Exponential dichotomy and dichotomy radius for difference equations

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Abstract

The aim of this paper is to provide a new approach concerning the characterization of exponential dichotomy of difference equations by means of admissible pair of sequence spaces. We classify the classes of input and output spaces, respectively, and deduce necessary and sufficient conditions for exponential dichotomy applicable for a large variety of systems. By an example we show that the obtained results are the most general in this topic. As an application we deduce a general lower bound for the dichotomy radius of difference equations in terms of input–output operators acting on sequence spaces which are invariant under translations.

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1. Introduction

The study of the exponential dichotomy of evolution equations using input–output techniques was intensively developed in the past few years (see [4–6,13,15–19]). In order to study the existence of exponential dichotomy one associates an input–output equation with the initial equation (see [4–6,10,15–19]). Thus, the asymptotic properties of the initial evolution equation may be expressed in terms of the solvability of an associated input–output equation between two spaces: the input space and the output space, which form the admissible pair. In this context, a natural question is which are the properties of the main classes where the input or the output space should belong to. For evolution families defined on the real line this question was answered for the case of admissibility with respect to integral equations (see [17,19]).

The central concern in the study of the exponential dichotomy is to obtain a splitting of the space at every moment such that the behavior on the corresponding subspaces is modelled by exponential decay backward and forward in time. This decomposition is expressed by the existence of a projection family. A notable property which individualizes the evolution equations on the real line is that the family of the dichotomy projections is uniquely determined by an initial condition.
This particularity led to interesting situations and to a wide applicability area, some of them being pointed out in the present paper. Our study is motivated by the recent development in the asymptotic theory of difference equations and by the open problems related to the robustness of exponential dichotomy of difference equations. Moreover we should note that the exponential dichotomy of an evolution family defined on the real line is equivalent to the existence of exponential dichotomy for the associated discrete family (see [15, Theorem 3.2]). This means that in the study of the exponential dichotomy the discrete case led to the most general conclusions, since no measurability or continuity conditions are needed. We mention that results of the type obtained in the present paper have valuable predecessors in the literature (see [1–3], Chapter III in [7], Sections 3.1 and 3.2 in [11]).

The aim of this paper is to provide a new and systematic study of the existence of exponential dichotomy in terms of the admissibility of general pairs of sequence spaces and to identify the classes of viable input spaces and output spaces, respectively. We consider the case of difference equations on the real line of the form

\[ x(n + 1) = A(n)x(n), \quad n \in \mathbb{Z}, \]  

where \((A(n))_{n \in \mathbb{Z}}\) is a sequence of bounded linear operators on a Banach space \(X\). When \(B\) is a Banach sequence space over \(\mathbb{Z}\) (see Section 2), we denote by \(B(\mathbb{Z}, X)\) the space consisting of all sequences \(s : \mathbb{Z} \to X\) such that the sequence \((\|s(n)\|_X)_{n \in \mathbb{Z}}\) belongs to \(B\). Let \(I\) and \(O\) be non-zero Banach sequence spaces that are invariant under translation. The pair \((O(\mathbb{Z}, X), I(\mathbb{Z}, X))\) is said to be admissible if for each \(s \in I(\mathbb{Z}, X)\) there exists a unique \(\gamma \in O(\mathbb{Z}, X)\) such that

\[ \gamma(n + 1) = A(n)\gamma(n) + s(n + 1), \quad n \in \mathbb{Z}. \]  

Here \(I(\mathbb{Z}, X)\) is the input space and \(O(\mathbb{Z}, X)\) is the output space.

One of the main results of this paper (see Theorem 3.5) shows that Eq. \((A)\) is uniformly exponentially dichotomic whenever there exist translation invariant Banach sequence spaces \(I\) and \(O\) such that \(\ell^1(\mathbb{Z}, \mathbb{R})\) is properly contained in \(I\) or \(\sup_{n \in \mathbb{N}} |X_{0,...,n}|O = \infty\) (see Definition 2.3 for the explanation of this notation), and the pair \((O(\mathbb{Z}, X), I(\mathbb{Z}, X))\) is admissible. Furthermore, we prove that the converse implication is true if, in addition, \(I \subset O\) and the coefficients \(A(n)\) are uniformly bounded in the operator norm. By an example, we motivate our hypotheses and show that the above characterization for exponential dichotomy is the most general in this topic. As particular cases we deduce the dichotomy theorems proved in [15,16] and also new characterizations of exponential dichotomy in terms of the admissibility of pairs of Orlicz sequence spaces.

A special application of the main results will be at the study of the exponential dichotomy robustness of difference equations (see Theorems 4.1 and 4.2). In order to analyze the asymptotic behavior of the perturbed system we introduce the concept of dichotomy radius, which is a natural generalization of the stability radius (see [8,9,14]). For the case when \(\sup_{n \in \mathbb{Z}} \|A(n)\|_X < \infty\), the dichotomy radius is defined as the largest \(r > 0\) such that \(\sup_{n \in \mathbb{Z}} \|D(n)\|_X < r\) implies that the perturbed equation

\[ x(n + 1) = (A(n) + D(n))x(n), \quad n \in \mathbb{Z}, \]  

remains uniformly exponentially dichotomic. The concept of dichotomy radius for hyperbolic semigroups was introduced in [6] (see Section 4). There the authors gave an estimation of the dichotomy radius in terms of the stability radius in the autonomous case and proved that for exponentially stable semigroups the dichotomy radius coincides with the stability radius (see Proposition 4.6). The concept of dichotomy radius for discrete variational systems was introduced in [18], where a lower bound for the dichotomy radius is also deduced in terms of input–output operators acting on \(\ell^p\) and \(c_0\)-spaces. In this paper, as a consequence of the main results, we will study the robustness of the exponential dichotomy of difference equations deducing lower bounds for the dichotomy radius. We will obtain very general estimations in terms of the norms of input–output operators acting on invariant under translation sequence spaces.

2. Banach sequence spaces

In this section we present some basic definitions and properties from the theory of Banach sequence spaces. Let \(\mathbb{Z}\) denote the set of the integers and let \(\mathbb{N}\) denote the set of natural integers. Let \(\mathcal{S}(\mathbb{Z}, \mathbb{R})\) be the linear space of all sequences \(s : \mathbb{Z} \to \mathbb{R}\).
Definition 2.1. A linear subspace $B \subset S(\mathbb{Z}, \mathbb{R})$ is called normed sequence space if there is a mapping $| \cdot |_B : B \rightarrow \mathbb{R}_+$ such that:

(i) $|s|_B = 0$ if and only if $s = 0$;
(ii) $|\alpha s|_B = |\alpha||s|_B$, for all $(\alpha, s) \in \mathbb{R} \times B$;
(iii) $s + \gamma|_B \leq |s|_B + |\gamma|_B$, for all $s, \gamma \in B$;
(iv) if $|s(j)| \leq |\gamma(j)|$, for all $j \in \mathbb{Z}$ and $\gamma \in B$, then $s \in B$ and $|s|_B \leq |\gamma|_B$.

If, moreover, $(B, | \cdot |_B)$ is complete, then $B$ is called Banach sequence space.

Remark 2.1. If $s_n \rightarrow s$ in $B$, then there is a subsequence $(s_{k_n}) \subset (s_n)$, which converges to $s$ pointwise (see [12]).

Definition 2.2. A Banach sequence space $(B, | \cdot |_B)$ is said to be invariant under translations if for every $s \in B$ and every $m \in \mathbb{Z}$, the sequence $s_m : \mathbb{Z} \rightarrow \mathbb{R}$, $s_m(j) = s(j - m)$ belongs to $B$ and $|s_m|_B = |s|_B$.

For every $A \subset \mathbb{Z}$ we denote by $\chi_A$ the characteristic function of the set $A$. We denote by $\mathcal{T}(\mathbb{Z})$ the class of all Banach sequence spaces which are invariant under translations and contain at least a non-zero sequence.

Remark 2.2. If $B \in \mathcal{T}(\mathbb{Z})$, then $\chi_A \subset B$, for every $A \subset \mathbb{Z}$.

Definition 2.3. Let $B \in \mathcal{T}(\mathbb{Z})$. The mapping $F_B : \mathbb{N}^* \rightarrow \mathbb{R}_+$, $F_B(n) = |\chi_{[0, \ldots, n-1]}|_B$ is called the fundamental function of the space $B$.

Example 2.1. Let $p \in [1, \infty)$. The linear space $\ell^p(\mathbb{Z}, \mathbb{R}) = \{s \in S(\mathbb{Z}, \mathbb{R}): \sum_{n=-\infty}^{\infty} |s(n)|^p < \infty\}$ is a Banach sequence space with respect to the norm $\|s\|_p := (\sum_{n=-\infty}^{\infty} |s(n)|^p)^{1/p}$ and belongs to $\mathcal{T}(\mathbb{Z})$.

