples, we have $U_2(K_2) = K_4$, $U_2(K_3) = K_6$, and $U_2(K_{1, 3}) = C_6$. See Fig. 1 for an example of a graph and its double vertex graph.

Recently, there have been some papers concerning this topic, for example [1–3,5,7]. In particular, paper [3] reviews the recent results. In this paper, we extend the definition of the double vertex graph of a graph to a directed graph, and establish some relationships between a directed graph and its double vertex directed graph.

We need some concepts and notations on directed graphs. Let $D = (V(D), A(D))$ denote a directed graph (or digraph) with vertex set $V(D)$ and arc set $A(D)$. Loops and multiple arcs are not permitted. A walk of length k (or k-walk) is a sequence $v_1v_2 \ldots v_kv_{k+1}$ of vertices such that there is an arc in D from $v_i$ to $v_{i+1}$ for $i = 1, 2, \ldots, k$. The walk is a path if the vertices $v_1, \ldots, v_k, v_{k+1}$ are distinct. The walk is closed if $v_{k+1} = v_1$, and a cycle is a closed walk in which $v_1, \ldots, v_k$ are distinct. For a vertex $v$ of D, we denote by $d^+_D(v)$ (respectively, $d^-_D(v)$) the outdegree (respectively, indegree) of $v$ in D. For two distinct vertices $u$ and $v$ of $D$, the distance from $u$ to $v$, denoted by $d_D(u, v)$, is the length of the shortest walk from $u$ to $v$. We agree that $d_D(u, u) = 0$ for any vertex $u$.

A digraph $D$ is called Hamiltonian if it contains a cycle through all the vertices of D. A digraph $D$ is called primitive if, for some positive integer k, there is a walk of length exactly $k$ from each vertex $u$ to each vertex $v$ (possibly $u$ again). If $D$ is primitive, the smallest such $k$ is called the exponent of $D$, denoted by $\text{exp}(D)$. It is well-known that (for example see [4]) $D$ is primitive if and only if D is strongly connected and the greatest common divisor of all the cycle lengths of D is 1, and if $\text{exp}(D) = k$, then there is a walk of length $k$ from each vertex $u$ to each vertex $v$, and thus a walk from $u$ to $v$ of every length greater than $k$.

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* Corresponding author.
E-mail addresses: ybgao@nuc.edu.cn (Y. Gao), ylshao@nuc.edu.cn (Y. Shao).

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2. Definition of the double vertex digraph of a digraph

Let \( D = (V, A) \) be a digraph of order \( n \) (\( n \geq 2 \)). The double vertex digraph, \( S_2(D) \), of \( D \) is the digraph whose vertex set consists of all ordered pairs of distinct vertices of \( V(D) \) such that there is an arc in \( S_2(D) \) from \((x, y)\) to \((u, v)\) if and only if \( x = u \) and there is an arc in \( D \) from \( y \) to \( v \), or \( y = v \) and there is an arc in \( D \) from \( x \) to \( u \).

We now give some examples of digraphs and their double vertex digraphs.

**Example 2.1.** The digraphs in Fig. 2 are the directed 4-cycle and its double vertex digraph.

**Example 2.2.** The digraphs in Fig. 3 are the double directed path of length 4 and its double vertex digraph.

In a general way, if \( P_n \) is the double directed path of length \( n \) (\( n \geq 2 \)), then the double vertex digraph, \( S_2(P_n) \), of \( P_n \) satisfies the following:

1. \( S_2(P_n) \) is a symmetric digraph with \( n(n - 1) \) vertices and \( 4(n - 1)(n - 2) \) arcs.
2. \( S_2(P_n) \) is not strongly connected, and has two strongly connected components.
3. There is no arc in \( S_2(P_n) \) such that its two vertices are in different strongly connected components of \( S_2(P_n) \).

**Example 2.3.** The digraphs in Fig. 4 are the star \( D \) of order 4 and its double vertex digraph \( S_2(D) \).
3. Basic properties of the double vertex digraph

In this section, we establish some basic relationships between a digraph and its double vertex digraph. The first two theorems are clear, and we omit the proofs.

**Theorem 3.1.** Let \( D \) be a digraph with \( n \) vertices and \( q \) arcs. Then the double vertex digraph \( S_2(D) \) of \( D \) has \( n(n-1) \) vertices and \( 2q(n-2) \) arcs.

**Theorem 3.2.** Let \( D \) be a symmetric digraph. Then the double vertex digraph \( S_2(D) \) of \( D \) is also symmetric.

**Theorem 3.3.** Let \( D \) be a bipartite digraph. Then the double vertex digraph \( S_2(D) \) of \( D \) is also a bipartite digraph.

**Proof.** Let \( D \) be a bipartite digraph and let \( A \) and \( B \) be the two partite sets of \( D \). Take \( X = \{(u, v) \mid u, v \in A \text{ or } u, v \in B\} \), and \( Y = V(S_2(D)) \setminus X \) (that is, \( Y = \{(u, v) \mid u \in A \text{ and } v \in B, \text{ or } u \in B \text{ and } v \in A\} \)). Clearly, \( X \) and \( Y \) are the two partite sets of \( S_2(D) \). Then \( S_2(D) \) is a bipartite digraph. \( \Box \)

**Corollary 3.4.** Let \( D \) be a bipartite digraph and the two partite sets of \( D \) have \( m \) and \( n \) vertices, respectively. Then \( S_2(D) \) is a bipartite digraph and the two partite sets of \( S_2(D) \) have \( m^2 + n^2 - m - n \) and \( 2mn \) vertices, respectively.

**Corollary 3.5.** Let \( D \) be a bipartite digraph and the two partite sets of \( D \) have \( m \) and \( n \) vertices, respectively. If \( 2mn \neq m^2 + n^2 - m - n \), then \( S_2(D) \) is not Hamiltonian.

**Theorem 3.6.** Let \( D \) be the double directed cycle of order \( n \geq 3 \). Then the double vertex digraph \( S_2(D) \) of \( D \) is a bipartite digraph.

**Proof.** If \( n \) is even, then \( D \) is a bipartite digraph, so \( S_2(D) \) is also a bipartite digraph by **Theorem 3.3**.

We now consider the case that \( n \) is odd. Let \( V(D) = \{1, 2, \ldots, n\} \). Take \( X = \{(i, j) \mid 1 \leq i < j \leq n, i + j \text{ is even}\} \cup \{(i, j) \mid 1 \leq j < i \leq n, i + j \text{ is even}\} \), and \( Y = V(S_2(D)) \setminus X \). Clearly, \( X \) and \( Y \) are the two partite sets of \( S_2(D) \). Then \( S_2(D) \) is a bipartite digraph. \( \Box \)

Note that if \( H \) is a subgraph of \( D \), then \( S_2(H) \) is also a subgraph of \( S_2(D) \). The following is clear.

**Corollary 3.7.** Let \( D \) be a digraph of order \( n \), and \( L(D) \) be the set of distinct cycle lengths of \( D \). If \( L(D) = \{2, n\} \), then \( S_2(D) \) is a bipartite digraph.

4. Strong connectivity of the double vertex digraph

In this section we study the relationship on strong connectivity between a digraph and its double vertex digraph. The main theorem of this section is **Theorem 4.6**. We first prove the following lemma.

**Lemma 4.1.** Let \( D \) be a digraph, and let \((u, v)\) and \((x, y)\) be two vertices (not necessarily distinct) of \( S_2(D) \). If there is a walk in \( S_2(D) \) from \((u, v)\) to \((x, y)\) with length \( m \), then there exist walks in \( D \) from \( u \) to \( x \), and \( v \) to \( y \), respectively, such that the sum of their lengths equals \( m \).

