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Viscous boundary value problems for symmetric systems with variable multiplicities

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Abstract

Extending investigations of Métivier and Zumbrun in the hyperbolic case, we treat stability of viscous shock and boundary layers for viscous perturbations of multidimensional hyperbolic systems with characteristics of variable multiplicity, specifically the construction of symmetrizers in the low-frequency regime where variable multiplicity plays a role. At the same time, we extend the boundary-layer theory to "real" or partially parabolic viscosities, Neumann or mixed-type parabolic boundary conditions, and systems with nonconservative form, in addition proving a more fundamental version of the Zumbrun–Serre–Rousset theorem, valid for variable multiplicities, characterizing the limiting hyperbolic system and boundary conditions as a nonsingular limit of a reduced viscous system. The new effects of viscosity are seen to be surprisingly subtle; in particular, viscous coupling of crossing hyperbolic modes may induce a destabilizing effect. We illustrate the theory with applications to magnetohydrodynamics.

Keywords: Boundary layers; Quasilinear systems; Hyperbolic–parabolic equations; Small viscosity limit; Spectral stability; Evans functions; Linear and nonlinear stability

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1. Introduction

This work is motivated by the stability analysis of boundary value problems and shock waves for viscous perturbations of multidimensional systems of conservation laws. In this analysis, three main steps are present. In the first step, one constructs simple waves $w((\nu \cdot x - \sigma t)/\varepsilon)$, or "profiles," which are exact solutions of the viscous equation, with viscosity of order ε. This amounts to solving an ordinary differential equation (the profile equation); the solutions describe the fast transition between the hyperbolic solution and the parabolic boundary conditions (boundary layers) or between two smooth hyperbolic solutions (shock layers). Next, given a profile, formal plane wave or spectral analysis yields necessary stability conditions in terms of Evans functions. The second step is to compute explicitly this function on specific examples and check the stability conditions. The third step is to prove the linear and nonlinear stability of solutions, assuming that the suitable Evans–Lopatinski condition is satisfied, in particular for curved fronts or boundaries and nonpiecewise constant hyperbolic solutions. This paper deals with the third step, with specific applications to magneto-hydrodynamics. The first and second steps are discussed for shock and boundary layers, respectively, in companion papers [9] and [8].

We concentrate on the construction of symmetrizers for the linearized equations, and more specifically in the so-called low-frequency regime, as they are the key point in the proof of stability estimates which eventually yield short-time existence and nonlinear stability theorems; see [14,15] for hyperbolic shocks and [5–7,20] for viscous perturbations. In these papers, it is proved that strong stability estimates hold, under the natural uniform Lopatinski condition, or Evans' condition, provided that the equations satisfy a structural condition, called *the block structure condition* (see [14,16] in the hyperbolic case and [20] for the viscous case). This condition is in some sense necessary for the construction of Kreiss' symmetrizers which are used to prove the stability estimates. It is satisfied in the case of inviscid Euler's equations of gas dynamics [14], but does not hold in other interesting examples such as the equations of magneto-hydrodynamics (MHD). So there is a real need for an extension of the analysis beyond the class of systems satisfying the block structure condition. This is done in [21] for hyperbolic systems and the main goal of this paper is to extend the analysis to viscous systems, in view of applications to MHD.

We carry out in passing several other useful generalizations of the basic boundary-layer analysis of [19,20], extending the theory to "real" or partially parabolic viscosities, Neumann or mixed-type parabolic boundary conditions, and systems with nonconservative form. In addition, we prove a more fundamental version of the Zumbrun–Serre–Rousset theorem, valid for variable multiplicities, characterizing the limiting hyperbolic system and boundary conditions as a nonsingular limit of a reduced viscous system as frequency goes to zero. Extensions to the shock case are given in [9].

Consider boundary value problems for hyperbolic systems

$$\partial_t u + \sum_{j=1}^{d} A_j \partial_j u \tag{1.1}$$

on $\{x_d \geqslant 0\}$ with boundary conditions on $\{x_d = 0\}$ which is assumed to be noncharacteristic. The plane wave analysis of such system leads to consider ordinary differential system in $z \geqslant 0$, depending on Fourier–Laplace frequencies $\zeta = (\tau, \eta, \gamma)$, with $\gamma > 0$,

$$\partial_z - H_0(\zeta), \quad H_0(\zeta) := -A_d^{-1}\left((i\tau + \gamma) + \sum_{j=1}^{d-1} i\eta_j A_j\right). \tag{1.2}$$

Viscous perturbations of (1.1) are systems of the form

$$\partial_t u + \sum_{j=1}^{d} A_j \partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j (B_{j,k} \partial_k u) \tag{1.3}$$

with natural structural conditions which are recalled below. The *low-frequency plane wave analysis* of such systems lead to consider perturbations of (1.2):

$$\partial_z - H(\zeta, \rho), \quad H(\zeta, \rho) := H_0(\zeta) + \rho H_1(\zeta, \rho) \tag{1.4}$$

depending smoothly on an additional parameter $\rho \geqslant 0$ (see Section 2 below or [19,20,26]).

In this paper, our main concern is the construction of symmetrizers $\Sigma(\zeta, \rho)$ for (1.4). The precise conditions we impose on Σ are given in Section 3. In particular, we focus on *smooth symmetrizers*, as they serve as symbols for pseudodifferential symmetrizers in the variable coefficient analysis.

When $\rho = 0$, $\Sigma_0(\zeta) = \Sigma(\zeta, 0)$ is a symmetrizer for $H_0(\zeta)$. Such symmetrizers were constructed first for strictly hyperbolic systems (1.1) by Kreiss [12] (see also [1]). Strict hyperbolicity is used at only one place: it implies that H_0 can be put in a normal form, which is called block structure in [14,16]. Therefore Kreiss' construction of symmetrizers extends immediately to systems which satisfy this *block structure condition*. In [21], it is proved that this condition is satisfied if and only if the symbol $A(\xi) := \sum \xi_j A_j$ of (1.1) is smoothly diagonalizable for $\xi \neq 0$, recovering known examples such as Euler's equation of gas dynamics or Maxwell's equations. The second important result in [21] it that the construction of symmetrizers is extended to a class of symmetric systems which are not smoothly diagonalizable for all $\xi \neq 0$: we demand that the "bad" multiple modes are *totally incoming or totally outgoing* (see Definitions 4.1 and 4.3 below), and this applies to inviscid MHD.

In the small viscosity case, the construction of symmetrizers is performed in [20], with application to the analysis of shocks in [4–7], assuming that the eigenvalues of $A(\xi)$ have constant multiplicity for $\xi \neq 0$. As mentioned above, this assumption rules out the case of MHD.

The main objective of this paper is to start the analysis of (1.3) or (1.4) when the constant multiplicity assumption is relaxed and in particular to investigate the construction of symmetrizers. It turns out that the influence of the viscosity is much more subtle than expected near multiple modes. In some cases, it may induce destabilizing effects. Let us list several new phenomena which can occur when there are multiple modes with nonconstant multiplicity.

• Smooth diagonalization of $A_0(\xi)$ implies a smooth block reduction for $H_0(\zeta)$. The perturbation ρH_1 in general couples the different blocks associated to a multiple eigenvalues (and this occurs for MHD). If the crossing eigenvalues do not have the same behavior with respect to the boundary (typically if they are not all incoming nor all outgoing), the spectral negative space $\mathbb{E}^-(\zeta, \rho)$ *is not continuous* (in general) at $\rho = 0$. This happens for slow shock waves in MHD. This phenomena is excluded when the eigenvalues have constant multiplicities; see [22] (in this case, since crossing eigenvalues are equal, they have the same behavior with respect to the boundary). Recall from [21] that the continuity of $\mathbb{E}^-$ is a *necessary condition* for Kreiss' construction of *smooth* symmetrizers, more precisely for the existence of what is called below, smooth K-families of symmetrizers. In any case, *the discontinuity of* $\mathbb{E}^-$ *is a major difficulty in the construction of symmetrizers*.

• As a consequence of the previous phenomenon, the *Evans function can be discontinuous* at $\rho = 0$. In the shock problem, the usual Evans function is in every case singular at $\rho = 0$ (see [28]), but the remark applies to the modified (or desingularized) Evans function introduced in [6,7] (see below).

• Because of the lack of continuity of the Evans function, it may happen that *the strong Lopatinski stability condition for the hyperbolic problem* (*at* $\rho = 0$) *does not imply the strong Evans stability condition for small* ρ. This is in sharp contrast with the known results obtained in the constant multiplicity case [6,7,20,24,26,28]. This is illustrated by an example in Section 7 and this can occur for MHD, for some ad hoc boundary condition. An interesting question is to know whether this can happen or not for physical boundary conditions, in particular for slow MHD shocks.

On the other hand, we prove in this paper the existence of smooth symmetrizers under a natural *generalized block structure condition* for (1.4). We also provide a geometrical characterization of this condition on the matrices A and B occurring in (1.3). Moreover, modes that are totally incoming or totally outgoing do not cause trouble in the analysis of $\mathbb{E}^-$ nor of the Evans function. They are easily handled in the case of symmetric systems as in [21]. For instance, an important outcome of the present paper is the following result. We refer to the next sections for precise definitions.