Example 2.2. The linear space $\ell^\infty(\mathbb{Z}, \mathbb{R}) = \{s \in S(\mathbb{Z}, \mathbb{R}): \sup_{n \in \mathbb{Z}} |s(n)| < \infty\}$ is a Banach space with respect to the norm $\|s\|_\infty := \sup_{n \in \mathbb{Z}} |s(n)|$ and $\ell^\infty(\mathbb{Z}, \mathbb{R}) \in \mathcal{T}(\mathbb{Z})$. If $c_0(\mathbb{Z}, \mathbb{R}) = \{s \in S(\mathbb{Z}, \mathbb{R}): \lim_{n \rightarrow \pm \infty} s(n) = 0\}$, then $c_0(\mathbb{Z}, \mathbb{R})$ is a closed linear subspace of $\ell^\infty(\mathbb{Z}, \mathbb{R})$.

Remark 2.3. If $p, q \in [1, \infty)$ with $p \leq q$, then $\ell^1(\mathbb{Z}, \mathbb{R}) \subset \ell^p(\mathbb{Z}, \mathbb{R}) \subset \ell^q(\mathbb{Z}, \mathbb{R}) \subset c_0(\mathbb{Z}, \mathbb{R})$.

Example 2.3 (Orlicz sequence spaces). Let $\varphi : \mathbb{R}_+ \rightarrow [0, \infty]$ be a nondecreasing left continuous function which is not identically 0 or $\infty$ on $(0, \infty)$. The *Young function* associated with $\varphi$ is $Y_\varphi(t) = \int_{0}^{t} \varphi(s) \, ds$, for all $t \geq 0$. For each $s \in S(\mathbb{Z}, \mathbb{R})$, let $M_\varphi(s) := \sum_{k=-\infty}^{\infty} Y_\varphi(|s(k)|)$. Then $\ell_\varphi(\mathbb{Z}, \mathbb{R}) := \{s \in S(\mathbb{Z}, \mathbb{R}): \exists c > 0 \text{ such that } M_\varphi(cs) < \infty\}$ is a Banach space with respect to the norm $|s|_\varphi := \inf\{c > 0: M_\varphi(s/c) \leq 1\}$. The space $\ell_\varphi(\mathbb{Z}, \mathbb{R})$ is called the Orlicz sequence space associated with $\varphi$. It is easy to see that $\ell_\varphi(\mathbb{Z}, \mathbb{R}) \in \mathcal{T}(\mathbb{Z})$.

Remark 2.4. The space $\ell^p(\mathbb{Z}, \mathbb{R})$ with $p \in [1, \infty)$ is a trivial example of Orlicz sequence space.

Lemma 2.1. If $B \in \mathcal{T}(\mathbb{Z})$, then $\ell^1(\mathbb{Z}, \mathbb{R}) \subset B \subset \ell^\infty(\mathbb{Z}, \mathbb{R})$.

Proof. Let $s \in \ell^1(\mathbb{Z}, \mathbb{R})$. For every $n \in \mathbb{N}$, let $s_n = s\chi_{[-n, \ldots, n]}$. We have that

$$|s_{n+p} - s_n|_B \leq \left( \sum_{k=-n-p}^{n-1} |s(k)| |\chi_{[k]}|_B + \sum_{k=n+1}^{n+p} |s(k)| |\chi_{[k]}|_B \right)$$

$$= |\chi_{[0]}|_B \left( \sum_{k=-n-p}^{n-1} |s(k)| + \sum_{k=n+1}^{n+p} |s(k)| \right), \quad \forall n \in \mathbb{N}, \forall p \in \mathbb{N}^*.$$
Since \( s \in \ell^1(\mathbb{Z}, \mathbb{R}) \) and \( B \) is a Banach space, we obtain that there is \( \delta \in B \) such that \( s_n \to \delta \) in \( B \). Using Remark 2.1 we deduce that \( \delta = s \), so \( s \in B \).

Let now \( s \in B \). Since \( F_B(1)|s(m)| = |\chi_{[m]}s(m)|_B \leq |s|_B \), for all \( m \in \mathbb{Z} \), we have that \( s \in \ell^\infty(\mathbb{Z}, \mathbb{R}) \). □

Lemma 2.2. Let \( B \in \mathcal{T}(\mathbb{Z}) \), \( n \in \mathbb{Z} \) and \( v > 0 \). Then, the sequences \( e^s_{n,v}, e^\mu_{n,v} : \mathbb{Z} \to \mathbb{R}_+ \) defined by

\[
e^s_{n,v}(j) = \begin{cases} e^{-v(j-n)}, & j \geq n, \\ 0, & j < n, \end{cases} \quad e^\mu_{n,v}(j) = \begin{cases} 0, & j > n, \\ e^{-v(n-j)}, & j \leq n \end{cases}
\]

belong to \( B \).

**Proof.** It is easy to see that \( e^s_{n,v}, e^\mu_{n,v} \in \ell^1(\mathbb{Z}, \mathbb{R}) \). Then, according to Lemma 2.1, \( e^s_{n,v}, e^\mu_{n,v} \in B \). □

Lemma 2.3. Let \( B \in \mathcal{T}(\mathbb{Z}) \) and \( v > 0 \). Then, for every \( s \in B \), the sequences \( u_s, v_s : \mathbb{Z} \to \mathbb{R}_+ \) defined by

\[
u_s(k) = \sum_{j=k+1}^\infty e^{-v(j-k)}s(j), \quad v_s(k) = \sum_{j=-\infty}^k e^{-v(j-k)}s(j)
\]

belong to \( B \).

**Proof.** From

\[|u_s(k)| \leq \sum_{j=-\infty}^k e^{-v(j-k)}|s(j)| = \sum_{i=0}^\infty e^{-vi}|s(k-i)| = \sum_{i=0}^\infty e^{-vi}|s_1(k)|, \quad \forall k \in \mathbb{Z},\]

using the invariance under translations of \( B \), we deduce that \( u_s \in B \) and \( |u_s|_B \leq [1/(1-e^{-v})]|s|_B \). In the same manner we obtain that \( v_s \in B \). □

Lemma 2.4. Let \( B \in \mathcal{T}(\mathbb{Z}) \). Then \( \sup_{n \in \mathbb{N}} F_B(n) < \infty \) if and only if \( c_0(\mathbb{Z}, \mathbb{R}) \subset B \).

**Proof.** Necessity. If \( L := \sup_{n \in \mathbb{N}} F_B(n) < \infty \), then let \( s \in c_0(\mathbb{Z}, \mathbb{R}) \). There is a strictly increasing sequence \( (k_n) \) such that \( |s(j)| / (1/n + 1) \leq |s(k_n)| \) and all \( n \in \mathbb{N} \). For every \( n \in \mathbb{N} \), let \( s_n = \chi_{[-k_n-1,...,0]}s \). Then, we have that \( |s_{n+p} - s_n|_B \leq 2L/(n+1) \), for all \( p \in \mathbb{N} \) and all \( p \in \mathbb{N}^* \). This shows that the sequence \( (s_n) \) is fundamental, so it is convergent. Let \( u \in B \) be such that \( s_n \to u \) in \( B \). Using Remark 2.1, we deduce that \( u = s \), so \( s \in B \). It follows that \( c_0(\mathbb{Z}, \mathbb{R}) \subset B \).

Sufficiency. If \( c_0(\mathbb{Z}, \mathbb{R}) \subset B \) then there is \( \delta > 0 \) such that \( |s|_B \leq \delta \|s\|_\infty \), for all \( s \in c_0(\mathbb{Z}, \mathbb{R}) \). In particular, this implies that \( F_B(n) = |\chi_{[0,...,n-1]}|_B \leq \delta \|\chi_{[0,...,n-1]}\|_\infty = \delta \), for all \( n \in \mathbb{N}^* \), and the proof is complete. □

Notations. In what follows we denote by \( \mathcal{W}(\mathbb{Z}) \) the class of all Banach sequence spaces \( B \in \mathcal{T}(\mathbb{Z}) \) with the property that \( \sup_{n \in \mathbb{N}} F_B(n) = \infty \) and by \( \mathcal{Y}(\mathbb{Z}) \) the class of all Banach sequence spaces \( B \in \mathcal{T}(\mathbb{Z}) \) with \( \ell^1(\mathbb{Z}, \mathbb{R}) \subset B \).