**Proof.** Let \( P \) be a walk in \( S_2(D) \) from \((u, v)\) to \((x, y)\) with length \( m \). Denote \( P = (u, v)(u_1, v_1)(u_2, v_2) \cdots (u_{m-1}, v_{m-1})(x, y) \), \( u_0 = u, u_m = x, v_0 = v, \) and \( v_m = y \). Consider the sequence of vertices

\[ u_0u_1u_2 \cdots u_{m-1}u_m. \] (4.1)
If there exists $0 \leq i \leq m - 1$ such that $u_i = u_{i+1}$, then we delete $u_i$ from (4.1). Carry on the above-mentioned operation again and again, we finally obtain a new sequence of vertices

$$u_1 u_2 \ldots u_i u_{m},$$

(4.2)

where $u_1 = u_0 = u$. It is not difficult to verify that (4.2) is a walk in $D$ from $u$ to $x$.

By a similar method, we can obtain a walk in $D$ from $v$ to $y$.

Note that for any two adjacent vertices $(u_i, v_i)$ and $(u_{i+1}, v_{i+1}), 0 \leq i \leq m - 1$, in $P$, it must be that either $u_i = u_{i+1}$ and $v_i \neq v_{i+1}$, or $u_i \neq u_{i+1}$ and $v_i = v_{i+1}$. Then the last half of the lemma holds.

**Corollary 4.2.** Let $D$ be a digraph. If $S_2(D)$ is strongly connected, then so is $D$.

**Proof.** Let $u$ and $v$ be any two distinct vertices of $D$. If $S_2(D)$ is strongly connected, then there is a walk in $S_2(D)$ from $(u, v)$ to $(v, u)$. By Lemma 4.1 there exist walks in $D$ from $u$ to $v$, and $v$ to $u$, respectively. It implies that $D$ is strongly connected. □

By Corollary 4.2, we can assume that $D$ is strongly connected for considering the strong connectivity of $S_2(D)$.

Let $D$ be a strongly connected digraph. Then $S_2(D)$ may be not strongly connected. As a example (or see Example 2.2), we let $D$ be a double directed path of length $n$ with vertex set $V(D) = \{1, 2, \ldots, n\}$. It is easy to see that for any $1 \leq i < j \leq n$, there does not exist a walk in $S_2(D)$ from $(i, j)$ to $(j, i)$. This implies that $S_2(D)$ is not strongly connected.

A more general natural question is: Given a strongly connected digraph $D$, under what circumstances is $S_2(D)$ strongly connected? In order to answer this problem, we first prove three lemmas.

**Lemma 4.3.** Let $D$ be a strongly connected digraph of order $n$ ($n \geq 3$), and not the double directed path of length $n$. Let $u, v, w$ be three distinct vertices of $D$. Then there is a walk in $S_2(D)$ from $(w, u)$ to $(w, v)$.

**Proof.** Since $D$ is strongly connected, for any two vertices $x$ and $y$, there exist walks in $D$ from $x$ to $y$ and from $y$ to $x$, respectively. Take $Γ = (u_1 u_2 \ldots u_m v)$ to be the shortest path in $D$ from $u$ to $v$.

If the path $Γ$ does not contain the vertex $w$, it is easy to see that the sequence of vertices

$$(w, u)(w, u_1)(w, u_2) \cdots (w, u_k)(w, v)$$

is a walk in $S_2(D)$ from $(w, u)$ to $(w, v)$.

We now assume that the path $Γ$ contains the vertex $w$, and $w = u_m$ with $1 \leq m \leq k$. If there exists a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$, noticing that the sequence of vertices

$$(u, w)(u, u_{m+1}) \cdots (u, u_k)(u, v)(u_1, v)(u_2, v) \cdots (w, v)$$

is a walk in $S_2(D)$ from $(u, w)$ to $(w, v)$, then there is a walk in $S_2(D)$ from $(w, u)$ to $(w, v)$. Thus we now only need to prove that there exists a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$. Denote $u_0 = u$ and $P = uu_1 u_2 \ldots u_{m-1} w$ to be the shortest path in $D$ from $u$ to $w$. Consider the following three cases.

Case 1. $D$ is a symmetric digraph.

Then for each vertex $D$, its outdegree and indegree are the same. Since $D$ is not the double directed path, there is a vertex $y$ of $D$ such that $d_0^-(y) = d_0^+(y) \geq 3$.

If there is $1 \leq i \leq m - 1$ such that $d_0^-(u_i) = d_0^+(u_i) \geq 3$, then there is a vertex $z \in V(D) \setminus V(P)$ such that there are arcs in $D$ from $u_i$ to $z$, and from $z$ to $u_i$, respectively. Thus the sequence of vertices

$$(w, u)(w, u_1) \cdots (w, u_i)(w, z)(u_{m-1}, z) \cdots (u_1, z)(u, z)(u, u_i) \cdots (u, u_m)(u, w)$$

is a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$.

If $d_0^-(u_1) = d_0^+(u_1) \geq 3$, then there are two distinct vertices $z_1, z_2 \in V(D) \setminus V(P)$ such that there are arcs in $D$ from $u$ to $z_i$, and from $z_i$ to $u$ for $i = 1, 2$, respectively. Then the sequence of vertices

$$(w, u)(w, z_1)(u_{m-1}, z_1) \cdots (u_1, z_1)(u, z_1)(z_2, u_1)(z_2, u_2) \cdots (z_2, u)(w, u)$$

is a walk in $S_2(D)$ from $(w, u)$ to $(w, u)$.

If $d_0^-(u_i) = d_0^+(u_i) = 2$ for $i = 1, 2, \ldots, m - 1$ and there is a vertex $y \in V(D) \setminus V(P)$ such that $d_0^-(y) = d_0^+(y) \geq 3$, letting $P_1 = uu_1 u_2 \ldots u_i y$ and $P_2 = wu_i u_2 \ldots w_i y$ be the shortest paths from $u$ to $y$, and from $w$ to $y$, respectively, then $\{u_i, u_2, \ldots, u_i, y\} \cup \{u_1, u_2, \ldots, u_{i-1}, w\} = φ$, or $\{w_i, u_2, \ldots, w_i, y\} \cap \{u_1, u_2, \ldots, u_{i-1}, w\} = φ$. Without loss of generality, we assume that $\{u_1, u_2, \ldots, u_i, y\} \cap \{u_1, u_2, \ldots, u_{i-1}, w\} = φ$. Note that $d_0^-(y) = d_0^+(y) \geq 3$. Then there are
two distinct vertices $y_1, y_2 \in V(D) \setminus V(P_1)$ such that there are arcs in $D$ from $y$ to $y_i$, and from $y_i$ to $y$ for $i = 1, 2$, respectively. Thus the sequence of vertices

\[(w, u)(w, u')(w, y)(w, y_1)(u_{m-1}, y_1) \cdots (u_1, y_1)(u, y_1)(y_1)(y_2, y_1)\]
\[(y_2, y)(y_2, y_2)(y_2, u)(y_2, u_1) \cdots (y_2, u_m)(y_2, w)(y, u)(u, u_1) \cdots (u, u_m)(u, w)\]

is a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$.