Theorem 1.1. *Suppose that the full system* (1.3) *is symmetric. Suppose in addition that the eigenvalues of the hyperbolic system* (1.1) *are either semi-simple with constant multiplicity or totally nonglancing in the sense of Definition* 4.3. *Then, there are K-families of symmetrizers for the associated reduced system* (1.4), *for* $\rho \geqslant 0$ *sufficiently small.*

As recalled in the next section, K-families of symmetrizers provide Kreiss symmetrizers for boundary value problems which satisfy a uniform Lopatinski stability condition. One important application and motivation is the following

Example 1.2. Fast Lax' shocks for MHD satisfy the assumptions of Theorem 1.1.

But we also have the following

Counterexample 1.3. Slow Lax' shocks for MHD do not satisfy the assumptions of Theorem 1.1.

Remark 1.4. Fast shocks with small magnetic field are perturbations of acoustic shocks of gas dynamics, whose stability has been studied by A. Majda [14]. Therefore, there are good reasons to think that the uniform Evans–Lopatinski condition is satisfied for Fast Lax' shocks for MHD, at least for perfect gases state laws and small magnetic field.

Remark 1.5. When the assumptions of Theorem 1.1 are not satisfied, or more generally when the generalized block structure fails, one could try to construct nonsmooth symmetrizers as in [21]. Counterexample 1.3 would be a good motivation for that. However, nonsmooth symmetrizer would require much more sophisticated pseudodifferential tools to handle variable coefficients.

Moreover, slow shocks are not so closely related to acoustic shocks, and it is not known whether the uniform Lopatinski condition is likely to be satisfied or not.

When all the eigenvalues have constant multiplicity, Theorem 1.1 is proved in [20] (see also [19]).[1] The construction is based on a reduction of (1.4) to a suitable block diagonal form. Blocks which correspond to totally nonglancing modes (incoming or outgoing) are treated using the symmetry of the system as in [21]. For other blocks, we discuss in detail in Section 4 the *generalized block structure condition* which is needed for the construction of Kreiss symmetrizers.

The symmetrizers are used in [4–7,20] to prove maximal stability estimates for boundary value problems. The Fourier multipliers $\Sigma(p,\zeta)$ serve as symbols for pseudo-differential symmetrizers. All the other steps in these papers, linearization, paralinearization, separation of frequencies, the high- and medium-frequency analysis, the conversion of the plane wave or symbolic calculus into an operator calculus via the use of a paradifferential calculus, are independent of the constant multiplicity assumption which was assumed there as a sufficient condition for the generalized block structure condition. Therefore, all these analyses are valid under the assumptions of Theorem 3.7.

As already mentioned, the main novelty of this paper with respect to previous works of the authors is the consideration of systems with variable multiplicity. To lighten the presentation, we will now we focus on *boundary layers* for noncharacteristic boundary value problems. The extension to classical, conservative Lax-type shocks requires only to incorporate the ideas already explained in detail in (for instance) [6,7]. (The treatment of nonconservative and or undercompressive shocks involves new issues, and is carried out in [9].) Similarly, we will concentrate only on the symbolic analysis for *constant–coefficient equations* and the construction of smooth Fourier–Laplace multipliers. The passage from these multipliers to linear and nonlinear stability estimates for variable coefficients is already performed in previous works (see [6,7,20]) and can be used as an independent black box.

2. Spectral stability

In this section, we recall the main steps of the spectral stability analysis of noncharacteristic boundary layers, refereeing to [2,3,6,7,19,20,26,27] for details and further references and applications to the similar analysis of shock profiles. In particular, we give a new proof of the Zumbrun–Serre lemma [24,28] which allows for variable multiplicities. Moreover, not only does it provide a comparison between the Evans function of the viscous equation and the Lopatinski determinant of the inviscid system, but it also shows the link between the equations themselves: for low frequencies, the viscous boundary value problem decouples into two boundary value problems, one of them being a *nonsingular perturbation* of the limiting hyperbolic boundary value problem. We will also recall from [7] the main arguments for the high-frequency regime.

2.1. Structural assumptions

Consider a system of equations

[1] The reduction to (1.4) is carried out for strictly parabolic viscosities in [19,20] and for partial viscosities in [7]. However, the form of H_1 is the same in each case (a consequence of Kawashima's genuine coupling condition, Assumption (H4) below).

$$\mathcal{L}_\varepsilon(u) := A_0(u)\partial_t u + \sum_{j=1}^{d} A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(B_{j,k}(u)\partial_k u\big) = 0. \tag{2.1}$$

When $\varepsilon = 0$, $\mathcal{L}_0$ is first order and assumed to hyperbolic; ε plays the role of a nondimensional viscosity and for $\varepsilon > 0$, the system is assumed to be parabolic or at least partially parabolic. Classical examples are the Navier–Stokes equations of gas dynamics, or the equations of magnetohydrodynamics (MHD).

The form of the equations is preserved under a change of unknowns $u = \Phi(\tilde{u})$ or multiplication on the left by a *constant* invertible matrix. To cover the case of partial viscosity and motivated by the examples of Navier–Stokes equations and MHD, we make the following assumption:

Assumption 2.1.

(H0) The matrices A_j and $B_{j,k}$ are C^∞ $N \times N$ real matrices of the variable $u \in \mathcal{U}^* \subset \mathbb{R}^N$. Moreover, for all $u \in \mathcal{U}^*$, the matrix $A_0(u)$ is invertible.

(H1) Possibly after a change of variables u and multiplying the system on the left by an invertible constant–coefficient matrix, there is $N' \in \{1, \dots, N\}$ and there are coordinates $(u^1, u^2) \in \mathbb{R}^{N-N'} \times \mathbb{R}^{N'}$ for the unknown and $(f^1, f^2) \in \mathbb{R}^{N-N'} \times \mathbb{R}^{N'}$ for the right-hand side such that the following block structure condition is satisfied:

$$A_0(u) = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix}, \qquad B_{jk}(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_{jk}^{22} \end{pmatrix}. \tag{2.2}$$

We refer to [7] or [27] for further comments and explanations. From now on we work with variables $u = (u^1, u^2) \in \mathcal{U}^*$ such that (2.2) holds. We set

$$\overline{A}_j = A_0^{-1} A_j, \qquad \overline{B}_{j,k} = A_0^{-1} B_{j,k}, \tag{2.3}$$

and systematically use the notation $M^{\alpha\beta}$ for the sub-blocks of a matrix M corresponding to the splitting $u = (u^1, u^2)$. Note that

$$\overline{B}_{j,k}(u) := A_0(u)^{-1} B_{jk}(u) = \begin{pmatrix} 0 & 0 \\ 0 & \overline{B}_{jk}^{22}(u) \end{pmatrix}. \tag{2.4}$$

The triangular form of the equations also reveals the importance of the $(1,1)$ block which plays a special role in the analysis:

$$L^{11}(u, \partial) = \sum_{j=0}^{d} A_j^{11}(u)\partial_j, \quad \text{or} \quad \overline{L}^{11}(u, \partial) = \big(A_0^{11}(u)\big)^{-1} L^{11}(u, \partial). \tag{2.5}$$

In this spirit, *the high-frequency principal part* of the equation is

$$\begin{cases} \overline{L}^{11}(u, \partial)u^1, \\ \partial_t u^2 - \varepsilon \overline{B}^{22}(u, \partial)u^2 \end{cases} \tag{2.6}$$

with $\overline{B}^{22}(u,\xi)=\sum_{j,k=1}^{d}\xi_j\xi_k\overline{B}^{2,2}_{j,k}(u)$. We refer to Lemma 7.3 for a more detailed account of this notion of principal part. The first natural hypothesis is that $L^{11}(u,\partial)$ is hyperbolic and $\partial_t-\overline{B}^{22}(u,\partial)$ is parabolic in the direction dt.

Assumption 2.2.

(H2) There is $c>0$ such that for all $u\in\mathcal{U}^*$ and $\xi\in\mathbb{R}^d$, the eigenvalues of $\overline{B}^{22}(u,\xi)$ satisfy $\operatorname{Re}\mu\geqslant c|\xi|^2$.
(H3) For all $u\in\mathcal{U}^*$ and all $\xi\in\mathbb{R}^d\setminus\{0\}$, $\overline{A}^{11}(u,\xi)=\sum_{j=1}^{d}\xi_j\overline{A}^{11}_j(u)$ has only real eigenvalues.

For the applications we have in mind such as Navier–Stokes and MHD, the operator $\overline{L}^{11}$ is a transport field and (H3) is trivially satisfied.

Next we assume that the inviscid equations are hyperbolic and that Kawashima's *genuine coupling* condition is satisfied for u, in some open subdomain $\mathcal{U}\subset\mathcal{U}^*$. Let

$$\overline{A}(u,\xi)=\sum_{j=1}^{d}\xi_j\overline{A}_j(u)\quad\text{and}\quad\overline{B}(u,\xi)=\sum_{j,k=1}^{d}\xi_j\xi_k\overline{B}_{j,k}(u).\tag{2.7}$$

Assumption 2.3.

(H4) There is $c>0$ such that for $u\in\mathcal{U}$ and $\xi\in\mathbb{R}^d$, the eigenvalues of $i\overline{A}(u,\xi)+\overline{B}(u,\xi)$ satisfy

$$\operatorname{Re}\mu\geqslant c\frac{|\xi|^2}{1+|\xi|^2}.\tag{2.8}$$

Remark 2.4. (H4) implies *hyperbolicity* of the inviscid equation: for all $u\in\mathcal{U}$ and $\xi\in\mathbb{R}^d\setminus\{0\}$ the eigenvalues of $\overline{A}(u,\xi)$ are real. The set $\mathcal{U}$ may be thought of as the "hyperbolic set" where interior, inviscid solutions are be constructed, and the larger $\mathcal{U}^*$ as the "hyperbolic–parabolic" set where exterior, boundary layer solutions are to be constructed, matching $\mathcal{U}$ to boundary values in a multi-scale expansion. In contrast with [7] and [27], we do not assume here that the eigenvalues of $\overline{A}$ have constant multiplicity. It is precisely the aim of this paper to substitute weaker conditions, allowing us to treat the case of MHD.