Remark 2.5. From Lemmas 2.4 and 2.1 we have that \( B \in \mathcal{T}(\mathbb{Z}) \setminus \mathcal{W}(\mathbb{Z}) \) if and only if \( c_0(\mathbb{Z}, \mathbb{R}) \subset B \subset \ell^\infty(\mathbb{Z}, \mathbb{R}) \).

Lemma 2.5. Let \( \ell^s_\phi(\mathbb{Z}, \mathbb{R}) \) be an Orlicz space. Then \( \ell_\phi(\mathbb{Z}, \mathbb{R}) \in \mathcal{W}(\mathbb{Z}) \) or \( \ell_\phi(\mathbb{Z}, \mathbb{R}) = \ell^\infty(\mathbb{Z}, \mathbb{R}) \).

**Proof.** If \( \ell^s_\phi(\mathbb{Z}, \mathbb{R}) \notin \mathcal{W}(\mathbb{Z}) \) we have that \( L := \sup_{n \in \mathbb{N}} F_\phi(n) < \infty \). Then \( (n+1)Y_\phi(1/L) = M_\phi(\chi_{[0,...,n]})/L \leq 1 \), for all \( n \in \mathbb{N} \), so \( Y_\phi(1/L) = 0 \). Let \( s \in \ell^\infty(\mathbb{Z}, \mathbb{R}) \) and \( v = s/[L(1 + \|s\|_\infty)] \). Since \( Y_\phi \) is nondecreasing we obtain that \( Y_\phi(|v(k)|) = 0 \), for all \( k \in \mathbb{Z} \). This implies that \( v \in \ell^s_\phi(\mathbb{Z}, \mathbb{R}) \), so \( s \in \ell_\phi(\mathbb{Z}, \mathbb{R}) \). It follows that \( \ell^\infty(\mathbb{Z}, \mathbb{R}) \subset \ell_\phi(\mathbb{Z}, \mathbb{R}) \). Using Lemma 2.1 we deduce the conclusion. □

Notations. Let \( (X, \| \cdot \|) \) be a real or complex Banach space. For every Banach sequence space \( B \in \mathcal{T}(\mathbb{Z}) \) we denote by \( B(\mathbb{Z}, X) \) the space of all sequences \( s : \mathbb{Z} \to X \) with the property that the mapping \( N_s : \mathbb{Z} \to \mathbb{R}_+^* \), \( N_s(m) = \|s(m)\| \) belongs to \( B \). \( B(\mathbb{Z}, X) \) is a Banach space with respect to the norm \( \|s\|_{B(\mathbb{Z}, X)} := |N_s|_B \).
3. Admissibility for difference equations

Let $X$ be a real or complex Banach space and let $I$ be the identity operator on $X$. The norm on $X$ and on $B(X)$—the Banach algebra of all bounded linear operators on $X$, will be denoted by $\| \cdot \|$. If $V$ is a Banach space, $S(V)$ denotes the set of all sequences $s : \mathbb{Z} \to V$.

Let $A \in S(B(X))$. Consider the linear system of difference equations

$$x(n + 1) = A(n)x(n), \quad n \in \mathbb{Z}. \quad \text{(A)}$$

Let $\Delta = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \geq n\}$ and let $\Phi = \{\Phi(m, n)\}_{(m, n) \in \Delta}$ be the evolution operator associated with (A), i.e.

$$\Phi(m, n) := \begin{cases} A(m - 1) \cdots A(n), & m > n, \\ I, & m = n. \end{cases}$$

**Remark 3.1.** The family $\Phi = \{\Phi(m, n)\}_{(m, n) \in \Delta}$ satisfies the evolution property, i.e. $\Phi(m, n)\Phi(n, k) = \Phi(m, k)$ for all $(m, n), (n, k) \in \Delta$.

**Definition 3.1.** The system (A) is said to be uniformly exponentially dichotomic if there are a family of projections $\{P(n)\}_{n \in \mathbb{Z}}$ and two constants $K \geq 1, \nu > 0$ such that the following properties hold:

(i) $A(n)P(n) = P(n + 1)A(n)$, for all $n \in \mathbb{Z}$;
(ii) $\|\Phi(m, n)x\| \leq Ke^{-\nu(m-n)}\|x\|$, for all $x \in \text{Im} P(n)$ and all $(m, n) \in \Delta$;
(iii) $\|\Phi(m, n)y\| \geq \frac{1}{K}e^{\nu(m-n)}\|y\|$, for all $y \in \text{Ker} P(n)$ and all $(m, n) \in \Delta$;
(iv) for every $n \in \mathbb{Z}$, the restriction $A(n) : \text{Ker} P(n) \to \text{Ker} P(n + 1)$ is an isomorphism.

Let $I, O : \mathcal{T}(\mathbb{Z})$. We associate with (A) the input–output system:

$$\gamma(n + 1) = A(n)\gamma(n) + s(n + 1), \quad n \in \mathbb{Z}, \quad \text{(SA)}$$

with $\gamma \in O(\mathbb{Z}, X)$ and $s \in I(\mathbb{Z}, X)$.

**Remark 3.2.** The space $I(\mathbb{Z}, X)$ is called the input space and the space $O(\mathbb{Z}, X)$ is called the output space.

**Definition 3.2.** The pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is said to be admissible for the system (SA) if for every $s \in I(\mathbb{Z}, X)$ there exists a unique $\gamma \in O(\mathbb{Z}, X)$ solution of the system (SA).

**Remark 3.3.** If the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the system (SA), then it makes sense to consider the linear operator $Q : I(\mathbb{Z}, X) \to O(\mathbb{Z}, X)$, $Q(s) = \gamma_s$, where $\gamma_s$ is the unique solution of the system (SA) corresponding to the input $s$. It is easy to verify that $Q$ is closed linear operator. Then $Q$ is bounded, so $\|Q(s)\|_{O(\mathbb{Z}, X)} \leq \|Q\|\|s\|_{I(\mathbb{Z}, X)}$.

For every $(n, x) \in \mathbb{Z} \times X$, we define the sequence:

$$s^x_n : \mathbb{Z} \to X, \quad s^x_n(k) = \begin{cases} \Phi(k, n)x, & k \geq n, \\ 0, & k < n. \end{cases}$$

For every $n \in \mathbb{Z}$, we consider the linear subspaces $X_s(n) = \{x \in X : s^x_n \in O(\mathbb{Z}, X)\}$ and $X_u(n) = \{x \in X : \exists \delta \in O(\mathbb{Z}, X) \text{ with } \delta(n) = x \text{ and } \delta(k) = A(k - 1)\delta(k - 1), \forall k \leq n\}$.

**Proposition 3.1.** For every $n \in \mathbb{Z}$, we have that

$$A(n)X_s(n) \subset X_s(n + 1) \quad \text{and} \quad A(n)X_u(n) = X_u(n + 1).$$

**Proof.** The inclusions $A(n)X_s(n) \subset X_s(n + 1)$ and $X_u(n + 1) \subset A(n)X_u(n)$ are immediate. Let $x \in X_u(n)$ and let $\delta \in O(\mathbb{Z}, X)$ with $\delta(n) = x$ and $\delta(k) = A(k - 1)\delta(k - 1)$, for all $k \leq n$. Denoting by $y = A(n)x$ we have that

$$\varphi : \mathbb{Z} \to X, \quad \varphi(k) = \begin{cases} \chi_{[n+1]}(k)y, & k \geq n + 1, \\ \delta(k), & k \leq n. \end{cases}$$
belongs to $O(\mathbb{Z}, X)$ and $\varphi(k) = A(k - 1)\varphi(k - 1)$, for all $k \leq n + 1$. This proves that $A(n)x \in X_u(n + 1)$, so the proof is complete. \hfill \square

**Proposition 3.2.** If the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the system $(S_A)$, then:

(i) $X_s(n) \cap X_u(n) = \{0\}$, for all $n \in \mathbb{Z}$;
(ii) $X_s(n) + X_u(n) = X$, for all $n \in \mathbb{Z}$.

**Proof.** (i) Let $n \in \mathbb{Z}$ and $x \in X_s(n) \cap X_u(n)$. Then, there is $\delta \in O(\mathbb{Z}, X)$ such that $\delta(n) = x$ and $\delta(k) = A(k - 1)\delta(k - 1)$, for all $k \leq n$. We consider

$$\gamma : \mathbb{Z} \rightarrow X, \quad \gamma(k) = \begin{cases} \Phi(k, n)x, & k \geq n, \\ \delta(k), & k < n. \end{cases}$$

Then $\gamma \in O(\mathbb{Z}, X)$ and it is a solution of the system $(S_A)$ corresponding to the input $s = 0$. It follows that $\gamma = 0$, so $x = 0$.