Case 2. $D$ is not a symmetric digraph, and there is a path $Q$ in $D$ from $w$ to $u$ such that $V(P) \neq V(Q)$. Let $Q = wv_1v_2 \cdots v_{m-1}v_m$ and $w' \in V(P) \cup V(Q)$ but $w' \notin V(P) \cap V(Q)$. Without loss of generality, we assume that $w' \in V(P)$, and say $w'' = u_1$, $1 \leq j \leq m - 1$. Then the sequence of vertices

\[(w, u)(w, u_1) \cdots (w, u_j)(w, u)(u, u_1) \cdots (u, u_m)(u, w)\]

is a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$.

Case 3. $D$ is not a symmetric digraph, and for each path $Q$ in $D$ from $w$ to $u$, $V(P) = V(Q)$. If there exist $0 \leq i, j \leq m$ with $i \leq j - 2$ such that there is an arc in $D$ from $u_i$ to $u_j$, let $Q_1 = wv_1 \cdots v_ju_i$ and $Q_2 = u_iu_{i+1} \cdots u_{j}$ be the shortest paths from $w$ to $u_j$, and from $u_i$ to $u$, respectively, it is clear that $u \notin V(Q_1)$ and $w \notin V(Q_2)$. Then the sequence of vertices

\[(w, u)(w, v_1)(w, u) \cdots (w, u_j)(w, u)(u, u_1) \cdots (u, u_m)(u, w)\]

is a walk in $S_2(D)$ from $(w, u)$ to $(u, w)$. Otherwise, $wv_{m-1} \cdots u_2u_1u$ is only one path in $D$ from $w$ to $u$. Note that $D$ is not symmetric. There is a $k$-cycle $\gamma$ in $D$ with $k \geq 3$ such that $\gamma$ has at least one vertex which is not in $P$. Let $\gamma = \gamma_1\gamma_2 \cdots \gamma_k$ and $x_j$ be not in $P$. Take $R = wv_1 \cdots w_{i}u$ to be the shortest walk in $D$ from $w$ to $u$ such that $R$ contains all vertices of $\gamma$, and let $w_j = x_j$.

If $[u, w] \cap \{w_1', w_2', \ldots, w_l'\} = \emptyset$, then the sequence of vertices

\[(w, u)(w, w_1')(w, u)(w_2', u)(w, u)(w_1', u)(w, u)(w_1', u)(w, w)\]

is a walk in $S_2(D)$ from $(u, w)$ to $(u, w)$.

If $[u, w] \cap \{w_1', w_2', \ldots, w_l'\} \neq \emptyset$, without loss of generality, we assume that $[u, w] \cap \{w_1', w_2', \ldots, w_l'\} = \{w\}$, and $w_i' = w$. Then the sequence of vertices

\[(w, u)(w, u_1) \cdots (w, u_{m-1})(w, w)(w, w)(w, w)(w, w)(w, w)\]
\[(w, u_1)(w, u_2)(w, u_{m-1})(u_{m-1}, w)(u_{m-1}, w)(u_1, w)(u, w)\]

is a walk in $S_2(D)$ from $(u, w)$ to $(u, w)$.

The lemma now follows. \hfill \Box

By a similar proof method to Lemma 4.3, we can obtain the following.

**Lemma 4.4.** Let $D$ be a strongly connected digraph of order $n$ ($n \geq 3$), and not the double directed path of length $n$. Let $u, v, w$ be three distinct vertices of $D$. Then there is a walk in $S_2(D)$ from $(u, w)$ to $(v, w)$.

**Lemma 4.5.** Let $D$ be a strongly connected digraph of order $n$ ($n \geq 3$), and not the double directed path of length $n$. Let $(u, v)$ and $(x, y)$ be two vertices of $S_2(D)$ with $u \neq x$ and $v \neq y$. Then there is a walk from $(u, v)$ to $(x, y)$ in $S_2(D)$.

**Proof.** By Lemmas 4.3 and 4.4, there is a walk from $(u, v)$ to $(x, y)$, and there is a walk from $(x, v)$ to $(x, y)$ in $S_2(D)$. Thus there is a walk from $(u, v)$ to $(x, y)$ in $S_2(D)$. \hfill \Box

Lemmas 4.3–4.5 and Example 2.2 immediately yield the following.

**Theorem 4.6.** Let $D$ be a strongly connected digraph of order $n$ ($n \geq 3$). Then $S_2(D)$ is strongly connected if and only if $D$ is not the double directed path of length $n$.

5. **Primitivity of the double vertex digraph**

In this section, we study the primitivity of a double vertex digraph. The example below expresses that the double vertex digraph $S_2(D)$ of a primitive digraph $D$ is not necessarily primitive.

**Example 5.1.** Let $D$ be a double directed cycle with $n \geq 3$ vertices. Clearly, $D$ is primitive if $n$ is odd. Since all the cycle lengths of $S_2(D)$ are even by Theorem 3.6, the double vertex digraph $S_2(D)$ is not primitive for any $n \geq 3$.

The following two lemmas give the relationship on the cycle lengths between a digraph and its double vertex digraph.
Lemma 5.2. Let D be a digraph of order n, and C = v_1v_2...v_k be a k-cycle of D. Then the following properties hold.

1. If k < n, then, for any x ∈ V(D) \ {v_1, v_2, . . . , v_k} and 1 ≤ i ≤ k, the vertex (x, v_i) (respectively, (v_i, x)) is in a k-cycle of S_2(D).
2. For 1 ≤ i, j ≤ k and i ≠ j, the vertex (v_i, v_j) is in a 2k-cycle of S_2(D).

Proof. (1) Let x be not in C. Then the sequence of vertices

(x, v_1)(x, v_2)· · ·(x, v_{k−1})(x, v_k)(x, x)

is a k-cycle in S_2(D) containing the vertex (x, v_i), and the sequence of vertices

(v_1, x)(v_2, x)· · ·(v_{k−1}, x)(v_k, x)(v_1, x)

is a k-cycle in S_2(D) containing the vertex (v_i, x).

(2) Without loss of generality, we assume that i < j. It is not difficult to verify that the sequence of vertices

(v_i, v_j)(v_i, v_{j+1})· · ·(v_i, v_k)(v_{i+1}, v_k)· · ·(v_{k−1}, v_k)(v_{k−1}, v_i)· · ·(v_{k−1}, v_j)(v_k, v_j)(v_1, v_j)· · ·(v_1, v_j)

is a 2k-cycle in S_2(D) containing the vertex (v_i, v_j).

The lemma now follows. □

Lemma 5.3. Let D be a digraph and S_2(D) be the double vertex digraph of D. If there is an r-cycle in S_2(D), then there is an r-cycle in D, or there are some cycles with lengths r_1, r_2, . . . , r_s, respectively, in D such that r_1 + r_2 + · · · + r_s = r.

Proof. By Lemma 4.1, the lemma is clear. □

Lemma 5.4. Let r_i and s_i be positive integers for i = 1, . . . , k and j = 1, . . . , l. If gcd(r_1, . . . , r_k, s_1 + · · · + s_l) = 1, then gcd(r_1, . . . , r_k, s_1, . . . , s_l) = 1.

Proof. If gcd(r_1, . . . , r_k, s_1, . . . , s_l) = t > 1, then t | r_i for i = 1, 2, . . . , k, and t | s_j for j = 1, . . . , l. So t | s_1 + · · · + s_l. It implies that gcd(r_1 + s_1, r_2 + s_2, . . . , r_k + s_k) ≥ t, a contradiction. The lemma holds. □

Theorem 5.5. Let D be a digraph of order n, and L(D) be the set of distinct cycle lengths of D. Then S_2(D) is primitive if and only if D is primitive and L(D) ≠ [2, n].