Symmetric systems play an important role, and symmetry will be an important assumption in some of our results. In particular, Assumption (H4) is satisfied when the following conditions are satisfied (see [10,11]):

Definition 2.5. The system (2.1) is said to be symmetric dissipative if there exists a real matrix $S(u)$, which depends smoothly on $u\in\mathcal{U}$, such that for all $u\in\mathcal{U}$ and all $\xi\in\mathbb{R}^d\setminus\{0\}$, the matrix $S(u)A_0(u)$ is symmetric definite positive, $S(u)A(u,\xi)$ is symmetric and the symmetric matrix $\operatorname{Re}S(u)B(u,\xi)$ is nonnegative with kernel of dimension $N-N'$.

We consider a boundary value problem for (2.1) and the model case of a half space, which is given by $\{x>0\}$, in some coordinates $(y_1,\dots,y_{d-1},x)$ for the space variables. We assume that the boundary is not characteristic both for the viscous and the inviscid equations. The principal term of the viscous equation is block diagonal as indicated in (2.6) The B^{22} block is noncharacteristic by (H2). Restricting $\mathcal{U}^*$ to a component where the profiles will take their values, the condition for the $\overline{A}^{11}$ block reads.

Assumption 2.6. $\mathcal{U}^*$ is connected and for all $u \in \mathcal{U}^*$, $\det A_d^{11}(u) \neq 0$.

For the inviscid equation, restricting $\mathcal{U}$ to the component where the hyperbolic solutions will take their value, the condition reads

Assumption 2.7. $\mathcal{U}$ is connected and for all $u \in \mathcal{U}$, $\det(A_d(u)) \neq 0$.

By Assumption (H3) and Remark 2.4, $\overline{A}_d^{11}(u)$ and $\overline{A}_d(u)$ have only real eigenvalues, which by Assumptions 2.7 and 2.6 never vanish. This leads to two important indices:

Notation 2.8. With assumptions as above, N_+ denotes the number of positive eigenvalues of $\overline{A}_d(u)$ for $u \in \mathcal{U}$ and N_+^1 the number of positive eigenvalues of $\overline{A}_d^{11}(u)$ for $u \in \mathcal{U}^*$. We also set $N_b = N' + N_+^1$.

The block structure (2.6) suggests that N_b is the correct number of boundary conditions for the well posedness of (2.1), for solutions with values in $\mathcal{U}^*$. Indeed, the high-frequency decoupling (2.6) suggests N' boundary conditions for u^2 and N_+^1 boundary conditions for u^1. On the other hand, N_+ is the correct number of boundary conditions for the inviscid equation for solutions with values in $\mathcal{U}$. Thus we supplement (2.1) with boundary conditions

$$\Upsilon\big(u, \varepsilon\partial_y u^2, \varepsilon\partial_x u^2\big)\big|_{x=0} = 0. \tag{2.9}$$

Without pretending to maximal generality, we assume that they decouple into zero-order boundary conditions for u^1 and zero-order and first-order conditions for u^2:

$$\begin{cases} \Upsilon_1\big(u^1\big)\big|_{x=0} = 0, \\ \Upsilon_2\big(u^2\big)\big|_{x=0} = 0, \\ \Upsilon_3\big(u, \varepsilon\partial_y u^2, \varepsilon\partial_x u^2\big)\big|_{x=0} = 0, \end{cases} \tag{2.10}$$

with

$$\Upsilon_3\big(u, \partial_y u^2, \partial_x u^2\big) = K_d \partial_x u^2 + \sum_{j=1}^{d-1} K_j(u)\partial_j u^2.$$

Assumption 2.9. Υ_1, Υ_2 and Υ_3 are smooth functions of their arguments with values in $\mathbb{R}^{N_+^1}$, $\mathbb{R}^{N'-N''}$ and $\mathbb{R}^{N''}$, respectively, where $N'' \in \{0, 1, \dots, N'\}$. Moreover, K_d has maximal rank N'' and for all $u \in \mathcal{U}^*$ the Jacobian matrices $\Upsilon_1'(u^1)$ and $\Upsilon_2'(u^2)$ have maximal rank N_+^1 and $N' - N''$, respectively.

2.2. Profiles and inviscid boundary conditions

To match constant solutions $\underline{u}$ of the inviscid problem to solutions satisfying the boundary conditions, one looks for exact solutions of (2.1), (2.9) of the form:

$$u_\varepsilon(t,y,x) = w\left(\frac{x}{\varepsilon}\right), \tag{2.11}$$

such that

$$\lim_{z\to+\infty} w(z) = \underline{u}. \tag{2.12}$$

The equation for w reads

$$\begin{cases} A_d(w)\partial_z w - \partial_z\big(B_{d,d}(w)\partial_z w\big) = 0, & z \geqslant 0, \\ \Upsilon\big(w, 0, \partial_z w^2\big)\big|_{z=0} = 0. \end{cases} \tag{2.13}$$

Solutions are called *layer profiles*. This equation can be written as a first order system for $U = (w, \partial_z w^2)$, which is nonsingular if and only if A_d^{11} is invertible (this indicates the strong link between Assumption 2.6 and the ansatz (2.11)):

$$\begin{aligned} \partial_z w^1 &= -\big(A_d^{11}\big)^{-1} A_d^{12} w^3, \\ \partial_z w^2 &= w^3, \\ \partial_z\big(B_{d,d} w^3\big) &= \big(A_d^{22} - A_d^{21}\big(A_d^{11}\big)^{-1} A_d^{12}\big) w^3, \end{aligned} \tag{2.14}$$

and the matrices are evaluated at $w = (w^1, w^2)$.

The natural limiting boundary conditions for the inviscid problem read

$$u|_{x=0} \in \mathcal{C}, \tag{2.15}$$

where $\mathcal{C}$ denotes the set of end points $\underline{u}$ such that there is a layer profile $w \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{U}^*)$ satisfying (2.12), (2.13). The properties of the set $\mathcal{C}$ as well as the stability analysis of (2.13) depend on the spectral properties of the linearized equations from (2.13) near $w(z)$. In particular we will discuss the notion of *transversality* for the profile w (see [19,20]). However, to avoid repetitions and prepare the multidimensional stability analysis, we enlarge the framework and consider the multidimensional linearized equations from the full system (2.1) near solutions (2.11).

For further use, it is convenient to enlarge the class of functions w: consider a function $C^\infty(\overline{\mathbb{R}}_+; \mathcal{U}^*)$ which converges at an exponential rate to and end state $\underline{u} \in \mathcal{U}$: there is $\delta > 0$ such that for all $k \in \mathbb{N}$

$$e^{\delta z}\big|\partial_z^k\big(w(z) - \underline{u}\big)\big| \in L^\infty(\overline{\mathbb{R}}_+). \tag{2.16}$$

We refer to such a function as a *profile*; it need not be a solution of (2.13), though it will be in applications. Note that solutions of (2.13), (2.12) satisfy the exponential convergence above.

Consider the linearized equations from (2.1), (2.9) around $u_\varepsilon = w(x/\varepsilon)$:

$$\mathcal{L}'_{u_\varepsilon}\dot{u} = \dot{f}, \qquad \Upsilon'(\dot{u}, \varepsilon\partial_y\dot{u}, \varepsilon\partial_x\dot{u})|_{x=0} = \dot{g}. \tag{2.17}$$

Here Υ' is the differential of Υ at $(w(0), 0, \partial_z w(0))$. $\mathcal{L}'_{u_\varepsilon}$ is a differential operator with coefficients that are smooth functions of $z := x/\varepsilon$. Factoring out ε^{-1} it also appears as an operator in $\varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x$:

$$\mathcal{L}'_{u_\varepsilon} = \frac{1}{\varepsilon} L\left(\frac{x}{\varepsilon}, \varepsilon\partial_t, \varepsilon\partial_y, \varepsilon\partial_x\right). \tag{2.18}$$

It has constant coefficients in (t, y), and following the usual theory of constant–coefficient evolution equations, one performs a Laplace–Fourier transform in (t, y), with frequency variables denoted by $\tilde{\gamma} + i\tilde{\tau}$ and $\tilde{\eta}$, respectively, yielding the systems

$$\frac{1}{\varepsilon} L\left(\frac{x}{\varepsilon}, \varepsilon(\tilde{\gamma} + i\tilde{\tau}), i\varepsilon\tilde{\eta}, \varepsilon\partial_x\right).$$