(ii) Let $n \in \mathbb{Z}$ and $x \in X_s(n) \cap X_u(n)$. We consider the sequences $s : \mathbb{Z} \rightarrow X$, $s(k) = \chi_{[n]}(k)$. Then $s \in I(\mathbb{Z}, X)$, so there exists a unique solution $\gamma \in O(\mathbb{Z}, X)$ corresponding to $s$. We observe that $\gamma(k) = \Phi(k, n)\gamma(n)$, for all $k \geq n$. Since $\gamma \in O(\mathbb{Z}, X)$ it follows that $\gamma(n) \in O(\mathbb{Z}, X)$, so $\gamma(n) \in X_s(n)$. Moreover, if
delimiters

$$\delta : \mathbb{Z} \rightarrow X, \quad \delta(k) = \begin{cases} \chi_{[n]}(k)(x - \gamma(n)), & k \geq n, \\ -\gamma(k), & k < n, \end{cases}$$

delimiters

then, we have that $\delta(k) = A(k - 1)\delta(k - 1)$, for all $k \leq n$. This shows that $x - \gamma(n) = \delta(n) \in X_u(n)$. Thus, $x = \gamma(n) + (x - \gamma(n)) \in X_s(n) + X_u(n)$, which completes the proof. \hfill \square

**Theorem 3.1.** If the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the system $(S_A)$, then there is $L > 0$ such that

(i) $\|\Phi(m, n)x\| \leq L\|x\|$, for all $x \in X_s(n)$ and all $(m, n) \in \Delta$;
(ii) $\|\Phi(m, n)y\| \geq \frac{1}{L}\|y\|$, for all $y \in X_u(n)$ and all $(m, n) \in \Delta$.

**Proof.** We set $L = \max\{(F_1(1)\|Q\|)/F_O(1), 1\}$, where $Q$ is the operator given by Remark 3.3.

(i) Let $n \in \mathbb{Z}$ and let $x \in X_s(n)$. We consider the sequences $s : \mathbb{Z} \rightarrow X$, $s(k) = \chi_{[n]}(k)x$ and

$$\gamma : \mathbb{Z} \rightarrow X, \quad \gamma(k) = \begin{cases} \Phi(k, n)x, & k \geq n, \\ 0, & k < n. \end{cases}$$

Since $x \in X_s(n)$ we have that $\gamma \in O(\mathbb{Z}, X)$. It is easy to see that $\gamma$ is the solution of $(S_A)$ corresponding to $s$, so $\gamma = Q(s)$.

Let $m \geq n$. From $\chi_{[m]}(j)\|\gamma(m)\| \leq \|\gamma(j)\|$, for all $j \in \mathbb{Z}$, we obtain that

$$\|\Phi(m, n)x\| = \|\gamma(m)\| \leq \frac{\|\gamma\|_{O(\mathbb{Z}, X)}}{F_O(1)} \leq \frac{\|Q\|}{F_O(1)} \|s\|_{I(\mathbb{Z}, X)} \leq L\|x\|.$$

(ii) Let $n \in \mathbb{Z}$ and $y \in X_u(n)$. Then there is $\delta \in O(\mathbb{Z}, X)$ with $\delta(n) = y$ and $\delta(k) = A(k - 1)\delta(k - 1)$, for all $k \leq n$. Let $m > n$. We consider the sequences $s : \mathbb{Z} \rightarrow X$, $s(k) = -\chi_{[m]}(k)\Phi(m, n)y$ and

$$\gamma : \mathbb{Z} \rightarrow X, \quad \gamma(k) = \begin{cases} \chi_{[m, \ldots, m - 1]}(k)\Phi(k, n)y, & k \geq n, \\ \delta(k), & k < n. \end{cases}$$

We have that $\gamma \in O(\mathbb{Z}, X)$, $s \in I(\mathbb{Z}, X)$ and an easy computation shows that $\gamma = Q(s)$. This implies that

$$\|\gamma\|_{O(\mathbb{Z}, X)} \leq \frac{\|Q\|}{F_O(1)}\|F_I(1)\|\Phi(m, n)y\| = \|Q\|F_O(1)\Phi(m, n)y\|.$$

In addition, $\chi_{[m]}(j)\|\gamma(n)\| \leq \|\gamma(j)\|$, for all $j \in \mathbb{Z}$, which implies that $F_O(1)\|y\| \leq \|\gamma\|_{O(\mathbb{Z}, X)}$. Then, using (3.1) we deduce that $\|\Phi(m, n)y\| \geq (1/L)\|y\|$ and the proof is complete. \hfill \square
Suppose that

\[ \| \Phi(m, n)x \| \leq Ke^{-\nu(m-n)}\|x\|, \quad \forall x \in X_s(n), \forall (m, n) \in \Delta. \]

**Proof.** Let \( L > 0 \) be given by Theorem 3.1.

**Case 1.** Suppose that \( O \in W(\mathbb{Z}) \). From Lemma 2.1 we have that \( \ell^1(\mathbb{Z}, X) \subset I(\mathbb{Z}, X) \), so there is \( \lambda > 0 \) such that \( \|u\|_{I(\mathbb{Z}, X)} \leq \lambda \|u\|_1 \), for all \( u \in \ell^1(\mathbb{Z}, X) \).

Let \( h \in \mathbb{N}^* \) be such that \( F_O(h) \geq e\lambda L^2\|Q\| \), where \( Q \) is given by Remark 3.3. Let \( n \in \mathbb{Z} \) and let \( x \in X_s(n) \). We distinguish two possible situations:

1. If \( \Phi(n+h, h)x \neq 0 \), then \( \Phi(k, n)x \neq 0 \), for all \( k \in \{n, \ldots, n+h\} \). We consider the sequences:

\[ s : \mathbb{Z} \to X, \quad s(k) = \chi_{[n+1, \ldots, n+h]}(k) \frac{\Phi(k, n)x}{\|\Phi(k, n)x\|}, \]

\[ y : \mathbb{Z} \to X, \quad y(k) = \left\{ \begin{array}{ll} \sum_{j=n}^{k} \chi_{[n+1, \ldots, n+h]}(j) \frac{\Phi(j, n)x}{\|\Phi(j, n)x\|}, & k \geq n, \\ 0, & k < n. \end{array} \right. \]

We have that \( \|s(j)\| = \chi_{[n+1, \ldots, n+h]}(j), \) for all \( j \in \mathbb{Z} \), so \( s \in \ell^1(\mathbb{Z}, X) \). Setting \( a = \sum_{j=n}^{n+h}(1/\|\Phi(j, n)x\|) \) we observe that \( y(k) = a\Phi(k, n)x \), for all \( k \geq n+h \). Since \( x \in X_s(n) \) it follows that \( y \in O(\mathbb{Z}, X) \). An easy computation shows that the pair \((y, s)\) satisfies the system \((S_A)\), so \( y = Q(s) \). This implies that

\[ \|y\|_{O(\mathbb{Z}, X)} \leq \|Q\| \|s\|_{\ell^1(\mathbb{Z}, X)} \leq \lambda \|Q\| \|s\|_{\ell^1(\mathbb{Z}, X)} = \lambda \|Q\| \cdot h. \tag{3.2} \]

In addition \( \|\Phi(n+2h, h)n\| \leq L \|\Phi(k, n)\| = (L/a)\|y(k)\| \), for all \( k \in \{n+1, \ldots, n+2h\} \) which implies that \( \|\Phi(n+2h, n)x\|_{\ell^1(\mathbb{Z})} \leq (L/a)\|y(k)\| \), for all \( k \in \mathbb{Z} \). Then we deduce that

\[ \|\Phi(n+2h, n)x\| F_O(h) \leq (L/a)\|y\|_{O(\mathbb{Z}, X)}. \tag{3.3} \]

From (3.2) and (3.3) it follows that \( \|\Phi(n+2h, n)x\| \leq (h/aeL) \). Observing that \( a \geq (h/L\|x\|) \) we obtain that \( \|\Phi(n+2h, n)x\| \leq (1/e)\|x\| \).

2. If \( \Phi(n+h, h)x = 0 \), then \( \Phi(n+2h, h)x = 0 \). Taking into account that \( h \) does not depend on \( x \) or \( n \), it follows that

\[ \|\Phi(n+2h, n)x\| \leq (1/e)\|x\|, \quad \forall x \in X_s(n), \forall n \in \mathbb{Z}. \tag{3.4} \]

Let \( \nu = 1/(2h) \) and \( K = Le \). Let \( (m, n) \in \Delta \) and \( x \in X_s(n) \). There are \( k \in \mathbb{N} \) and \( j \in \{0, \ldots, 2h-1\} \) such that \( m = n+2k+j \). Using Proposition 3.1 and relation (3.4) we have that

\[ \|\Phi(m, n)x\| \leq L \|\Phi(n+2k, n)x\| \leq Le^{-k}\|x\| \leq Ke^{-\nu(m-n)}\|x\|. \]

**Case 2.** Suppose that \( I \in \mathcal{H}(\mathbb{Z}) \). From Lemma 2.1 we have that \( O(\mathbb{Z}, X) \subset \ell^\infty(\mathbb{Z}, X) \), so there is \( r > 0 \) such that \( \|y\|_{\ell^\infty} \leq r \|y\|_{O(\mathbb{Z}, X)}, \) for all \( y \in O(\mathbb{Z}, X) \).