Proof. Necessity. Let S_2(D) be primitive. Then S_2(D) is strongly connected and the greatest common divisor of all the cycle lengths of S_2(D) is 1. By Corollary 4.2 and Lemmas 5.3 and 5.4, D is also strongly connected, and the greatest common divisor of all the cycle lengths of D is 1. It implies that D is primitive, and so the necessity holds.

Sufficiency. Let D be primitive. Then D is strongly connected and the greatest common divisor of all the cycle lengths of D is 1. Clearly, S_2(D) is strongly connected by Theorem 4.6. Let L(D) = {r_1, r_2, . . . , r_k} with r_1 < r_2 < · · · < r_k, and L(S_2(D)) be the set of distinct cycle lengths of S_2(D). If r_k < n, then by Lemma 5.2, L(D) ⊆ L(S_2(D)), and so the greatest common divisor of all the cycle lengths of D is 1. If r_k = n, then r_k−1 ≥ 2 from the hypothesis, and {r_1, r_2, . . . , r_k−1} ⊆ L(S_2(D)) by Lemma 5.2. Let C_1 and C_2 be the n-cycle and r_k−1-cycle of D. Note that each vertex of C_2 is a vertex of C_1. Without loss of generality, we assume that C_1 = v_1v_2...v_n, C_2 = u_1u_2...u_{r_k−1}, u_{r_k−1} = v_n−1 and v_n ∉ V(C_2). It is easy to see that the sequence of vertices

(v_1, v_{n−1})(v_1, v_{n−1})· · ·(v_{n−2}, v_{n−1})(v_{n−2}, u_1)(v_{n−1}, u_1)(v_n, u_1)(v_n, u_2)· · ·(v_n, u_{r_k−1−1})(v_n, v_{n−1})

is a cycle in S_2(D) with length n + r_{k−1}, that is, n + r_{k−1} ∈ L(S_2(D)). Thus {r_1, r_2, . . . , r_{k−1}, r_k + r_{k−1}} ⊆ L(S_2(D)). Since gcd(r_1, r_2, . . . , r_{k−1}, r_k) = 1, we have that gcd(r_1, r_2, . . . , r_{k−1}, r_k + r_{k−1}) = 1. It implies that the greatest common divisor of all the cycle lengths of S_2(D) is 1. Then S_2(D) is primitive, and so the sufficiency holds. □

6. A special digraph — Wielandt digraph

The Wielandt digraph (as in Fig. 5) of order n ≥ 3 is the digraph with vertices 1, 2, . . . , n consisting of the cycle 1 → 2 → 3 · · · → n → 1 and the arc n → 2. It is known [6] that, up to isomorphism, the Wielandt digraph of order n has the largest exponent of primitive digraphs on n vertices, and that this exponent is n^2 − 2n + 2.

In this section, we consider the primitivity and the exponent of the double vertex digraph S_2(W_n) of the Wielandt digraph W_n. The main theorem of this section is Theorem 6.8.

We need some notations and techniques of graph theory. Let {s_1, s_2, . . . , s_p} be a set of relatively prime positive integers. The Frobenius number, φ(s_1, s_2, . . . , s_p), is the least integer such that the equation x_1s_1 + x_2s_2 + · · · + x_ps_p = m has a nonnegative integral solution x_1, x_2, . . . , x_p for all m ≥ φ(s_1, s_2, . . . , s_p).

Let D be a digraph, and L(D) the set of distinct cycle lengths of D. For any x, y ∈ V(D) and R = {a_1, a_2, . . . , a_r} ⊆ L(D) with gcd(a_1, a_2, . . . , a_r) = 1, the relative distance d_R(x, y) from x to y is defined to be the length of the shortest walk from x to y which meets at least one cycle of each length a_i, i = 1, 2, . . . , r. We have Lemma 6.1 obviously.
Lemma 6.1. Let $D$ be the primitive digraph, and $R = \{a_1, a_2, \ldots, a_r\} \subseteq L(D)$ with $\gcd(a_1, a_2, \ldots, a_r) = 1$. For any $x, y \in V(D)$, and $m \geq d_R(x, y) + \phi(a_1, a_2, \ldots, a_r)$, there is a walk in $D$ from $x$ to $y$ with length $m$.

Lemma 6.2. Let $W_n$ be the Wielandt digraph of order $n \geq 4$. Then the following properties hold.

1. For any $2 \leq i \leq n$, the vertices $(1, i)$ and $(i, 1)$ are in some $(n - 1)$-cycles of $S_2(W_n)$.
2. For any $2 \leq i, j \leq n$ and $i \neq j$, the vertex $(i, j)$ is in a $(2n - 2)$-cycle of $S_2(W_n)$.
3. All vertices of $S_2(W_n)$ are in some $(2n - 1)$-cycles and $2n$-cycles of $S_2(W_n)$.
4. Each of the $(2n - 1)$-cycles and $2n$-cycles of $S_2(W_n)$ contains at least one vertex of some $(n - 1)$-cycle of $S_2(W_n)$.

Proof. (1) Take

$$C_1 = (1, 2)(1, 3) \cdots (1, n)(1, 2),$$

$$C_2 = (2, 1)(3, 1) \cdots (n, 1)(2, 1).$$

Then $C_1$ and $C_2$ are two $(n - 1)$-cycles of $S_2(W_n)$, and they contain all vertices $(1, i)$ and $(i, 1)$ for $i = 2, 3, \ldots, n$. Then (1) holds.

(2) Since $2 \rightarrow 3 \rightarrow \cdots \rightarrow n \rightarrow 2$ is an $(n - 1)$-cycle of $W_n$, it is clear by Lemma 5.2 that for any $2 \leq i, j \leq n$ and $i \neq j$, the vertex $(i, j)$ is in a $(2n - 2)$-cycle of $S_2(W_n)$.

(3) Let $(i, j)$ be a vertex of $S_2(W_n)$. Without loss of generality, we assume $i < j$.

If $i = 1$, then the sequences of vertices

$$(1, j)(1, j + 1) \cdots (1, n)(2, n) \cdots (n - 1, n)(n - 1, 2)(n, 2)(1, 2) \cdots (1, j),$$

and

$$(1, j)(1, j + 1) \cdots (1, n)(2, n) \cdots (n - 1, n)(n - 1, 2)(n, 2)(1, 2) \cdots (1, j)$$

are a $(2n - 1)$-cycle and a $2n$-cycle, respectively, in $S_2(W_n)$ containing the vertex $(i, j)$.

If $i > 1$, then the sequences of vertices

$$(i, j)(i, j + 1) \cdots (i, n)(i, 1)(i + 1, 1) \cdots (n, 1)(n, 2)(n, 3)(2, 3) \cdots (2, j)(3, j) \cdots (i, j),$$

and

$$(i, j)(i, j + 1) \cdots (i, n)(i, 1)(i + 1, 1) \cdots (n, 1)(n, 2)(1, 2)(1, 3) \cdots (1, j)(2, j) \cdots (i, j),$$

are a $(2n - 1)$-cycle and a $2n$-cycle, respectively, in $S_2(W_n)$ containing the vertex $(i, j)$. Then (3) holds.

(4) Let $D$ be the digraph obtained from $W_n$ by deleting the vertex $1$. If $S_2(D)$ has an $r$-cycle, then $r = k(n - 1)$ by Lemma 5.3, where $k$ is a positive integer. So $S_2(D)$ has no $(2n - 1)$-cycles and $2n$-cycles. Thus each of the $(2n - 1)$-cycles and $2n$-cycles of $S_2(W_n)$ contains at least one vertex having the form $(1, i)$ or $(i, 1)$ $(2 \leq i \leq n)$. So (4) holds from (1). □

We remark that Lemma 6.2 implies $\{n - 1, 2n - 2, 2n - 1, 2n\} \subseteq L(S_2(W_n))$.