Next, we introduce explicitly the fast variable $z = x/\varepsilon$, rescale the frequency variables as $\zeta = (\tau, \eta, \gamma) = \varepsilon(\tilde{\tau}, \tilde{\eta}, \tilde{\gamma})$, and multiply the equation by ε, revealing the equation

$$L(z, \gamma + i\tau, i\eta, \partial_z)u = f, \qquad \Upsilon'(u, i\eta u, \partial_z u)|_{z=0} = g, \tag{2.19}$$

$$L = -\mathcal{B}(z)\partial_z^2 + \mathcal{A}(z, \zeta)\partial_z + \mathcal{M}(z, \zeta), \tag{2.20}$$

with in particular, $\mathcal{B}(z) = B_{d,d}(w(z))$ and $\mathcal{A}^{11}(z, \zeta) = A_d^{11}(w(z))$. We do not give here the explicit form of $\mathcal{A}$ and $\mathcal{M}$. Using (H2) and Assumption 2.2, the equation is written as a first order system

$$\partial_z U = \mathcal{G}(z, \zeta)U + F, \qquad \Gamma(\zeta)U|_{z=0} = g, \tag{2.21}$$

where

$$U = {}^t\big(u, \partial_z u^2\big) = \big(u^1, u^2, \partial_z u^2\big) \in \mathbb{C}^{N+N'}, \tag{2.22}$$

$$F = \big(\big(\mathcal{A}^{11}(z)\big)^{-1} f^1, 0, \big(\mathcal{B}^{22}(z)\big)^{-1}\big(-f^2 + \mathcal{A}^{21}(z)\big(\mathcal{A}^{11}(z)\big)^{-1} f^1\big)\big). \tag{2.23}$$

The analysis of this equation depends on the size of the frequencies ζ. When ζ is large, the character of the equations is dominated by the high-frequency principal part (2.6), and we use a slowly-varying-coefficients analysis (related to the "tracking lemmas" of [26,27]) based on the relatively slow rate of change of coefficients compared to the size of the frequency; see [6,7] and Section 7 below. For small or bounded frequencies ζ, we use the conjugation lemma of [20]. The condition (2.16) implies that there is $\delta > 0$ and an end state matrix $G(\underline{u}, \zeta)$, depending on the endstate $\underline{u}$ of w, such that

$$\partial_z^k\big(\mathcal{G}(z, \zeta) - G(\underline{u}, \zeta)\big) = O\big(e^{-\delta z}\big). \tag{2.24}$$

Lemma 2.10. *Given $\underline{\zeta} \in \mathbb{R}^{d+1}$, there is a smooth invertible matrix $\Phi(z,\zeta)$ for $z \in \overline{\mathbb{R}}_+$ and ζ in a neighborhood of $\underline{\zeta}$, such that* (2.19) *is equivalent to*

$$\partial_z \widetilde{U} = G(\underline{u},\zeta)\widetilde{U} + \widetilde{F}, \qquad \widetilde{\Gamma}(\zeta)\widetilde{U}|_{z=0} = g, \tag{2.25}$$

with $U = \Phi(z,\zeta)\widetilde{U}$, $F = \Phi(z,\zeta)\widetilde{F}$ and $\widetilde{\Gamma}(\zeta) = \Gamma(\zeta)\Phi(0,\zeta)$. In addition, Φ and Φ^{-1} converge the identity matrix at an exponential rate when $z \to \infty$.

Moreover, if the coefficients of the operator and w depend smoothly on extra parameters p (such as the end state $\underline{u}$), then Φ can also be chosen to depend smoothly on p, on a neighborhood of a given $\underline{p}$.

Remark 2.11. The linearized profile equations from (2.13) around w, are exactly (2.19) at the frequency $\zeta = 0$. In particular, Lemma 2.10 implies that these equations are conjugated to constant–coefficient equations, via the conjugation by $\Phi(\cdot, 0)$.

Next we investigate the spectral properties of the matrix G. Below, $\mathbb{R}^{d+1}_+$ denotes the open half space $\{\zeta = (\tau,\eta,\gamma)\colon \gamma > 0\}$ and $\overline{\mathbb{R}}^{d+1}_+$ its closure $\{\gamma \geqslant 0\}$. We also introduce the matrices

$$P_0(u) := \left(B^{22}\right)^{-1}\left(A_d^{22} - A_d^{21}\left(A_d^{11}\right)^{-1}A_d^{12}\right), \tag{2.26}$$

$$H_0(u,\zeta) := -\left(A_d(u)\right)^{-1}\left((i\tau+\gamma)A_0(u) + \sum_{j=1}^{d-1} i\eta_j A_j(u)\right). \tag{2.27}$$

Lemma 2.12.

(i) *For $u \in \mathcal{U}$, $P_0(u)$ has no eigenvalue on the imaginary axis. We denote by N_-^2 the number of its eigenvalues in $\{\operatorname{Re}\mu < 0\}$.*

(ii) *For $u \in \mathcal{U}$ and $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$, $G(u,\zeta)$ has no eigenvalue on the imaginary axis. The number of its eigenvalues, counted with their multiplicity, in $\{\operatorname{Re}\mu < 0\}$ is equal to $N_+ + N_-^2 = N_b := N' + N_+^1$.*

(iii) *For a given $\underline{u} \in \mathcal{U}$, there are smooth matrices $V(u,\zeta)$ on a neighborhood of $(\underline{u}, 0)$ such that*

$$V^{-1}GV = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} \tag{2.28}$$

with $H(u,\zeta)$ of dimension $N \times N$, $P(u,\zeta)$ of dimension $N' \times N'$, and

(a) *the eigenvalues of P satisfy $|\operatorname{Re}\mu| \geqslant c$ for some $c > 0$,*

(b) *there holds*

$$H(u,\zeta) = H_0(u,\zeta) + O\left(|\zeta|^2\right), \tag{2.29}$$

(c) *at $\zeta = 0$, V has a triangular form*

$$V(u,0) = \begin{pmatrix} \mathrm{Id} & \overline{V} \\ 0 & \mathrm{Id} \end{pmatrix}. \tag{2.30}$$

Proof. (i) Take $u \in \mathcal{U}$. If $v^2 \in \ker P_0(u)$, then ${}^t(-(A_d^{11})^{-1} A_d^{12} v^2, v^2) \in \ker A_d$, implying that 0 is not an eigenvalue of P_0. Similarly, if $i\xi$ is an eigenvalue of P then 0 is an eigenvalue of $i\xi \overline{A}_d + \xi^2 \overline{B}_d$, which is impossible by (H4) if $\xi \neq 0$ is real.

(ii) Direct computations show that $G(u,\zeta) = G_d(u,\zeta)^{-1} M(u,\zeta)$ with

$$G_d(u,\zeta) = \begin{pmatrix} -\widetilde{A}_d & \widetilde{B}_d \\ J & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} \widetilde{M} & 0_{N\times N'} \\ 0_{N'\times N} & \mathrm{Id}_{N'\times N'} \end{pmatrix}$$

with, in the splitting $u = (u^1, u^2)$,

$$\widetilde{B}_d(u) = \begin{pmatrix} 0_{N-N'\times N'} \\ \overline{B}^{22}_{d,d}(u) \end{pmatrix}, \qquad J = \begin{pmatrix} 0_{N'\times N-N'} & \mathrm{Id}_{N'\times N'} \end{pmatrix},$$

and

$$\begin{cases} \widetilde{A}(u,\zeta) = A_d(u) - \displaystyle\sum_{j=1}^{d-1} i\eta_j \big(B_{j,d}(u) + B_{d,j}(u)\big), \\ \widetilde{M}(u,\zeta) = (i\tau + \gamma) A_0(u) + \displaystyle\sum_{j=1}^{d-1} i\eta_j A_j(u) + \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(u). \end{cases}$$

In particular, $i\xi$ is an eigenvalue of $G(u,\zeta)$ if and only if $\gamma + i\tau$ is an eigenvalue of $i\overline{A}(\eta,\xi) + \overline{B}(\eta,\xi)$, which, by (H4), implies either that $\gamma < 0$ if ξ is real and $(\eta,\xi) \neq 0$ or that $\zeta = 0$.

Thus $G(u,\zeta)$ has no eigenvalues on the imaginary axis and the number $\widetilde{N}$ of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is constant for $u \in \mathcal{U}$ and $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$. That this number is equal to $N_b = N_+^1 + N'$ is a consequence of the high-frequency analysis in Lemma 7.3 below (see also Lemma 1.7 in [27]).

(iii) Because $\widetilde{M}(u,0) = 0$ and $\widetilde{A}(u,0) = A_d(u)$, there holds

$$G(u,0) = \begin{pmatrix} 0_{N\times N} & \begin{pmatrix} -(A_d^{11})^{-1} A_d^{12} \\ \mathrm{Id}_{N'\times N'} \end{pmatrix} \\ 0_{N'\times N} & P_0(u) \end{pmatrix}. \tag{2.31}$$

Since P_0 is invertible, G can be smoothly conjugated to a block diagonal matrix as in (2.28), with V satisfying (2.30) and $H(u,0) = 0$. More precisely, the matrix $\overline{V}$ is

$$\overline{V} = \begin{pmatrix} -(A_d^{11})^{-1} A_d^{12} P_0^- 1 \\ P_0^{-1} \end{pmatrix}. \tag{2.32}$$

The expansion (2.29) can be easily obtained by standard perturbation expansions, and we refer to Lemma 4.23 below for a more precise version.