Let \( \alpha \in I(\mathbb{Z}, X) \). Since \( I \) is invariant under translations, we may suppose that there is \( h \in \mathbb{N}^* \) such that

\[ \sum_{j=1}^{h} \|\alpha(j)\| \geq er L\|Q\|\|\alpha\|_{I(\mathbb{Z}, X)}. \tag{3.5} \]

Let \( n \in \mathbb{Z} \) and \( x \in X_s(n) \). We consider the sequences

\[ s : \mathbb{Z} \to X, \quad s(k) = \chi_{[n+1, \ldots, n+h]}(k) \frac{\alpha(k-n)\|\Phi(k, n)x\|}{\|\Phi(k, n)x\|}, \]

\[ y : \mathbb{Z} \to X, \quad y(k) = \left\{ \begin{array}{ll} (\sum_{j=n}^{k} \chi_{[n+1, \ldots, n+h]}(j)\|\alpha(j-n)\|)\Phi(k, n)x, & k \geq n, \\ 0, & k < n. \end{array} \right. \]
We have that \( s \) has finite support, so \( s \in I(\mathbb{Z}, X) \). Using similar arguments as in Case 1, we deduce that \( \gamma \in O(\mathbb{Z}, X) \). An easy computation shows that \( \gamma = Q(s) \), so
\[
\|\gamma\|_{\infty} \leq r \|\gamma\|_{O(\mathbb{Z}, X)} \leq r \|Q\|_{\|s\|_{I(\mathbb{Z}, X)}}.
\] (3.6)
From \( \|s(k)\| \leq L \|x\|_{\|\alpha(k-n)\|} \), for all \( k \in \mathbb{Z} \), we have that \( \|s\|_{I(\mathbb{Z}, X)} \leq L \|x\|_{\|\alpha\|_{I(\mathbb{Z}, X)}} \). Then, using (3.6) it follows that
\[
\sum_{j=1}^{h} \|\alpha(j)\| \|\Phi(n + h, n)x\| = \|\gamma(n + h)\| \leq \|\gamma\|_{\infty} \leq r L \|Q\|_{\|\alpha\|_{I(\mathbb{Z}, X)}} \|x\|.
\] (3.7)
From (3.5) and (3.7) we have that \( \|\Phi(n + h, n)x\| \leq (1/e)\|x\| \). Since \( h \) does not depend on \( n \) or \( x \), we deduce that
\[
\|\Phi(n + h, n)x\| \leq (1/e)\|x\|, \quad \forall x \in X_s(n), \quad \forall n \in \mathbb{Z}.
\]
Using similar arguments as in Case 1, we obtain the conclusion. \( \square \)

**Corollary 3.1.** If the pair \((O(\mathbb{Z}, X), I(\mathbb{Z}, X))\) is admissible for the system \((S_\lambda)\) and \( O \in \mathcal{W}(\mathbb{Z}) \) or \( I \in \mathcal{H}(\mathbb{Z}) \), then the subspace \( X_s(n) \) is closed, for all \( n \in \mathbb{Z} \).

**Proof.** Let \( K, \nu > 0 \) be given by Theorem 3.2. Let \( n \in \mathbb{Z} \) and \( (x_j) \subset X_s(n) \) with \( x_j \to x \) as \( j \to \infty \). Let \( M > 0 \) be such that \( \|x_j\| \leq M \), for all \( j \in \mathbb{N} \). From Theorem 3.2, we have that
\[
\|\Phi(k, n)x_j\| \leq Ke^{-\nu(k-n)} \|x_j\| \leq MK e^{-\nu(k-n)}, \quad \forall k \geq n, \quad \forall j \in \mathbb{N}.
\]
As \( j \to \infty \) from the above inequality, we obtain that
\[
\|\Phi(k, n)x\| \leq MK e^{-\nu(k-n)}, \quad \forall k \geq n.
\] (3.8)
From (3.8) and Lemma 2.2, it follows that \( x \in X_s(n) \), so \( X_s(n) \) is closed. \( \square \)

**Theorem 3.3.** If the pair \((O(\mathbb{Z}, X), I(\mathbb{Z}, X))\) is admissible for the system \((S_\lambda)\) and \( O \in \mathcal{W}(\mathbb{Z}) \) or \( I \in \mathcal{H}(\mathbb{Z}) \), then there are \( K, \nu > 0 \) such that
\[
\|\Phi(m, n)x\| \geq \frac{1}{K} e^{\nu(m-n)} \|x\|, \quad \forall x \in X_u(n), \quad \forall (m, n) \in \Delta.
\]
**Proof.** Let \( L > 0 \) be given by Theorem 3.1.

**Case 1.** Suppose that \( O \in \mathcal{W}(\mathbb{Z}) \). From Lemma 2.1 we have that there is \( \lambda > 0 \) such that \( \|u\|_{I(\mathbb{Z}, X)} \leq \lambda \|u\|_1 \), for all \( u \in \ell^1(\mathbb{Z}, X) \).

Let \( h \in \mathbb{N}^* \) be such that \( F_O(h) \geq e \lambda L^2 \|Q\| \), where \( Q \) is the operator given by Remark 3.3. Let \( n \in \mathbb{Z} \) and \( x \in X_u(n) \setminus \{0\} \). Then from Theorem 3.1, we have that \( \Phi(k, n)x \neq 0 \), for all \( k \geq n \). Since \( x \in X_u(n) \), there is \( \delta(n) = x \) and \( \delta(k) = A(k-1)\delta(k-1) \), for all \( k \leq n \).

We consider the sequences:
\[
s: \mathbb{Z} \to X, \quad s(k) = -\chi_{[n+h+1,\ldots,n+2h]}(k) \frac{\Phi(k, n)x}{\|\Phi(k, n)x\|},
\]
\[
\gamma: \mathbb{Z} \to X, \quad \gamma(k) = \sum_{j=k+1}^{\infty} \frac{\chi_{[n+h+1,\ldots,n+2h]}(j)}{a\delta(k)} \Phi(k, n)x, \quad k \geq n,
\]
where \( a = \sum_{j=n+h+1}^{n+2h} (1/\|\Phi(j, n)x\|) \). We have that \( s \in I(\mathbb{Z}, X), \gamma \in O(\mathbb{Z}, X) \) and an easy computation shows that \( \gamma = Q(s) \). This implies that
\[
\|\gamma\|_{O(\mathbb{Z}, X)} \leq \|Q\|_{\|s\|_{I(\mathbb{Z}, X)}} \leq \lambda \|Q\|_{\|s\|_1} = \lambda \|Q\|_h.
\] (3.9)
Observing that $\gamma(k) = a\Phi(k,n)x$, for all $k \in \{n + 1, \ldots, n + h\}$, from Theorem 3.1 we deduce that $\|\gamma(k)\| \geq (a/L)\|x\|$, for all $k \in \{n + 1, \ldots, n + h\}$, so

\[ \chi(n+1,\ldots,n+h)(k)(a/L)\|x\| \leq \|\gamma(k)\|, \quad \forall k \in \mathbb{Z}, \]

which implies that

\[ F_\mathcal{O}(h)(a/L)\|x\| \leq \|\gamma\|_{\mathcal{O}(\mathbb{Z},X)}. \tag{3.10} \]

From relations (3.9), (3.10) and taking into account the way how $h$ was chosen we deduce that

\[ e\|x\| \leq (h/aL). \tag{3.11} \]

Moreover, from $\|\Phi(n + 2h,n)x\| \geq (1/L)\|\Phi(k,n)x\|$, for all $k \in \{n + h + 1, \ldots, n + 2h\}$, we have that $\|\Phi(n + 2h,n)x\| \geq (h/aL)$. Then, from (3.11) we obtain that $\|\Phi(n + 2h,n)x\| \geq e\|x\|$. Taking into account that $h$ does not depend on $n$ or $x$, it follows that

\[ \|\Phi(n + 2h,n)x\| \geq e\|x\|, \quad \forall x \in X_\mu(n), \quad \forall (m,n) \in \Delta. \]

Let $v = 1/(2h)$ and let $K = eL$. Let $(m,n) \in \Delta$ and $x \in X_\mu(n)$. Then there are $k \in \mathbb{N}$ and $j \in \{0, \ldots, 2h - 1\}$ such that $m = n + 2kh + j$. Then, we have that

\[ \|\Phi(m,n)x\| \geq \frac{1}{L}\|\Phi(n + 2kh,n)x\| \geq \frac{1}{L}e^j\|x\| \geq \frac{1}{K}e^j\|\Phi(n - n)\|. \]

**Case 2.** Suppose that $I \in \mathcal{H}(\mathbb{Z})$. From Lemma 2.1 we have that $O(\mathbb{Z},X) \subset \ell^\infty(\mathbb{Z},X)$, so there is $r > 0$ such that $\|\gamma\|_\infty \leq r\|\gamma\|_{O(\mathbb{Z},X)}$, for all $\gamma \in O(\mathbb{Z},X)$.