Lemma 6.3. For $n \geq 4$, $\gcd(n - 1, 2n - 1, 2n) = 1$, and

$$\phi(n - 1, 2n - 1, 2n) = \begin{cases} n^2 - 2n + 1, & \text{if } n \text{ is odd}, \\ n^2 - 3n + 2, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since $\gcd(n - 1, 2n - 1) = 1$, it is clear that $\gcd(n - 1, 2n - 1, 2n) = 1$.

Note that $\phi(n - 1, 2n - 1, 2n) \leq \phi(n - 1, 2n - 1) = 2n^2 - 6n + 4$. Denote

$$h(n) = \begin{cases} n^2 - 2n + 1, & \text{if } n \text{ is odd}, \\ n^2 - 3n + 2, & \text{if } n \text{ is even}. \end{cases}$$
We first prove that the equation
\[ x_1(n - 1) + x_2(2n - 1) + x_3(2n) = m \]  
has a nonnegative integral solution \(x_1, x_2, x_3\) for all integers \(h(n) \leq m \leq 2n^2 - 5n\).

Case 1. \(m = 2n^2 - kn\) with \(5 \leq k \leq n + 1\).

Since \((k - 1)(n - 1) + (n - k + 1)(2n - 1) = 2n^2 - kn\), Eq. (6.1) has a nonnegative integral solution \(x_1 = k - 1, x_2 = n - k + 1, x_3 = 0\).

Case 2. \(m = 2n^2 - kn - y\) with \(k\) even, \(6 \leq k \leq n + 1\) and \(1 \leq y \leq n - \frac{k}{2}\). Clearly, \(y(2n - 1) + (n - y - \frac{k}{2})(2n) = 2n^2 - kn - y\). Note \(n - y - \frac{k}{2} \geq 0\). Then Eq. (6.1) has a nonnegative integral solution \(x_1 = 0, x_2 = y, x_3 = n - y - \frac{k}{2}\).

Case 3. \(m = 2n^2 - kn - y\) with \(k\) even, \(6 \leq k \leq n + 1\) and \(n - \frac{k}{2} + 1 \leq y \leq n - 1\). Clearly, \((2y + k - 2n)(n - 1) + (2n - y - k)(2n - 1) = 2n^2 - kn - y\). Note \(2y + k - 2n \geq 2\) and \(2n - y - k \geq 0\). Then Eq. (6.1) has a nonnegative integral solution \(x_1 = 2y + k - 2n, x_2 = 2n - y - k, x_3 = 0\).

Case 4. \(m = 2n^2 - kn - y\) with \(k\) odd, \(5 \leq k \leq n + 1\) and \(1 \leq y \leq n - \frac{k-1}{2}\). Clearly, \((n - 1) + (y - 1)(2n - 1) + (n - y - \frac{k-1}{2})(2n) = 2n^2 - kn - y\). Note \(n - y - \frac{k-1}{2} \geq 0\). Then Eq. (6.1) has a nonnegative integral solution \(x_1 = 1, x_2 = y - 1, x_3 = n - y - \frac{k-1}{2}\).

Case 5. \(m = 2n^2 - kn - y\) with \(k\) odd, \(5 \leq k \leq n + 1\) and \(n - \frac{k-1}{2} + 1 \leq y \leq n - 1\). Clearly, \((2y + k - 2n)(n - 1) + (2n - y - k)(2n - 1) = 2n^2 - kn - y\). Note \(2y + k - 2n \geq 3\) and \(2n - y - k \geq 0\). Then Eq. (6.1) has a nonnegative integral solution \(x_1 = 2y + k - 2n, x_2 = 2n - y - k, x_3 = 0\).

Case 6. \(m = n^2 - 2n - y\) with \(n\) even and \(0 \leq y \leq n - 2\). If \(y\) is even, noticing that \(y(n - 1) + \frac{n^2 - y - 2}{2}(2n) = n^2 - 2n - y\), then Eq. (6.1) has a nonnegative integral solution \(x_1 = y, x_2 = 0, x_3 = -n + y - 2\). If \(y\) is odd, noticing that \((y - 1)(n - 1) + (2n - 1) + \frac{n^2 - y - 3}{2}(2n) = n^2 - 2n - y\), then Eq. (6.1) has a nonnegative integral solution \(x_1 = y - 1, x_2 = 1, x_3 = -n + y - 2\).

Combining Cases 1–6, we have that Eq. (6.1) has a nonnegative integral solution \(x_1, x_2, x_3\) for all integers \(h(n) \leq m \leq 2n^2 - 5n\). Notice \(2n^2 - 5n \geq 2n^2 - 6n + 4\). It implies that \(\phi(n - 1, 2n - 1, 2n) \leq h(n)\).

Next, we prove that when \(n\) is odd, \(\phi(n - 1, 2n - 1, 2n) = h(n) = n^2 - 2n + 1\), that is, Eq. (6.1) has no nonnegative integral solution \(x_1, x_2, x_3\) for \(m = n^2 - 2n\).

If there exist nonnegative integers \(x_1, x_2, x_3\) such that
\[ x_1(n - 1) + x_2(2n - 1) + x_3(2n) = n^2 - 2n, \]
then
\[ (x_1 + 2x_2 + 2x_3)n - (x_1 + x_3) = n(n - 2). \]

Thus the number \(x_1 + x_3\) is zero or a multiple of \(n\). If \(x_1 + x_2 = 0\), then \(x_1 = 0\) and \(x_2 = 0\). So \(2x_3 = n - 2\), a contradiction to the fact that \(n\) is odd. We now assume that \(x_1 + x_2 = kn\), where \(k\) is a positive integer. Thus
\[ kn + x_2 + 2x_3 - k = n - 2. \]  
(6.2)

Note that \(kn + x_2 + 2x_3 \leq n - 1\). Then (6.2) does not hold. Therefore \(\phi(n - 1, 2n - 1, 2n) = h(n) = n^2 - 2n + 1\) when \(n\) is odd.

Lastly, we prove that when \(n\) is even, \(\phi(n - 1, 2n - 1, 2n) = h(n) = n^2 - 3n + 2\), that is, Eq. (6.1) has no nonnegative integral solution \(x_1, x_2, x_3\) for \(m = n^2 - 3n + 1\).

If there exist the nonnegative integers \(x_1, x_2, x_3\) such that
\[ x_1(n - 1) + x_2(2n - 1) + x_3(2n) = n^2 - 3n + 1, \]
then
\[ (x_1 + 2x_2 + 2x_3)n - (x_1 + x_2 + 1) = n(n - 3). \]

Thus the number \(x_1 + x_2 + 1\) must be a multiple of \(n\). We assume that \(x_1 + x_2 + 1 = kn\), where \(k\) is a positive integer. Thus
\[ kn + x_2 + 2x_3 - (k + 1) = n - 3. \]  
(6.3)

Note that \(kn + x_2 + 2x_3 \leq (k + 1) \geq n - 2\). Then (6.3) does not hold. Therefore \(\phi(n - 1, 2n - 1, 2n) = h(n) = n^2 - 3n + 2\) when \(n\) is even. \(\square\)

**Lemma 6.4.** Let \(W_n\) be the Wielandt digraph of order \(n \geq 4\), and
\[ l(n) = \begin{cases} n^2 - 3, & \text{if } n \text{ is even}, \\ n^2 - 1, & \text{if } n \text{ is odd}. \end{cases} \]

Let \((u, v)\) and \((x, y)\) be two vertices (not necessarily distinct) of \(S_2(W_n)\), and \(P\) be a walk in \(S_2(W_n)\) from \((u, v)\) to \((x, y)\) with length \(d(P)\). If \(0 \leq d(P) \leq 3n - 5\) with \(d(P) \neq 2n - 4\) and \(d(P) \neq 2n - 1\), then there is a walk in \(S_2(W_n)\) from \((u, v)\) to \((x, y)\) with length \(l(n)\).
Proof. We consider the following five cases.