For ζ small, the number of eigenvalues of P in $\{\operatorname{Re}\mu < 0\}$ is equal to N_-^2, and for $\gamma > 0$, the number of eigenvalues of $H_0(u,\zeta)$ in the negative half space is constant, by hyperbolicity, and equal to N_+. This implies that $\widetilde{N} = N_+ + N_-^2$. □

Similarly, one considers the linearized equations from the inviscid hyperbolic problem $\mathcal{L}_0(u) = 0$ around the constant solution $\underline{u}$:

$$\mathcal{L}'_{0,\underline{u}}\dot{u} = \dot{f}. \tag{2.33}$$

After performing a Laplace–Fourier transform, this equation reads

$$L_0(\underline{u}, \gamma + i\tau, i\eta, \partial_x)u = f \tag{2.34}$$

or, with H_0 defined at (2.27),

$$\partial_x u = H_0(\underline{u}, \zeta)u + A_d^{-1}(\underline{u})f. \tag{2.35}$$

An important property for profiles is the notion of *transversality* (see [20] or [19] for the case of total viscosity). It concerns the linearized equations from (2.11) around w. As mentioned in Remark 2.11, they correspond exactly to the first order system (2.19) with $\zeta = 0$. We abbreviate the homogeneous problem as

$$\begin{cases} L(z, 0, \partial_z)\dot{w} = 0, & z \geqslant 0, \\ \Upsilon'(\dot{w}, 0, \partial_z \dot{w}^2)\big|_{z=0} = 0. \end{cases} \tag{2.36}$$

A corollary of Lemmas 2.10 and 2.12 is that the solutions of the homogeneous equation $L(z, 0, \partial_z)\dot{w} = 0$ form a space of dimension $N + N'$, parametrized by $(u_H, u_P) \in \mathbb{C}^N \times \mathbb{C}^{N'}$:

$$\dot{w}(z) = \Phi_H(z)u_H + \Phi_P(z)e^{zP_0(\underline{u})}u_P \tag{2.37}$$

where the matrices $\Phi_H(z)$ and $\Phi_P(z)$ are smooth and bounded on $\mathbb{R}_+$ and $\Phi_H(z) \to \mathrm{Id}$ as $z \to +\infty$. The solution is bounded if and only if u_P belongs to the negative space $\mathbb{E}^-(P_0(\underline{u}))$ of $P_0(\underline{u})$, that is the invariant space of $P_0(\underline{u})$ associated to the spectrum lying in $\{\operatorname{Re}\mu < 0\}$; thus the space $\mathcal{S}$ of bounded solutions has dimension $N + N_-^2$. The space of solutions that tend to zero at infinity, denoted by $\mathcal{S}_0$, has dimension N_-^2, corresponding to the conditions $u_H = 0$ and $u_P \in \mathbb{E}^-(P_0(\underline{u}))$.

The boundary conditions in (2.36) read

$$\underline{\Gamma}_H u_H + \underline{\Gamma}_P u_P := \underline{\Gamma}(\dot{w}, \partial_z \dot{w}^2)\big|_{z=0} = 0. \tag{2.38}$$

Definition 2.13. The profile w is said to be transversal if

(i) there is no nontrivial solution $\dot{w} \in \mathcal{S}_0$ which satisfies the boundary conditions $\underline{\Gamma}(\dot{w}, \partial_z \dot{w}^2)|_{z=0} = 0$,
(ii) the mapping $\dot{w} \mapsto \underline{\Gamma}(\dot{w}, \partial_z \dot{w}^2)|_{z=0}$ from $\mathcal{S}$ to $\mathbb{C}^{N_b}$ has rank N_b.

Equivalently, it means that $\ker \underline{\Gamma}_P \cap \mathbb{E}^-(P_0(\underline{u})) = \{0\}$ and that the rank of the matrix $(\underline{\Gamma}_H, \underline{\Gamma}_P)$ from $\mathbb{C}^N \times \mathbb{E}^-(P_0(\underline{u}))$ to $\mathbb{C}^{N_b}$ is N_b.

If the profile satisfies condition (i), there is a decomposition

$$\mathbb{C}^{N_b} = \mathbb{F}_H \oplus \mathbb{F}_{0,P}, \quad \mathbb{F}_{0,P} := \underline{\Gamma}_P \mathbb{E}^-\big(P_0(\underline{u})\big) \tag{2.39}$$

with $\dim \mathbb{F}_H = N_+$ and $\dim \mathbb{F}_{0,P} = N_-^2$. Denote by π_H and π_P the projections associated to this splitting.

For $\dot{w} \in \mathcal{S}$ given by (2.37), one can eliminate u_P from the boundary conditions (2.38) and write them

$$\underline{\Gamma}_{\mathrm{red}} u_H = 0, \qquad u_P = R_{0,P} u_H, \tag{2.40}$$

with

$$\underline{\Gamma}_{\mathrm{red}} := \pi_H \underline{\Gamma}_H, \qquad R_{0,P} := -(\underline{\Gamma}_P)^{-1} \pi_P \underline{\Gamma}_H \tag{2.41}$$

and $(\underline{\Gamma}_P)^{-1}$ is the inverse of the mapping $\underline{\Gamma}_P$ from $\mathbb{E}^-(P_0(\underline{u}))$ to $\mathbb{F}_{0,P}$.

With these notations, (ii) means that $\underline{\Gamma}_{\mathrm{red}}$ has rank N_+. Its kernel $\ker \underline{\Gamma}_{\mathrm{red}}$ is the space of $\dot{u} \in \mathbb{R}^d$ such that there is a solution of $\dot{w}$ of (2.36) with end point $\dot{u}$. It has dimension $N - N_+$.

Remark 2.14. When w is a layer profile, solution of (2.13), the transversality condition implies that near the end point $\underline{u}$, the set $\mathcal{C}$ in (2.15) which describes the limiting hyperbolic conditions is a smooth manifold of dimension $N_- = N - N_+$ and $\ker \underline{\Gamma}_{\mathrm{red}}$ is the tangent space to $\mathcal{C}$ at $\underline{u}$. Therefore, the natural boundary condition for the linearized hyperbolic equation, and in particular for (2.33), are

$$\underline{\Gamma}_{\mathrm{red}} u = h. \tag{2.42}$$

2.3. Evans functions and Lopatinski determinant

For a given $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$, we now investigate the well-posedness of Eq. (2.19) or equivalently (2.21) or (2.25). Introduce the space $\mathbb{E}^-(\zeta)$ of initial conditions $(u(0), \partial_z u^2(0))$ (or equivalently $U(0)$) such that the corresponding solution of $L(z,\zeta,\partial_z)u = 0$ (or $\partial_z U - \mathcal{G}(z,\zeta)U = 0$) is exponentially decaying at $+\infty$. Lemmas 2.10 and 2.12 show that

$$\mathbb{E}^-(\zeta) = \Phi(0,\zeta)\mathbb{E}^-\big(G(\underline{u},\zeta)\big) \tag{2.43}$$

where we use the following notations:

Notation 2.15. Given a square matrix M, $\mathbb{E}^-(M)$ (respectively $\mathbb{E}^+(M)$) denotes the invariant space of M associated to the spectrum of M contained in $\{\operatorname{Re}\mu < 0\}$ (respectively $\{\operatorname{Re}\mu > 0\}$).

In particular, by Lemma 2.12, $\mathbb{E}^-(\zeta)$ is a smooth vector bundle for $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$ and $\dim(\mathbb{E}^-(\zeta)) = N_b$.

The problems (2.19), (2.21) or (2.25) are well posed if and only if

$$\mathbb{E}^-(\zeta) \cap \ker \Gamma(\zeta) = \{0\} \quad \text{or} \quad \mathbb{E}^-\big(G(\underline{u},\zeta)\big) \cap \ker \widetilde{\Gamma}(\zeta) = \{0\}. \tag{2.44}$$

Note that, because the rank of Γ is at most N_b and the dimension of $\mathbb{E}^-$ is N_b, this condition implies and is equivalent to

$$\mathbb{C}^{N+N'} = \mathbb{E}^-(\zeta) \oplus \ker \Gamma(\zeta) \quad \text{or} \quad \mathbb{C}^{N+N'} = \mathbb{E}^-\big(G(\underline{u},\zeta)\big) \oplus \ker \widetilde{\Gamma}(\zeta). \tag{2.45}$$

The *Evans function* is defined as

$$D(\zeta) = \left|\det{}_{N+N'}\left(\mathbb{E}^-(\zeta), \ker\Gamma(\zeta)\right)\right| \tag{2.46}$$

where, for subspaces $\mathbb{E}$ and $\mathbb{F}$ of $\mathbb{C}^n$, $\det_n(\mathbb{E}, \mathbb{F})$ is equal to 0 if $\dim\mathbb{E} + \dim\mathbb{F} \neq n$ and is the $n \times n$ determinant formed by orthonormal bases in $\mathbb{E}$ and $\mathbb{F}$ if $\dim\mathbb{E} + \dim\mathbb{F} = n$.

Remark 2.16. The definition of the determinant above depends on choices of bases. Note that changing bases in $\mathbb{E}$ and $\mathbb{F}$ changes the determinant by a complex number of modulus one, thus leaves $|\det(\mathbb{E}, \mathbb{F})|$ invariant. But it also depends on the choice of a scalar product on $\mathbb{C}^n$. Changing the scalar products (or changing of bases in $\mathbb{C}^n$) changes the function $\det(\mathbb{E}, \mathbb{F})$ to a new function $\widetilde{\det}(\mathbb{E}, \mathbb{F})$ such that $c|\det(\mathbb{E}, \mathbb{F})| \leqslant |\widetilde{\det}(\mathbb{E}, \mathbb{F})| \leqslant c^{-1}|\det(\mathbb{E}, \mathbb{F})|$ where $c > 0$ is independent of the spaces $\mathbb{E}$ and $\mathbb{F}$. We will denote by

$$\det \approx \widetilde{\det} \quad \text{or} \quad D \approx \widetilde{D} \tag{2.47}$$

this property. In particular, the definition of D is independent of the choice of orthonormal bases in $\mathbb{E}^-$ and $\ker\Gamma$ and all the uniform stability conditions stated below are independent of the choice of the scalar product.