Let $\beta \in I(\mathbb{Z},X) \setminus \ell^1(\mathbb{Z},X)$. Since $I(\mathbb{Z},X)$ is invariant under translations, we may assume that there is $h \in \mathbb{N}^*$ such that

\[ b := \sum_{j=1}^h \|\beta(j)\| \geq erL\|\beta\|_{I(\mathbb{Z},X)}. \tag{3.12} \]

Let $n \in \mathbb{Z}$ and $x \in X_\mu(n)$. Since $x \in X_\mu(n)$ there is $\delta \in O(\mathbb{Z},X)$ such that $\delta(n) = x$ and $\delta(k) = A(k - 1)\delta(k - 1)$, for all $k \leq n$.

We consider the sequences:

\[ s : \mathbb{Z} \to X, \quad s(k) = -\chi_{[n+1,\ldots,n+h]}(k)\|\beta(k - n)\|\Phi(k,n)x, \]

\[ \gamma : \mathbb{Z} \to X, \quad \gamma(k) = \left\{ \begin{array}{ll} \sum_{j=k+1}^{n+h} \chi_{[n + 1, \ldots, n + h]}(j)\|\beta(j - n)\|\Phi(k,n)x, & k \geq n, \\ 0, & k < n. \end{array} \right. \]

Then, we have that $s \in I(\mathbb{Z},X)$, $\gamma \in O(\mathbb{Z},X)$ and the pair $(\gamma, s)$ satisfies the system $(S_A)$, so $\gamma = Q(s)$. Since $\|\Phi(n + h,n)x\| \geq (1/L)\|\Phi(k,n)x\|$, for all $k \in \{n + 1, \ldots, n + h\}$, we deduce that

\[ |s(k)| = \chi_{[n+1,\ldots,n+h]}(k)\|\beta(k - n)\|\Phi(k,n)x \leq L\|\beta(k - n)\|\Phi(n + h,n)x, \quad \forall k \in \mathbb{Z}. \]

This implies that $\|s\|_{I(\mathbb{Z},X)} \leq L\|\beta\|_{I(\mathbb{Z},X)}\|\Phi(n + h,n)x\|$, so we obtain that

\[ b\|x\| = \|\gamma(n)\| \leq r\|\gamma\|_{O(\mathbb{Z},X)} \leq r\|Q\|\|s\|_{I(\mathbb{Z},X)} \leq rL\|Q\|\|\beta\|_{I(\mathbb{Z},X)}\|\Phi(n + h,n)x\|. \tag{3.13} \]

From (3.12) and (3.13) it follows that $\|\Phi(n + h,n)x\| \geq e\|x\|$. Taking into account that $h$ does not depend on $n$ or $x$, it follows that $\|\Phi(n + h,n)x\| \geq e\|x\|$, for all $x \in X_\mu(n)$ and all $(m,n) \in \Delta$. Using similar arguments as in Case 1 we obtain the conclusion. \(\Box\)

**Corollary 3.2.** If the pair $(O(\mathbb{Z},X), I(\mathbb{Z},X))$ is admissible for the system $(S_A)$ and $O \in \mathcal{W}(\mathbb{Z})$ or $I \in \mathcal{H}(\mathbb{Z})$, then the subspace $X_\mu(n)$ is closed, for all $n \in \mathbb{Z}$. 
**Theorem 3.4.** If the pair $(O(\mathbb{Z}), I(\mathbb{Z}, X))$ is admissible for the system $(S_A)$ and $O \in \mathcal{W}(\mathbb{Z})$ or $I \in \mathcal{H}(\mathbb{Z})$, then the system $(A)$ is uniformly exponentially dichotomic.

**Proof.** From Proposition 3.2, Corollaries 3.1 and 3.2, we deduce that $X_s(n) \oplus X_u(n) = X$, for all $n \in \mathbb{Z}$. For every $n \in \mathbb{Z}$, let $P(n) = \text{Im} P(n) = X_s(n)$ and Ker $P(n) = X_u(n)$. Then, it is easy to verify that $A(n)P(n) = P(n + 1)A(n)$, for all $n \in \mathbb{Z}$. In addition, from Proposition 3.1 and Theorem 3.3, we have that for every $n \in \mathbb{Z}$, the restriction $A(n)|_{\text{Ker} P(n)} : \text{Ker} P(n) \rightarrow \text{Ker} P(n + 1)$ is an isomorphism. Finally, from Theorems 3.2 and 3.3, we obtain the conclusion. □

**Definition 3.3.** The system $(A)$ is said to be uniformly bounded if $\sup_{n \in \mathbb{Z}} \|A(n)\| < \infty$.

**Lemma 3.1.** If the system $(A)$ is uniformly bounded and it is uniformly exponentially dichotomic with respect to the family of projections $\{P(n)\}_{n \in \mathbb{Z}}$, then $\sup_{n \in \mathbb{Z}} \|P(n)\| < \infty$.

**Proof.** See [15, Proposition 2.1]. □

The second main result of this section is:

**Theorem 3.5.** Let $I, O \in T(\mathbb{Z})$ with $O \in \mathcal{W}(\mathbb{Z})$ or $I \in \mathcal{H}(\mathbb{Z})$. The following assertions hold:

(i) if the pair $(O(\mathbb{Z}), I(\mathbb{Z}, X))$ is admissible for the system $(S_A)$, then the system $(A)$ is uniformly exponentially dichotomic;
(ii) if $(A)$ is uniformly bounded and $I \subset O$, then $(A)$ is uniformly exponentially dichotomic if and only if the pair $(O(\mathbb{Z}), I(\mathbb{Z}, X))$ is admissible for the system $(S_A)$.

**Proof.** (i) This follows from Theorem 3.4.

(ii) Necessity. Suppose that the system $(A)$ is uniformly exponentially dichotomic with respect to the family of projections $\{P(n)\}_{n \in \mathbb{Z}}$. From Lemma 3.1 it follows that $L := \sup_{n \in \mathbb{Z}} \|P(n)\| < \infty$. 
Let $s \in I(\mathbb{Z}, X)$. We consider the sequence
\[
\gamma : \mathbb{Z} \to X, \quad \gamma(n) = \sum_{k=-\infty}^{n} \Phi(n,k)P(k)s(k) - \sum_{k=n+1}^{\infty} \Phi(k,n)^{-1}(I - P(k))s(k),
\]
where for every $(k,n) \in \Delta$, $\Phi(k,n)^{-1}$ denotes the inverse of the operator $\Phi(k,n) : \text{Ker} P(n) \to \text{Ker} P(k)$. If $K, \nu > 0$ are given by Definition 3.1, then
\[
\|\gamma(n)\| \leq LK \sum_{k=-\infty}^{n} e^{-\nu(n-k)}\|s(k)\| + K(L+1) \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}\|s(k)\|, \quad \forall n \in \mathbb{Z}.
\]
Now, according to Lemma 2.3, we deduce that $\gamma \in O(\mathbb{Z}, X)$. An easy computation shows that $\gamma$ is solution of the system $(SA)$ corresponding to the input $s$.