Case 1. \(d(P) = 0\).

Clearly, \(u = x\) and \(v = y\). If \(n\) is odd, then the walk that starts at vertex \((u, v)\), goes around a \(2n\)-cycle \(\frac{1}{2}(n - 1)\) times, and an \((n - 1)\)-cycle, is a walk from \((u, v)\) to \((u, v)\) with length \(n^2 - 1\). If \(n\) is even, then the walk that starts at vertex \((u, v)\), goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(n - 4)\) times, and an \((n - 1)\)-cycle \(2\) times, is a walk from \((u, v)\) to \((u, v)\) with length \(n^2 - 3\).

Case 2. \(1 \leq d(P) \leq n - 2\).

If \(n\) is even and \(d(P)\) is odd, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(n - d(P) - 3)\) times, and a \((2n - 2)\)-cycle \(\frac{1}{2}(d(P) + 3)\) times (or an \((n - 1)\)-cycle \((d(P) + 3)\) times), is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If both \(n\) and \(d(P)\) are even, and \(d(P) \neq n - 2\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(n - d(P) - 4)\) times, and an \((n - 1)\)-cycle \((d(P) + 2)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If \(n\) is even and \(d(P) = n - 2\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(n - 2)\) times, and an \((n - 1)\)-cycle, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If \(n\) is odd and \(d(P)\) is even, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(n - d(P) - 1)\) times, and an \((n - 1)\)-cycle \((d(P) + 1)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\).

If both \(n\) and \(d(P)\) are odd, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(n - d(P) - 2)\) times, and an \((n - 1)\)-cycle \((d(P)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\).

Case 3. \(d(P) = n - 1\).

If \(n\) is odd, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(n - 1)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\). If \(n\) is even, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(n - d(P) - 2)\) times, and an \((n - 1)\)-cycle \((d(P)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

Case 4. \(1 \leq d(P) \leq 2n - 2\) and \(d(P) \neq 2n - 4\).

If \(n\) is odd, and \(d(P)\) is odd, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(2n - d(P) - 3)\) times, and an \((n - 1)\)-cycle \((d(P) - n)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\).

If \(n\) is odd, and \(d(P)\) is even, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 2)\)-cycle \(\frac{1}{2}(d(P) - n + 1)\) times (or an \((n - 1)\)-cycle \((d(P) - n + 1)\) times), and a \(2n\)-cycle \(\frac{1}{2}(2n - d(P) - 2)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\).

If \(n\) is even, and \(d(P) = 2n - 2\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, and a \(2n\)-cycle \(\frac{1}{2}(n - 4)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If \(n\) is even, and \(d(P) = 2n - 3\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(n - 2)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If \(n\) is even, and \(d(P) \neq 2n - 5\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(2n - d(P) - 5)\) times, and an \((n - 1)\)-cycle \((d(P) - n + 2)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If \(n\) is odd, and \(d(P)\) is even with \(n \leq d(P) \leq 2n - 6\), then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(3n - d(P) - 4)\) times, and an \((n - 1)\)-cycle \((d(P) - 2n)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

Case 5. \(2n \leq d(P) \leq 3n - 5\).

If both \(n\) and \(d(P)\) are odd, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(3n - 3 - d(P))\) times, and an \((n - 1)\)-cycle \((d(P) - 2n + 1)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 1\).

If \(n\) is odd, and \(d(P)\) is even, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \(2n\)-cycle \(\frac{1}{2}(3n - d(P) - 5)\) times, and a \((2n - 2)\)-cycle \(\frac{1}{2}(d(P) - 2n + 3)\) times (or an \((n - 1)\)-cycle \((d(P) - 2n + 3)\) times), is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

If both \(n\) and \(d(P)\) are even, then the walk that starts at vertex \((u, v)\), follows \(P\) to vertex \((x, y)\) and along the way goes around a \((2n - 1)\)-cycle, a \(2n\)-cycle \(\frac{1}{2}(3n - d(P) - 6)\) times, and a \((2n - 2)\)-cycle \(\frac{1}{2}(d(P) - 2n + 2)\) times, is a walk from \((u, v)\) to \((x, y)\) with length \(n^2 - 3\).

This completes the proof. □

In order to give the exponent of \(S_2(W_n)\), we still need the following three lemmas.
Lemma 6.5. Let $D$ be a digraph, and $C = v_1v_2\ldots v_kv_1$ be a $k$-cycle of $D$ with $k \geq 3$. Then for any $1 \leq i, j \leq k$ and $i \neq j$, there is a path in $S_2(D)$ from $(v_i, v_j)$ to $(v_j, v_i)$ with length $k$.

Proof. Without loss of generality, we assume that $i < j$. If $i < j - 1$, then the sequence of vertices

$$(v_i, v_j)(v_{i+1}, v_j)\cdots (v_{j-1}, v_j)$$

is a path in $S_2(D)$ from $(v_i, v_j)$ to $(v_j, v_i)$ with length $k$, where $v_{k+1} = v_1$.

If $i = j - 1$, then the sequence of vertices

$$(v_i, v_j)(v_{i+1}, v_j)\cdots (v_{j-1}, v_{j+1})(v_j, v_{j+1})$$

is a path in $S_2(D)$ from $(v_i, v_j)$ to $(v_j, v_i)$ with length $k$, where $v_{k+1} = v_1$ and $v_{k+2} = v_2$. □

Lemma 6.6. Let $W_n$ be the Wielandt digraph of order $n \geq 4$, $x, y$ and $z$ be three distinct vertices, and their relative positions on the $n$-cycle counter-clockwise be $x, y, z$. Then there are two paths in $S_2(W_n)$ from $(x, z)$ to $(x, y)$ with lengths $k$ and $k + 1$, respectively, such that $n \leq k \leq 2n - 2$.

Proof. Consider the following four cases.

Case 1. $1 \not\in \{x, y, z\}$.

By Lemma 6.5 there are two paths in $S_2(W_n)$ from $(x, z)$ to $(z, x)$ with lengths $n - 1$ and $n$, respectively. Note that the sequence of vertices

$$(z, x)(z, x + 1)\cdots (z, y)(z + 1, y)\cdots (x, y)$$

is a path in $S_2(D)$ from $(z, x)$ to $(x, y)$ with length $d_{W_n}(z, y)$. Clearly, $2 \leq d_{W_n}(z, y) \leq n - 2$. Then there are two paths in $S_2(W_n)$ from $(x, z)$ to $(x, y)$ with lengths $k$ and $k + 1$, respectively, such that $n + 1 \leq k \leq 2n - 2$.

Case 2. $x = 1$.