Remark 2.17. If the coefficients of the operator and the profile depend smoothly on parameters p, then the Evans function is also a smooth function of the parameters.

These notations being settled, the *weak stability* condition, which is a necessary condition for well posedness in Sobolev spaces of (2.17), reads:

Definition 2.18. Given a profile w, the linearized equation (2.17) satisfies the weak spectral stability condition if $D(\zeta) \neq 0$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$.

The next lemma is useful and elementary.

Lemma 2.19. *Suppose that* $\mathbb{E} \subset \mathbb{C}^n$ *and* $\Gamma : \mathbb{C}^n \mapsto \mathbb{C}^m$, *with* $\operatorname{rank}\Gamma = \dim\mathbb{E} = m$. *If* $|\det(\mathbb{E}, \ker\Gamma)| \geqslant c > 0$, *then there is* C, *which depends only on* c *and* $|\Gamma^*(\Gamma\Gamma^*)^{-1}|$ *such that*

$$\forall U \in \mathbb{E} \quad |U| \leqslant C|\Gamma U|.$$

Conversely, if this estimate is satisfied then $|\det(\mathbb{E}, \ker\Gamma)| \geqslant c$ *where* $c > 0$ *depends only on* C *and* $|\Gamma|$.

Proof. Let $\pi = \Gamma^*(\Gamma\Gamma^*)^{-1}\Gamma$ denote the orthogonal projector on $(\ker\Gamma)^\perp$. Diagonalizing the hermitian form $(\pi e, \pi e)$, yields orthonormal bases $\{e_j\}$ and $\{f_j\}$ in $\mathbb{E}$ and $(\ker\Gamma)^\perp$, respectively, such that $\pi e_j = \lambda_j f_j$ with $0 < \lambda_j \leqslant 1$. Take any basis $\{g_k\}$ of $\ker\Gamma$. Expressing the e_j in the base $\{f_k, g_l\}$, implies that $|\det(\mathbb{E}, \ker\Gamma)| = \prod \lambda_j$. Since $\lambda_j \leqslant 1$ for all j, if this determinant is larger than or equal to $c > 0$, then $\min\lambda_j \geqslant c$ and for all $e \in \mathbb{E}$

$$c|e| \leqslant |\pi e| \leqslant \left|\Gamma^*\left(\Gamma\Gamma^*\right)^{-1}\right||\Gamma e|.$$

Conversely, if the estimate is satisfied, then $|e| \leqslant C|\Gamma||\pi e|$ since $\Gamma e = \Gamma \pi e$ for all $e \in \mathbb{E}$. Therefore $\lambda_j C|\Gamma| \geqslant 1$ and the determinant is at least equal to $(C|\Gamma|)^{-m}$. □

There are analogous definitions for the linearized hyperbolic problem (2.33) with boundary conditions (2.42). For $\gamma > 0$, $H_0(\underline{u}, \zeta)$ has no eigenvalues on the imaginary axis, as a consequence of the hyperbolicity assumption (see Remark 2.4). The *Lopatinski determinant* is defined for $\zeta \in \mathbb{R}^{d+1}_+ := \{\gamma > 0\}$ by

$$D_{\text{Lop}}(\zeta) = \left|\det\left(\mathbb{E}^-\left(H_0(\underline{u}, \zeta), \ker \underline{\Gamma}_{\text{red}}\right)\right)\right|. \tag{2.48}$$

By homogeneity of H_0, this determinant is homogeneous of degree zero in ζ and one can restrict attention to $\zeta \in S^d = \{|\zeta| = 1\}$.

Definition 2.20. The linearized equation (2.33), (2.42) satisfies the weak spectral stability condition if $D_{\text{Lop}}(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^{d+1}_+$.

2.4. Uniform spectral stability and maximal estimates

The weak stability conditions and the reduction to constant coefficients of Lemma 2.10 guarantee the well posedness of (2.19) for fixed $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$ and in particular estimates of the form

$$\|u\|_{L^2} + \left\|\partial_z u^2\right\|_{L^2} + \left|u(0)\right| + \left|\partial_z u^2(0)\right| \leqslant C(\zeta)\left(\|f\|_{L^2} + |g|\right). \tag{2.49}$$

The next step in the study of (2.17), is to perform an inverse Fourier–Laplace transform and thus requires suitable estimates for the solutions of (2.19), with a precise description of the constants in the estimate above.

By continuity in ζ, the weak stability condition implies that the estimate (2.49) is satisfied with a uniform constant C when ζ remains in a compact subset of $\overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$. Thus the true question is to get a detailed behavior of the estimate when $\zeta \to 0$ and when $|\zeta| \to \infty$.

2.4.1. Low and medium frequencies

Consider first the *low-frequency* case. Following [20], the uniform stability condition reads:

Definition 2.21. Given a profile w, the uniform spectral stability condition for low frequencies is satisfied when there are $c > 0$ and $\rho_0 > 0$ such that $D(\zeta) \geqslant c$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $0 < |\zeta| \leqslant \rho_0$.

By Assumption 2.9, the rank of $\Gamma(\zeta)$ is always N_b, and the norms of $\Gamma(\zeta)$ and $(\Gamma\Gamma^*)^{-1}$ are uniformly bounded for ζ bounded. Thus, by Lemma 2.19, the low-frequency uniform stability condition holds if and only if there are C and $\rho_0 > 0$ such that

$$\forall \zeta \in \overline{\mathbb{R}}^{d+1}_+, \quad 0 < |\zeta| \leqslant \rho_0, \qquad \forall U \in \mathbb{E}^-(\zeta): \quad |U| \leqslant C\left|\Gamma(\zeta)U\right|. \tag{2.50}$$

Following [20], the expected *maximal estimates* for low and medium frequencies for the solutions of (2.19) read

$$\varphi\|u\|_{L^2(\mathbb{R}_+)} + \left\|\partial_z u^2\right\|_{L^2(\mathbb{R}_+)} + \left|u(0)\right| + \left|\partial_z u^2(0)\right| \leqslant C\left(\frac{1}{\varphi}\|f\|_{L^2(\mathbb{R}_+)} + |g|\right) \tag{2.51}$$

where $\varphi = (\gamma + |\zeta|^2)^{\frac{1}{2}}$ with C independent of $\zeta \in \overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$, $|\zeta| \leqslant \rho_0$. Note that for fixed $|\zeta| > 0$, this estimate implies (2.49).

The estimates (2.51) correspond to estimates for the solutions of the first order system (2.21):

$$\varphi \| U^1 \|_{L^2(\mathbb{R}_+)} + \| U^2 \|_{L^2(\mathbb{R}_+)} + |U(0)| \leqslant C\left(\frac{1}{\varphi} \|F\|_{L^2(\mathbb{R}_+)} + |g|\right) \tag{2.52}$$

where $U = (U^1, U^2) \in \mathbb{C}^N \times \mathbb{C}^{N'}$. For the constant–coefficient system (2.25) the expected estimates read:

$$\varphi \| \widetilde{U}^1 \|_{L^2(\mathbb{R}_+)} + \| \widetilde{U}^2 \|_{L^2(\mathbb{R}_+)} + |\widetilde{U}(0)| \leqslant C\left(\frac{1}{\varphi} \|\widetilde{F}\|_{L^2(\mathbb{R}_+)} + |g|\right). \tag{2.53}$$

Lemma 2.22. *The estimates* (2.52) *imply* (2.53) *which imply* (2.51).

Proof. (See [20].) Clearly, (2.51) is a particular case of (2.52) applied to source terms F of the special form (2.23). Moreover, using the conjugation Lemma 2.10, there holds $U = O(1)\widetilde{U}$ and $\widetilde{U} = O(1)U$ and similar estimates for F and $\widetilde{F}$. Moreover,

$$U^1 = O(1)\widetilde{U}, \qquad U^2 = O\big(e^{-\theta z}\big)\widetilde{U}^1 + O(1)\widetilde{U}^2$$

with $\theta > 0$. We use the inequality

$$\| e^{-\theta z} \widetilde{U}^1 \|_{L^2} \lesssim |\widetilde{U}^1(0)| + \| \partial_z \widetilde{U}^1 \|_{L^2}.$$

Moreover, the form of $G(\underline{u}, \zeta)$ at $\zeta = 0$ shows that

$$\partial_z \widetilde{U}^1 = O\big(|\zeta|\big)\widetilde{U}^1 + O(1)\widetilde{U}^2 + \widetilde{F}^1.$$

Therefore,

$$\| U^2 \|_{L^2} \lesssim \| \widetilde{U}^2 \|_{L^2} + |\widetilde{U}^1(0)| + |\zeta| \| \widetilde{U}^1 \|_{L^2} + \| \widetilde{F}^1 \|_{L^2}.$$

Since $|\zeta| \leqslant \varphi$, this shows that (2.53) implies (2.52). □

For ζ in a compact subset of $\overline{\mathbb{R}}^{d+1}_+ \setminus \{0\}$, all these estimates are true under the weak stability condition (see e.g. [20]). Note also (taking $f = 0$ in (2.51)) that the uniform stability condition (2.50) is necessary for the validity of the maximal estimate. *The main subject of this paper is to prove that the uniform stability condition implies the maximal estimate* (2.51) *for low frequencies, under structural assumptions on the system weaker than in* [6,7,20], allowing for instance to consider MHD.