Let $\tilde{\gamma} \in O(\mathbb{Z}, X)$ be a solution of $(SA)$ corresponding to the input $s$. Setting $\delta = \tilde{\gamma} - \gamma$, we have that $\delta \in O(\mathbb{Z}, X)$ and $\delta(m) = \Phi(m,n)\delta(n)$, for all $(m,n) \in \Delta$. Let $\delta_1(n) = P(n)\delta(n)$ and $\delta_2(n) = (I - P(n))\delta(n)$, for all $n \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$. Then we have that
\[
\|\delta_1(k)\| = \|\Phi(k,j)P(j)\delta(j)\| \leq Ke^{-\nu(k-j)}\|P(j)\delta(j)\| \leq KLe^{-\nu(k-j)}\|\delta\|_\infty,
\]
for all $j \leq k$. For $j \to -\infty$ we deduce that $\delta_1(k) = 0$. Moreover,
\[
\|\delta_2(k)\| \leq Ke^{-\nu(j-k)}\|\delta_2(j)\| \leq K(1 + L)e^{-\nu(j-k)}\|\delta\|_\infty, \quad \forall j \geq k.
\]
Hence, as $j \to \infty$ it follows that $\delta_2(k) = 0$. Since $k \in \mathbb{Z}$ was arbitrary, we obtain that $\delta(k) = 0$, for all $k \in \mathbb{Z}$. This shows that $\tilde{\gamma} = \gamma$, so $\gamma$ is uniquely determined. In conclusion, the pair $(O(\mathbb{Z}, X), I(\mathbb{Z}, X))$ is admissible for the system $(SA)$.

Sufficiency. This follows from (i). $\square$

The natural question arises whether the result given by Theorem 3.5 is the most general in this topic and whether the hypotheses on the structure of the underlying sequence spaces are indeed necessary. The answers are positive and will be illustrated by the following example.

**Example 3.1.** Let $H$ be a separable Hilbert space and let $\{b_k\}_{k \in \mathbb{Z}}$ be an orthonormal basis on $H$. For every $h \in H$ and every $k \in \mathbb{Z}$ we set $h_k = (h, b_k)$. The norm on $H$ is given by $\|h\|_H = (\sum_{k \in \mathbb{Z}} h_k^2)^{1/2}$, for each $h = \sum_{k \in \mathbb{Z}} h_k b_k$. For every $n \in \mathbb{Z}$, we define the operator
\[T(n) : H \to H, \quad T(n)(h) = \sum_{k=n}^{\infty} h_kb_k.\]
Then $\|T(n)\| = 1$, for every $n \in \mathbb{Z}$ and $T(m)T(n) = T(m)$, for every $m > n$. We consider
\[a_n = \begin{cases} 1 + e^{-n}, & n \in \mathbb{Z} \setminus \mathbb{N}, \\ 2, & n \in \mathbb{N}. \end{cases}\]
Then $(a_n)_{n \in \mathbb{Z}}$ is a decreasing sequence with $\lim_{n \to -\infty} a_n = \infty$.

Let $X = H \times H$ with the norm $\|x, y\|_X = \|x\|_H + \|y\|_H$. For every $n \in \mathbb{N}$, we define the operator
\[A(n) : X \to X, \quad A(n)(x, y) = (\frac{a_{n+1}}{a_n}T(n)x, ey).\]
We consider the linear system of difference equations
\[x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}, \tag{A}\]
and the associated input–output system
\[\gamma(n+1) = A(n)\gamma(n) + s(n+1), \quad \forall n \in \mathbb{Z}. \tag{SA}\]
Denoting by
\[F(m,n)x = \begin{cases} \frac{a_m}{a_n}T(m-1)x, & m > n, \\ x, & m = n, \end{cases}\]
we have that the evolution operator $\Phi = \{\Phi(m, n)\}_{(m, n) \in \Delta}$ associated with (A) is

$$\Phi(m, n) : X \to X, \quad \Phi(m, n)(x, y) = (F(m, n)x, e^{m-n}y).$$

Let $I, O \in \mathcal{T}(\mathbb{Z})$ be such that $O \notin \mathcal{W}(\mathbb{Z})$ and $I \notin \mathcal{W}(\mathbb{Z})$. Then, from Lemma 2.1, $I = \ell^1(\mathbb{Z}, \mathbb{R})$ and according to Remark 2.5 we have that $c_0(\mathbb{Z}, \mathbb{R}) \subset O \subset \ell^\infty(\mathbb{Z}, \mathbb{R})$.

**Step 1.** We prove that the pair $(O(Z, X), \ell^1(Z, X))$ is admissible for the system $(S_A)$. Indeed, let $s \in \ell^1(Z, X)$. Then $s = (u, v)$ with $u, v \in \ell^1(Z, H)$. We define the sequences

$$\varphi : Z \to H, \quad \varphi(n) = \sum_{k=-\infty}^{n} F(n, k)u(k),$$

$$\delta : Z \to H, \quad \delta(n) = -\sum_{k=n+1}^{\infty} e^{-(k-n)}v(k).$$

Since $u, v \in \ell^1(Z, H)$ we have that $\varphi$ and $\delta$ are correctly defined. It is easy to see that $\delta \in \ell^1(Z, H)$. In particular $\delta \in c_0(Z, H)$, so $\delta \in O(Z, H)$.

Let

$$w_n = \sum_{k=-\infty}^{n-1} \frac{a_n}{a_k} u(k), \quad n \in \mathbb{N}.$$  

Since $u \in \ell^1(Z, H)$ we have that $(w_n)$ is a convergent sequence in $H$. We set $w = \lim_{n \to \infty} w_n$. We observe that $\varphi(n) = u(n) + T(n-1)w_n$, for every $n \in Z$. Then, for each $n \in \mathbb{N}$, we have that

$$\|\varphi(n)\| \leq \|u(n)\| + \|T(n-1)\| \|w_n - w\| + \|T(n-1)w\| = \|u(n)\| + \|w_n - w\| + \|T(n-1)w\|.$$  

This shows that $\varphi(n) \to 0$ as $n \to \infty$. Moreover, taking into account that

$$\|\varphi(n)\| \leq \sum_{k=-\infty}^{n} \|u(k)\|, \quad \forall n \in Z,$$

and $u \in \ell^1(Z, H)$, we deduce that $\varphi(n) \to 0$ as $n \to -\infty$. It follows that $\varphi \in c_0(Z, H)$, so $\varphi \in O(Z, H)$. This shows that $\gamma = (\varphi, \delta) \in O(Z, X)$. It is easy to verify that the pair $(\gamma, s)$ satisfies Eq. $(S_A)$.

To prove the uniqueness of $\gamma$, let $\gamma' = (\varphi', \delta') \in O(Z, X)$ be a solution of $(S_A)$ corresponding to the input $s$. Setting $\alpha = \gamma - \varphi$ and $\beta = \delta - \delta$, we have that $\alpha(m) = F(m, n)\alpha(n)$, for all $(m, n) \in \Delta$ and $\beta(m) = e^{m-n}\beta(n)$, for all $(m, n) \in \Delta$.

Let $m \in Z$. From

$$\|\alpha(m)\| \leq \frac{a_m}{a_n} \|\alpha(n)\| \leq \frac{a_m}{a_n} \|\alpha\|_{\infty}, \quad \forall n \leq m,$$

since $a_n \to \infty$ as $n \to -\infty$, we deduce that $\alpha(m) = 0$. From

$$\|\beta(m)\| = e^{-(k-m)}\|\beta(k)\| \leq e^{-(k-m)}\|\beta\|_{\infty}, \quad \forall k \geq m,$$

it immediately follows that $\beta(m) = 0$. Since $m \in Z$ was arbitrary, we obtain that $\gamma$ is uniquely determined, so the pair $(O(Z, X), \ell^1(Z, X))$ is admissible for the system $(S_A)$.

**Step 2.** We prove that the system (A) is not uniformly exponentially dichotomic. Indeed, suppose by contrary that the system $(A)$ is uniformly exponentially dichotomic. Let $\{P(n)\}_{n \in \mathbb{Z}}$ be the family of projections and let $K, v > 0$ be two constants given by Definition 3.1. According to Proposition 2.2 in [15], the family of projections is uniquely determined and $\text{Im} P(n) = \{x \in X : \sup_{m \geq n} \|\Phi(m, n)x\| < \infty\}$, for every $n \in \mathbb{Z}$. This implies that $\text{Im} P(n) = H \times \{0\}$, for every $n \in \mathbb{Z}$. Then, from

$$\|\Phi(m, n)x\|_X \leq Ke^{-v(m-n)}\|x\|_X, \quad \forall x \in \text{Im} P(n), \forall (m, n) \in \Delta,$$
we deduce that
\[ \| F(m, n)h \|_H \leq K e^{-\nu(m-n)}\| h \|_H, \quad \forall h \in H, \forall (m, n) \in \Delta. \]
In particular, for \( h = b_m \) it follows that \( a_m/a_n \leq K e^{-\nu(m-n)} \), for all \( m > n \), which is absurd.
In conclusion, the pair \((O(Z, X), \ell^1(Z, X))\) is admissible for the system \((S_A)\), but for all that the system \((A)\) is not uniformly exponentially dichotomic.

**Remark 3.4.** The above example shows that in Theorem 3.5 the assumption \( I, O \in T(\mathbb{Z}) \) with \( O \in \mathcal{W}(\mathbb{Z}) \) or \( I \in \mathcal{H}(\mathbb{Z}) \) is essential.