It is easy to see that the sequences of vertices

$$(x, z)\cdots (x, n)(2, n)(3, n)(3, 2)(4, 2)\cdots (n, 2)(x, 2)\cdots (x, y)$$

and

$$(x, z)\cdots (x, n)(2, n)(3, n)(3, 1)(3, 2)(4, 2)\cdots (n, 2)(x, 2)\cdots (x, y)$$

are two paths in $S_2(D)$ from $(x, z)$ to $(x, y)$ with lengths $n - 1 + d_{W_n}(z, y)$ and $n + d_{W_n}(z, y)$, respectively. Note that $1 \leq d_{W_n}(z, y) \leq n - 2$. Then the lemma holds.

Case 3. $y = 1$.

It is easy to see that the sequences of vertices

$$(x, z)\cdots (n, z)(n, z + 1)\cdots (n, n - 1)(2, n - 1)(2, n)(2, y)(3, y)\cdots (x, y)$$

and

$$(x, z)\cdots (n, z)(n, z + 1)\cdots (n, n - 1)(1, n - 1)(2, n - 1)(2, n)(2, y)(3, y)\cdots (x, y)$$

are two paths in $S_2(D)$ from $(x, z)$ to $(x, y)$ with lengths $n - 1 + d_{W_n}(z, y)$ and $n + d_{W_n}(z, y)$, respectively. Note that $2 \leq d_{W_n}(z, y) \leq n - 1$. Then the lemma holds.

Case 4. $z = 1$.

It is easy to see that the sequences of vertices

$$(x, z)(x + 1, z)\cdots (n, z)(n, 2)(n, 3)(2, 3)\cdots (2, y)\cdots (x, y)$$

and

$$(x, z)(x + 1, z)\cdots (n, z)(n, 2)(n, 3)(1, 3)(2, 3)\cdots (2, y)\cdots (x, y)$$

are two paths in $S_2(D)$ from $(x, z)$ to $(x, y)$ with lengths $n - 1 + d_{W_n}(z, y)$ and $n + d_{W_n}(z, y)$, respectively. Note that $2 \leq d_{W_n}(z, y) \leq n - 1$. Then the lemma holds.

This completes the proof. □

By a similar method to the proof of Lemma 6.6, the following lemma is clear.

Lemma 6.7. Let $W_n$ be the Wielandt digraph of order $n \geq 4$, $x, y$ and $z$ be three distinct vertices, and their relative positions on the $n$-cycle counter-clockwise be $x, y, z$. Then there are two paths in $S_2(W_n)$ from $(x, y)$ to $(z, y)$ with lengths $k$ and $k + 1$, respectively, such that $n \leq k \leq 2n - 2$.

We now give the main theorem of this section.
Theorem 6.8. Let $W_n$ be the Wielandt digraph of order $n \geq 4$. Then $S_2(W_n)$ is primitive, and
\[
\exp(S_2(W_n)) = \begin{cases} 
  n^2 - 3, & \text{if } n \text{ is even,} \\
  n^2 - 1, & \text{if } n \text{ is odd.}
\end{cases}
\] (6.4)

Proof. The primitivity of $S_2(W_n)$ is clear by Theorem 4.6 and Lemma 6.2.
Take $R = \{n - 1, 2n - 1, 2n\}$, and
\[
l(n) = \begin{cases} 
  n^2 - 3, & \text{if } n \text{ is even,} \\
  n^2 - 1, & \text{if } n \text{ is odd.}
\end{cases}
\]

We first prove that for each ordered pair $(u, v), (x, y)$ of vertices of $S_2(W_n)$, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$. Consider the following three cases.

Case 1. $|u, v, x, y| = 2$.
Subcase 1.1. $u = x$ and $v = y$.
By Lemma 6.4, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(u, v)$ with length $l(n)$.
Subcase 1.2. $u = y$ and $v = x$.
Clearly, $d_R((u, v), (v, u)) \leq n$ by Lemma 6.5. By Lemmas 6.1 and 6.3, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(v, u)$ with length $l(n)$.

Case 2. $|u, v, x, y| = 3$.
Subcase 2.1. $u = x$ and $v \neq x$ or $v = y$ and $u \neq x$.
We consider the following cases according to their relative positions on the n-cycle of $W_n$.
Subcase 2.1.1. The vertices $u, v$ and $y$ on the n-cycle counter-clockwise are $u, v, y$.
Let $P$ be the shortest path in $S_2(W_n)$ from $(u, v)$ to $(u, y)$. Clearly, the length $d(P)$ of $P$ satisfies that $1 \leq d(P) \leq n - 2$. By Lemma 6.4, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(u, y)$ with length $l(n)$.
Subcase 2.1.2. The vertices $u, v$ and $y$ on the n-cycle counter-clockwise are $u, v, x$.
From the proof of Lemma 6.6, $d_R((u, v), (u, y)) \leq 2n - 2$. By Lemmas 6.1 and 6.3, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(u, y)$ with length $l(n)$.

Subcase 2.2.2. The vertices $u, v$ and $x$ on the n-cycle counter-clockwise are $u, v, x$.
Then $d_R((u, v), (u, x)) \leq d_R((u, v), (v, u)) + d_R((v, u), (u, x)) \leq n + (n - 2) = 2n - 2$. By Lemmas 6.1 and 6.3, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(u, x)$ with length $l(n)$.

Case 3. $|u, v, x, y| = 4$.
We consider the following cases according to their relative positions on the n-cycle of $W_n$.
Subcase 3.1. Four vertices $u, v, x$ and $y$ on the n-cycle counter-clockwise are $u, x, v, y$.
Let $P = P_1 + P_2$, where $P_1$ and $P_2$ are the shortest paths from $(u, v)$ to $(x, v)$, and $(x, v)$ to $(x, u)$, respectively. Then $P$ is a walk from $(u, v)$ to $(x, u)$, where $1 \in \{u, v, x, y\}$, then the length of $P$ does not exceed $n - 1$, so $d_R((u, v), (x, y)) \leq n - 1$ and there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$ by Lemmas 6.1 and 6.3. If $1 \notin \{u, v, x, y\}$, then the length of $P$ does not exceed $n - 2$, and so, by Lemma 6.4, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$.

Subcase 3.2. Four vertices $u, v, x$ and $y$ on the n-cycle counter-clockwise are $u, y, x, v$.
Let $P = P_1 + P_2$, where $P_1$ and $P_2$ are the shortest paths in $S_2(W_n)$ from $(u, v)$ to $(x, v)$, and $(x, v)$ to $(x, y)$, respectively. We use $d(P)$, $d(P_1)$ and $d(P_2)$ to denote the lengths of $P, P_1$ and $P_2$, respectively. Then $P$ is a walk from $(u, v)$ to $(x, y)$ with length $d(P) = d(P_1) + d(P_2)$. It is not difficult to verify that $1 \leq d(P_1) \leq n - 3$. By Lemma 6.6, we can choose $P_2$ such that $n + 1 \leq d(P) \leq 3n - 5, d(P) \neq 2n - 1$ and $d(P) \neq 2n - 4$. By Lemma 6.4, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$.

Subcase 3.3. Four vertices $u, v, x$ and $y$ on the n-cycle counter-clockwise are $u, x, v, y$.
Clearly, $d_R((u, v), (u, y)) \leq d_R((u, v), (x, y)) \leq n - 2$. Then by Lemma 6.4, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$.

Subcase 3.4. Four vertices $u, v, x$ and $y$ on the n-cycle counter-clockwise are $u, y, x, v$.
Then $d_R((u, v), (x, y)) \leq d_R((u, v), (v, u)) + d_R((v, u), (x, y)) \leq n + (n - 2) = 2n - 2$. By Lemmas 6.1 and 6.3, there is a walk in $S_2(W_n)$ from $(u, v)$ to $(x, y)$ with length $l(n)$.