2.4.2. High frequencies

For the *high-frequency* analysis, the maximal estimates that are proven in [7] concern homogeneous boundary conditions ($g = 0$) and read

$$\begin{aligned}&(1+\gamma)\|u^1\|_{L^2(\mathbb{R}_+)} + \Lambda\|u^2\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)}\\&\quad + (1+\gamma)^{\frac12}|u^1(0)| + \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|\partial_z u^2(0)|\\&\leqslant C\big(\|f^1\|_{L^2(\mathbb{R}_+)} + \Lambda^{-1}\|f^2\|_{L^2(\mathbb{R}_+)}\big),\end{aligned} \tag{2.54}$$

where Λ is the natural parabolic weight

$$\Lambda(\zeta) = \big(1+\tau^2+\gamma^2+|\eta|^4\big)^{\frac14}. \tag{2.55}$$

The balance between the weights for u^1 and for u^2 is subtle: these components are decoupled in the high-frequency principal system (2.6) and the weights depend on their actual coupling through the nondiagonal terms and the boundary conditions. Here we see the importance of the form (2.10) of the boundary conditions. Their linearized version, $\Upsilon'(u, i\eta u^2, \partial_z u^2) = g$ reads

$$\begin{cases}\Gamma_1 u^1(0) := \Upsilon_1'\big(w^1(0)\big)\cdot u^1(0) = g^1,\\ \Gamma_2 u^2(0) := \Upsilon_2'\big(w^2(0)\big)\cdot u^2(0) = g^2,\\ \Gamma_3(\zeta)\big(u^2(0), \partial_z u^2(0)\big) := K_d\partial_z u^2(0) + K_{\mathrm{tg}}(\eta)u^2(0) = g^3,\end{cases} \tag{2.56}$$

with

$$K_{\mathrm{tg}} = \sum_{j=1}^{d-1} i\eta_j K_j\big(w(0)\big). \tag{2.57}$$

The complete maximal estimate with nonvanishing boundary source terms g, reads

$$\begin{aligned}&(1+\gamma)\|u^1\|_{L^2(\mathbb{R}_+)} + \Lambda\|u^2\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)}\\&\quad + (1+\gamma)^{\frac12}|u^1(0)| + \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|\partial_z u^2(0)|\\&\leqslant C\big(\|f^1\|_{L^2(\mathbb{R}_+)} + \Lambda^{-1}\|f^2\|_{L^2(\mathbb{R}_+)}\big) + C\big((1+\gamma)^{\frac12}|g^1| + \Lambda^{\frac12}|g^2| + \Lambda^{-\frac12}|g^3|\big)\end{aligned} \tag{2.58}$$

with C independent of $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ large. Taking $f = 0$, this implies the following necessary condition: there are C and $\rho_1 > 0$ such that

$$\begin{aligned}&\forall\zeta \in \overline{\mathbb{R}}^{d+1}_+,\quad |\zeta| \geqslant \rho_1,\quad \forall U = \big(u^1, u^2, u^3\big) \in \mathbb{E}^-(\zeta):\\&(1+\gamma)^{\frac12}|u^1| + \Lambda^{\frac12}|u^2| + \Lambda^{-\frac12}|u^3|\\&\quad\leqslant C\big((1+\gamma)^{\frac12}|\Gamma_1 u^1| + \Lambda^{\frac12}|\Gamma_2 u^2| + \Lambda^{-\frac12}\big|\Gamma_3(\zeta)\big(u^2, u^3\big)\big|\big).\end{aligned} \tag{2.59}$$

This can be reformulated in terms of a *rescaled Evans function* (see [20]): In $\mathbb{C}^{N+N'}$ and $\mathbb{C}^{N_b}$ introduce the mappings

$$J_\zeta\big(u^1, u^2, u^3\big) := \big((1+\gamma)^{\frac{1}{2}} u^1, \Lambda^{\frac{1}{2}} u^2, \Lambda^{-\frac{1}{2}} u^3\big),$$
$$J_\zeta\big(g^1, g^2, g^3\big) := \big((1+\gamma)^{\frac{1}{2}} g^1, \Lambda^{\frac{1}{2}} g^2, \Lambda^{-\frac{1}{2}} g^3\big). \tag{2.60}$$

Note that $J_\zeta \Gamma(\zeta) U = \Gamma^{\mathrm{sc}}(\zeta) J_\zeta U$ with

$$\Gamma^{\mathrm{sc}} U = \big(\Gamma_1 u^1, \Gamma_2 u^2, K_d u^3 + \Lambda^{-1} K_{\mathrm{tg}}(\eta) u^2\big). \tag{2.61}$$

Thus (2.59) reads

$$\forall U \in J_\zeta \mathbb{E}^-(\zeta): \quad |U| \leqslant C \big| J_\zeta \Gamma(\zeta) J_\zeta^{-1} U \big|. \tag{2.62}$$

Introducing the *rescaled Evans function*

$$D^{\mathrm{sc}}(\zeta) = \big|\det\big(J_\zeta \mathbb{E}^-(\zeta), J_\zeta \ker \Gamma(\zeta)\big)\big|, \tag{2.63}$$

we see that this stability condition is equivalent to the following definition:

Definition 2.23. Given a profile w, the linearized equation (2.17) satisfies the uniform spectral stability condition for high frequencies when there are $c > 0$ and $\rho_1 > 0$ such that $D^{\mathrm{sc}}(\zeta) \geqslant c$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $|\zeta| \geqslant \rho_1$.

Note that for ζ in bounded sets, J_ζ and J_ζ^{-1} are uniformly bounded and $D(\zeta) \approx D^{\mathrm{sc}}(\zeta)$, thus the condition $D^{\mathrm{sc}}(\zeta) \neq 0$ is nothing but a reformulation of the weak stability condition.

By Lemma 2.19, the high-frequency uniform stability is equivalent to (2.59). In Section 7, we will recall from [7] that *the uniform spectral stability implies the high-frequency maximal estimates* (2.58)*, under structural assumptions on the system that are satisfied in many examples*, including Navier–Stokes and MHD.

Remark 2.24. The structural assumptions we refer to are connected with well-posedness of the initial-value problem for the viscous equations. For shock waves, they by themselves guarantee spectral stability and maximal estimates [7]. For boundary-value problems, they reduce spectral stability to well-posedness of the frozen-coefficient boundary-value problem at the boundary; see [8,20] for further discussion.

2.4.3. The inviscid case

There are analogous definitions for the linearized hyperbolic problem (2.33) with boundary conditions (2.42). Recall that the Lopatinski determinant is defined at (2.48). Definition 2.20 of weak stability is strengthened as follows.

Definition 2.25. The linearized equation (2.33), (2.42) satisfies the uniform spectral stability condition when there are $c > 0$ such that $D_{\mathrm{Lop}}(\zeta) \geqslant c$ for all $\zeta \in S^d_+ := S^d \cap \{\gamma > 0\}$.

This uniform stability condition is equivalent to a uniform estimate for all $\zeta \in S^d_+$:

$$\forall u \in \mathbb{E}^-\big(H_0(\underline{u}, \zeta)\big): \quad |u| \leqslant C|\underline{\Gamma}_{\mathrm{red}} u|. \tag{2.64}$$

The expected maximal estimates for solutions of (2.33), (2.42) are

$$\gamma^{\frac{1}{2}} \|u\|_{L^2} + \big|u(0)\big| \leqslant C\big(\gamma^{-\frac{1}{2}} \|f\|_{L^2} + |h|\big) \tag{2.65}$$

with C independent of $\zeta \in \mathbb{R}^{d+1}_+$.

2.5. *The Zumbrun–Serre–Rousset theorem and the reduced low-frequency problem*

In this section, we extend the previous results of [28] and [24] which link the low-frequency uniform stability of the viscous regularizations and the uniform stability of the limiting inviscid problem. First, we recall that the transversality of the profile is a necessary condition.