In what follows we will give several consequences of Theorem 3.5.

**Corollary 3.3.** Let \( O \in \{\ell^\infty(Z, \mathbb{R}), c_0(Z, \mathbb{R})\} \) and let \( B \in \mathcal{H}(\mathbb{Z}) \). The following assertions hold:

(i) if the pair \((O(Z, X), B(Z, X))\) is admissible for the system \((S_A)\), then the system \((A)\) is uniformly exponentially dichotomic;
(ii) if \((A)\) is uniformly bounded and \( B \subset O \), then \((A)\) is uniformly exponentially dichotomic if and only if the pair \((O(Z, X), B(Z, X))\) is admissible for the system \((S_A)\).

**Remark 3.5.** For the cases when \((O, B)\) is one of the pairs \((\ell^\infty(Z, \mathbb{R}), c_0(Z, \mathbb{R}))\), \((\ell^\infty(Z, \mathbb{R}), \ell^\infty(Z, \mathbb{R}))\), \((c_0(Z, \mathbb{R}), c_0(Z, \mathbb{R}))\) a different proof of Corollary 3.3 was given in [15] (see Theorems 2.3 and 2.5 in [15]).

**Corollary 3.4.** Let \( \ell_\psi(Z, \mathbb{R}), \ell_\psi(Z, \mathbb{R}) \) be two Orlicz sequence spaces such that \((\ell_\psi(Z, \mathbb{R}), \ell_\psi(Z, \mathbb{R})) \neq (\ell^\infty(Z, \mathbb{R}), \ell^1(Z, \mathbb{R}))\). The following assertions hold:

(i) if the pair \((\ell_\psi(Z, \mathbb{R}), \ell_\psi(Z, \mathbb{R}))\) is admissible for the system \((S_A)\), then the system \((A)\) is uniformly exponentially dichotomic;
(ii) if \((A)\) is uniformly bounded and \( \ell_\psi(Z, \mathbb{R}) \subset \ell_\psi(Z, \mathbb{R}) \), then \((A)\) is uniformly exponentially dichotomic if and only if the pair \((\ell_\psi(Z, \mathbb{R}), \ell_\psi(Z, \mathbb{R}))\) is admissible for the system \((S_A)\).

**Remark 3.6.** For the cases \( \ell_\psi(Z, \mathbb{R}) = \ell^p(Z, \mathbb{R}), \ell_\psi(Z, \mathbb{R}) = \ell^q(Z, \mathbb{R}) \) with \( p, q \in [1, \infty) \) a different proof of Corollary 3.4 was given in [16] (see Theorem 2.3 in [16]).

**Corollary 3.5.** Let \( W \in \mathcal{T}(\mathbb{Z}) \). The following assertions are equivalent:

(i) if the pair \((W(Z, X), W(Z, X))\) is admissible for the system \((S_A)\), then the system \((A)\) is uniformly exponentially dichotomic;
(ii) if the system \((A)\) is uniformly bounded, then the system \((A)\) is uniformly exponentially dichotomic if and only if the pair \((W(Z, X), W(Z, X))\) is admissible for the system \((S_A)\).

**Proof.** From Lemma 2.1 we have that either \( W \in \mathcal{H}(\mathbb{Z}) \) or \( W = \ell^1(Z, \mathbb{R}) \) (and in this case \( W \in \mathcal{W}(\mathbb{Z}) \)). By applying Theorem 3.5 we obtain the conclusion. \( \square \)

### 4. Dichotomy radius of nonautonomous difference equations

Consider the system of difference equations
\[ x(n+1) = A(n)x(n), \quad n \in \mathbb{Z}, \quad (A) \]
with \( A \in \ell^\infty(Z, \mathcal{B}(X)) \). In what follows we suppose that the system \((A)\) is uniformly exponentially dichotomic.

For every \( D \in \ell^\infty(Z, \mathcal{B}(X)) \) we consider the perturbed system
\[ y(n+1) = (A(n) + D(n))y(n), \quad n \in \mathbb{Z}. \]
The main question is whether the system \((A + D)\) remains uniformly exponentially dichotomic and if so, which are the conditions that the perturbation structure should verify. In order to answer this question it makes sense to introduce the following concept.

**Definition 4.1.** The dichotomy radius of the system \((A)\) is

\[
r_{\text{dich}}(A) := \sup \{ r > 0 : \forall D \in \ell^\infty(\mathbb{Z}, \mathcal{B}(X)) \text{ with } \|D\|_\infty < r \Rightarrow (A + D) \text{ is uniformly exponentially dichotomic} \}.
\]

We associate with the system \((A)\) the input–output system

\[
\gamma(n + 1) = A(n)\gamma(n) + s(n + 1), \quad n \in \mathbb{Z}.
\]

\[\text{(S_A)}\]

Let \(W \in T(\mathbb{Z})\). Since \((A)\) is uniformly exponentially dichotomic from Corollary 3.5(ii) it follows that the pair \((W(\mathbb{Z}), X)\) is admissible for the system \((S_A)\). Then it makes sense to consider the input–output operator

\[
Q_W : W(\mathbb{Z}, X) \to W(\mathbb{Z}, X), \quad Q_W(s) = \gamma_s
\]

where \(\gamma_s\) is the unique solution of \((S_A)\) corresponding to the input \(s\).

**Lemma 4.1.** The operator \(Q_W\) is bounded and invertible.

**Proof.** It is easy to verify that \(Q_W\) is a closed linear operator, so it is bounded. If \(s \in \text{Ker} Q_W\), then \(\gamma_s = 0\). This implies that \(s = 0\), so \(Q_W\) is injective.

Let \(\gamma \in W(\mathbb{Z}, X)\). We consider the sequence \(s : \mathbb{Z} \to X, s(n) = \gamma(n) - A(n - 1)\gamma(n - 1)\). From

\[
\|s(n)\| \leq \|\gamma(n)\| + \|A\|_\infty \|\gamma(n - 1)\|, \quad \forall n \in \mathbb{Z},
\]

we obtain that \(s \in W(\mathbb{Z}, X)\) and \(\|s\|_{W(\mathbb{Z}, X)} \leq (1 + \|A\|_\infty)\|\gamma\|_{W(\mathbb{Z}, X)}\). Since the pair \((\gamma, s)\) satisfies the system \((S_A)\) we obtain that \(\gamma = Q_W(s)\), so \(Q_W\) is surjective. It follows that \(Q_W\) is invertible and

\[
Q_W^{-1} : W(\mathbb{Z}, X) \to W(\mathbb{Z}, X), \quad (Q_W^{-1}\gamma)(n) = \gamma(n) - A(n - 1)\gamma(n - 1).
\]

The main result of this section is:

**Theorem 4.1.** The dichotomy radius of the system \((A)\) satisfies the following property

\[
r_{\text{dich}}(A) \geq \left(1/\|Q_W\|\right).
\]

**Proof.** Let \(D \in \ell^\infty(\mathbb{Z}, \mathcal{B}(X))\) with \(\|D\|_\infty < 1/\|Q_W\|\). We consider the input–output system

\[
\varphi(n + 1) = \left(A(n) + D(n)\right)\varphi(n) + s(n + 1), \quad \forall n \in \mathbb{Z}.
\]

\[\text{(S_{A+D})}\]

We define the operator

\[
H_W : W(\mathbb{Z}, X) \to W(\mathbb{Z}, X), \quad (H_W\gamma)(n) = \gamma(n) - \left(A(n - 1) + D(n - 1)\right)\gamma(n - 1).
\]

We have that \(H_W\) is correctly defined and is a bounded linear operator. From

\[
\|\left(Q_W^{-1} - H_W\right)(\gamma)(n)\| \leq \|D\|_\infty \|\gamma(n - 1)\|, \quad \forall n \in \mathbb{Z},
\]

we obtain that \(\|Q_W^{-1} - H_W\| \leq \|D\|_\infty < 1/\|Q_W\|\). This implies that \(H_W\) is invertible.

Let \(s \in W(\mathbb{Z}, X)\). Then \(\gamma_s = H_W^{-1}(s)\) is the unique solution of the system \((S_{A+D})\) corresponding to the input \(s\). This implies that the pair \((W(\mathbb{Z}), X)\) is admissible for the system \((S_{A+D})\). From Corollary 3.5 it follows that the system \((A + D)\) is uniformly exponentially dichotomic and the proof is complete. \(\square\)

**Theorem 4.2.** A lower bound for the dichotomy radius of the system \((A)\) is

\[
r_{\text{dich}}(A) \geq \sup_{W \in T(\mathbb{Z})} \frac{1}{\|Q_W\|}.
\]
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References