Subcase 3.5. Four vertices $u, v, x$ and $y$ on the n-cycle counter-clockwise are $u, v, y, x$.
Let $P = P_1 + P_2$, where $P_1$ is the shortest path in $S_2(W_n)$ from $(u, v)$ to $(y, x)$, and $P_2$ is a path in $S_2(W_n)$ from $(u, y)$ to $(x, y)$. We use $d(P), d(P_1)$ and $d(P_2)$ to denote the lengths of $P, P_1$ and $P_2$, respectively. Then $P$ is a walk from $(u, v)$ to $(x, y)$.
with length \( d(P) = d(P_1) + d(P_2) \). It is not difficult to verify that \( 1 \leq d(P_1) \leq n - 3 \). By Lemma 6.7, we can choose \( P_2 \) such that \( n + 1 \leq d(P) \leq 3n - 5 \), \( d(P) \not\equiv 2n - 1 \) and \( d(P) \not\equiv 2n - 4 \). By Lemma 6.4, there is a walk in \( S_2(W_n) \) from \( (u, v) \) to \( (x, y) \) with length \( l(n) \).

Subcase 3.6. Four vertices \( u, v, x \) and \( y \) on the \( n \)-cycle counter-clockwise are \( u, v, x \).

Let \( P = P_1 + P_2 \), where \( P_1 \) and \( P_2 \) are the shortest paths in \( S_2(W_n) \) from \( (u, v) \) to \( (u, y) \), and \( (u, y) \) to \( (x, y) \), respectively. Then \( P \) is a walk from \( (u, v) \) to \( (x, y) \). We denote by \( d(P) \) the length of \( P \). If \( 1 \in \{ u, v, x, y \} \), then \( d_\delta((u, v), (x, y)) = d(P) \leq 2n - 4 \). By Lemmas 6.1 and 6.3, there is a walk in \( S_2(W_n) \) from \( (u, v) \) to \( (x, y) \) with length \( l(n) \). If \( 1 \not\in \{ u, v, x, y \} \), then \( d(P) \not\equiv 2n - 6 \) so there is a walk in \( S_2(W_n) \) from \( (u, v) \) to \( (x, y) \) with length \( l(n) \) from Lemma 6.4.

To combine the above discussions, we have that \( \exp(S_2(W_n)) \leq l(n) \).

On the other hand, we show that there does not exist a walk in \( S_2(W_n) \) from \( (1, 2) \) to \( (n, 1) \) with length \( n^2 - 2 \) for odd \( n \) and from \( (1, 2) \) to \( (n, n - 1) \) with length \( n^2 - 4 \) for even \( n \) respectively.

Case 1. \( n \) is odd.

Assume to the contrary that there exists a walk in \( S_2(W_n) \) from \( (1, 2) \) to \( (n, 1) \) with length \( n^2 - 2 \). By Lemma 4.1, there exist walks \( P_1 \) and \( P_2 \) in \( W_n \) from \( 1 \) to \( n \) and from \( 2 \) to \( 1 \), respectively, such that \( d(P_1) + d(P_2) = n^2 - 2 \). Since the length of the only path in \( W_n \) from \( 1 \) to \( n \) is \( n - 1 \), and the length of the only path in \( W_n \) from \( 2 \) to \( 1 \) is \( n - 1 \), there are nonnegative integers \( a_i \) and \( b_i \), \( i = 1, 2 \), such that

\[
\begin{align*}
\begin{cases}
  d(P_1) &= n - 1 + a_1 n + b_1(n - 1), \\
  d(P_2) &= n - 1 + a_2 n + b_2(n - 1).
\end{cases}
\end{align*}
\]

(6.5)

So

\[ n^2 - 2 = n - 1 + a_1 n + b_1(n - 1) + n - 1 + a_2 n + b_2(n - 1), \]

that is,

\[ n^2 - 2n = (a_1 + a_2)n + (b_1 + b_2)(n - 1). \]

(6.6)

Note that \( \phi(n, n - 1) = n^2 - 3n + 2 \). Then

\[ \phi(n, n - 1) - 1 = (a_1 + a_2)n + (b_1 + b_2 - 1)(n - 1). \]

If \( b_1 + b_2 \geq 1 \), then it contradicts the definition of \( \phi(n, n - 1) \). So \( b_1 = b_2 = 0 \). From (6.6) and (6.5), \( a_1 + a_2 = n - 2 \), \( P_1 \) consists of the only path from \( 1 \) to \( n \) and \( a_1 \) \( n \)-cycles, and \( P_2 \) consists of the only path from \( 2 \) to \( 1 \) and \( a_2 \) \( n \)-cycles. From the Wielandt digraph \( W_n \), it is easy to see that \( a_1 = a_2 \). But \( a_1 \) and \( a_2 \) have different parity from the fact that \( n - 2 \) is odd and \( a_1 + a_2 = n - 2 \). It is a contradiction. Thus there does not exist a walk in \( S_2(W_n) \) from \( (1, 2) \) to \( (n, 1) \) with length \( n^2 - 2 \).

Case 2. \( n \) is even.

Assume to the contrary that there exists a walk in \( S_2(W_n) \) from \( (1, 2) \) to \( (n, n - 1) \) with length \( n^2 - 4 \). By Lemma 4.1, there exist walks \( P_3 \) and \( P_4 \) in \( W_n \) from \( 1 \) to \( n \) and from \( 2 \) to \( n - 1 \), respectively, such that \( d(P_3) + d(P_4) = n^2 - 4 \). Since the length of the only path in \( W_n \) from \( 1 \) to \( n \) is \( n - 1 \), and the length of the only path in \( W_n \) from \( 2 \) to \( n - 1 \) is \( n - 3 \), there are nonnegative integers \( a_i \) and \( b_i \), \( i = 3, 4 \), such that

\[
\begin{align*}
\begin{cases}
  d(P_3) &= n - 1 + a_3 n + b_3(n - 1), \\
  d(P_4) &= n - 3 + a_4 n + b_4(n - 1).
\end{cases}
\end{align*}
\]

(6.7)

So

\[ n^2 - 4 = n - 1 + a_3 n + b_3(n - 1) + n - 3 + a_4 n + b_4(n - 1), \]

that is,

\[ n^2 - 2n = (a_3 + a_4)n + (b_3 + b_4)(n - 1). \]

(6.8)

Then

\[ \phi(n, n - 1) - 1 = (a_3 + a_4)n + (b_3 + b_4 - 1)(n - 1). \]

If \( b_3 + b_4 \geq 1 \), then it contradicts the definition of \( \phi(n, n - 1) \). So \( b_3 = b_4 = 0 \). From (6.8) and (6.7), \( a_3 + a_4 = n - 2 \), \( P_3 \) consists of the only path from \( 1 \) to \( n \) and \( a_3 \) \( n \)-cycles, and \( P_4 \) consists of the only path from \( 2 \) to \( n - 1 \) and \( a_4 \) \( n \)-cycles. From the Wielandt digraph \( W_n \), it is easy to see that \( a_3 = a_4 \). But \( a_3 \) and \( a_4 \) have the same parity from the fact that \( n - 2 \) is even and \( a_3 + a_4 = n - 2 \). It is a contradiction. Thus there does not exist a walk in \( S_2(W_n) \) from \( (1, 2) \) to \( (n, n - 1) \) with length \( n^2 - 4 \).

Combining Cases 1 and 2, \( \exp(S_2(W_n)) \geq l(n) \). Therefore \( \exp(S_2(W_n)) = l(n) \). The theorem holds. \( \square \)

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References