Proposition 2.26. *Given a profile w, if the low-frequency uniform spectral stability condition is satisfied, then w is transversal.*

Proof. Lemma 2.12 implies that for $\zeta \neq 0$ small enough, $\widetilde{U}$ is a solution of (2.25) if and only if ${}^t(u_H, u_P) = V^{-1}(\zeta)\widetilde{U}$ satisfies

$$\partial_z u_H = H(\underline{u}, \zeta) u_H + f_H, \tag{2.66}$$

$$\partial_z u_P = P(\underline{u}, \zeta) u_P + f_P, \tag{2.67}$$

$$\Gamma_H(\zeta) u_H(0) + \Gamma_P(\zeta) u_P(0) := \widetilde{\Gamma}(\zeta)\widetilde{U}(0) = g, \tag{2.68}$$

where ${}^t(f_H, f_P) = V^{-1}(\zeta)\widetilde{F}$ and Γ_H (respectively Γ_P) denotes the restriction of $\widetilde{\Gamma} V$ to $\mathbb{C}^N \times \{0\}$ (respectively $\{0\} \times \mathbb{C}^{N'}$). In particular,

$$\mathbb{E}^-\big(G(\underline{u}, \zeta)\big) = V(\zeta)\big(\mathbb{E}^-\big(H(\underline{u}, \zeta)\big) \oplus \mathbb{E}^-\big(P(\underline{u}, \zeta)\big)\big).$$

With (2.50), this shows that the low-frequency uniform stability condition holds if and only if there are C and $\rho_0 > 0$ such that for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $0 < |\zeta| \leqslant \rho_0$

$$\begin{aligned} &\forall u_H \in \mathbb{E}^-\big(H(\underline{u}, \zeta)\big), \quad \forall u_P \in \mathbb{E}^-\big(P(\underline{u}, \zeta)\big): \\ &|u_H| + |u_P| \leqslant C\big|\Gamma_H(\zeta) u_H + \Gamma_P(\zeta) u_P\big|. \end{aligned} \tag{2.69}$$

In particular,

$$\forall u_P \in \mathbb{E}^-\big(P(\underline{u}, \zeta)\big): \quad |u_P| \leqslant C\big|\Gamma_P(\zeta) u_P\big|. \tag{2.70}$$

By Lemma 2.12, $\mathbb{E}^-(P(\underline{u}, \zeta))$ is a smooth bundle for ζ in a neighborhood of 0. Moreover, $\widetilde{\Gamma}(\zeta)$ and $\Gamma_P(\zeta)$ are smooth around the origin. This implies that $|u_P| \leqslant C|\Gamma_P(0) u_P|$ on $\mathbb{E}^-(P(\underline{u}, 0))$, implying that condition (i) of Definition 2.13 is satisfied.

Since $\dim(\mathbb{E}^-(G(\zeta))) = \operatorname{rank} \widetilde{\Gamma}(\zeta) = N_b$, (2.69) implies that for all $h \in \mathbb{C}^{N_b}$ and all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $0 < |\zeta| \leqslant \rho_0$, there is $\widetilde{U}(\zeta) = V(\zeta)(u_H(\zeta), u_P(\zeta))$ in $\mathbb{E}^-(\zeta) \subset V(\zeta)(\mathbb{C}^N \oplus \mathbb{E}^-(P(\zeta)))$ such that $\widetilde{\Gamma}(\zeta)\widetilde{U}(\zeta) = h$ and $|\widetilde{U}(\zeta)| \leqslant c|h|$. By compactness and continuity, letting ζ tend to zero, implies that there is $\widetilde{U} = V(0)(u_H, u_P)$ in $V(0)(\mathbb{C}^N \oplus \mathbb{E}^-(P(0)))$ such that $\widetilde{\Gamma}(0)\widetilde{U} = h$, showing that condition (ii) of Definition 2.13 is also satisfied. □

Suppose that the profile w is transversal. Then, by (i) of Definition 2.13 and Remark 2.11, $\Gamma_P(\zeta)$ is an isomorphism from $\mathbb{E}^-(P(\underline{u}, \zeta))$ to its image $\mathbb{F}_{0,P}$ when $\zeta = 0$; by continuity this extends to a neighborhood of the origin and the decomposition (2.39) valid at $\zeta = 0$, extends smoothly on a neighborhood of the origin:

$$\mathbb{C}^{N_b} = \mathbb{F}_H \oplus \mathbb{F}_P(\zeta), \quad \mathbb{F}_P(\zeta) := \Gamma_P(\zeta)\mathbb{E}^-\big(P(\underline{u}, \zeta)\big). \tag{2.71}$$

Denote by $\pi_H(\zeta)$ and $\pi_P(\zeta)$ the projections associated to this splitting and define the *reduced boundary operator* as

$$\Gamma_{\mathrm{red}}(\zeta) := \pi_H(\zeta)\Gamma_H(\zeta), \tag{2.72}$$

as well as the *reduced boundary value problem*

$$\partial_z u_H - H(\underline{u}, \zeta)u_H = f_H, \qquad \Gamma_{\mathrm{red}}(\zeta)u_H(0) = h. \tag{2.73}$$

The *reduced Evans function* is

$$D_{\mathrm{red}}(\zeta) = \big|\det\big(\mathbb{E}^-\big(H(\underline{u}, \zeta)\big), \ker \Gamma_{\mathrm{red}}(\zeta)\big)\big|. \tag{2.74}$$

Definition 2.27. The reduced uniform stability condition is satisfied if $D_{\mathrm{red}}(\zeta) \geqslant c > 0$ for all $\zeta \in \overline{\mathbb{R}}^{d+1} \setminus \{0\}$ with $|\zeta|$ small enough.

This is equivalent to the condition

$$\forall u \in \mathbb{E}^-\big(H(\underline{u}, \zeta)\big): \quad |u| \leqslant C\big|\Gamma_{\mathrm{red}}(\zeta)u\big|, \tag{2.75}$$

for $\zeta \in \overline{\mathbb{R}}^{d+1} \setminus \{0\}$ small.

Theorem 2.28. *Given a profile w, the linearized equation* (2.19) *satisfies the low-frequency uniform spectral stability condition if and only if*

(i) *w is transversal,*
(ii) *the reduced problem* (2.73) *satisfies the reduced uniform stability condition.*

Proof. We have already shown that the low-frequency uniform stability requires that w is transversal. Moreover, using the splitting (2.71), we see that the uniform stability conditions (2.50) or (2.69) are equivalent to

$$|u_H| + |u_P| \leqslant C\big(|\Gamma_{\mathrm{red}} u_H| + |\Gamma_P u_P + \pi_P \Gamma_H u_H|\big) \tag{2.76}$$

for all $u_H \in \mathbb{E}^-(H)$ and $u_P \in \mathbb{E}^-(P)$ (to lighten notations we have omitted the ζ dependance). Since Γ_P is surjective from $\mathbb{E}^-(P)$ onto $\mathbb{F}_P$, for all $u_H \in \mathbb{E}^-(H)$ there is $u_P \in \mathbb{E}^-(P)$ such that $\Gamma_P u_P = -\pi_P \Gamma_H u_H$ and (2.76) implies (2.75).

Conversely, if the profile is transverse, the estimate (2.70) is valid at $\zeta = 0$ and extend by continuity to ζ in a neighborhood of 0. With (2.75), this clearly implies (2.76). □

It remains to link the reduced uniform stability condition to the uniform (Lopatinski) stability condition for the hyperbolic boundary value problem, that is for the problem (2.33) with boundary conditions (2.42). Note that these boundary conditions are given by $\underline{\Gamma}_{\mathrm{red}} = \Gamma_{\mathrm{red}}(0)$ (see Remark 2.14).

Because H vanishes at $\zeta = 0$, it is natural to use polar coordinates:

$$\zeta = \rho\check{\zeta}, \quad \rho = |\zeta|, \ \check{\zeta} \in S^d. \tag{2.77}$$

In these coordinates

$$H(\underline{u}, \zeta) = \rho \check{H}(\underline{u}, \check{\zeta}, \rho), \quad \check{H}(\underline{u}, \check{\zeta}, \rho) = H_0(\underline{u}, \check{\zeta}) + O(\rho). \tag{2.78}$$

Changing z to $\check{z} = \rho z$, $u(z)$ to $\check{u}(\check{z})$ and $f(z)$ to $\rho \check{f}(\check{z})$ the reduced problem (2.73) is equivalent to

$$\partial_{\check{z}} \check{u}_H - H(\underline{u}, \check{\zeta}, \rho)\check{u}_H = \check{f}_H, \qquad \Gamma_{\mathrm{red}}(\zeta)\check{u}_H(0) = h, \tag{2.79}$$

which, for $\rho = 0$, is exactly the inviscid problem (2.35) (2.42). We are thus led to a *nonsingular* perturbation problem.

Clearly, for $\zeta \in \overline{S}^d_+ := S^d \cap \{\check{\gamma} \geqslant 0\}$, there holds $\mathbb{E}^-(H(\underline{u}, \zeta)) = \mathbb{E}^-(\check{H}(\underline{u}, \check{\zeta}, \rho))$ and $D_{\mathrm{red}}(\zeta) = \check{D}(\check{\zeta}, \rho)$ with

$$\check{D}(\check{\zeta}, \rho) = \left|\det\left(\mathbb{E}^-\left(\check{H}(\underline{u}, \check{\zeta}, \rho), \ker \Gamma_{\mathrm{red}}(\rho\check{\zeta})\right)\right)\right|. \tag{2.80}$$

Remark 2.29. For $\check{\gamma} > 0$, $H_0(\underline{u}, \check{\zeta})$ has no eigenvalues on the imaginary axis, as a consequence of hyperbolicity (see Remark 2.4). By perturbation, this property holds true for $\check{H}(\underline{u}, \check{\zeta}, \rho)$ for ρ small enough (depending on $\check{\gamma} > 0$). This shows that the vector bundle $\mathbb{E}^-(\check{H}(\underline{u}, \check{\zeta}, \rho))$ which was defined on $\overline{S}^d_+ \times]0, \rho_0]$ has a smooth extension to $\in S^+ \times [0, \rho_0]$, as well as $\check{D}$. Comparing with the definition of the Lopatinski determinant (2.48), we see that

$$D_{\mathrm{Lop}}(\check{\zeta}) = \check{D}(\check{\zeta}, 0), \quad \text{for } \check{\gamma} > 0. \tag{2.81}$$

The next theorem, combined with Theorem 2.28, extends Rousset's theorem [24] (see also [28] for shocks).

Theorem 2.30. *Given a transverse profile w, if the reduced uniform spectral stability condition is satisfied, then the linearized hyperbolic problem* (2.33), (2.42) *satisfies reduced uniform stability condition.*

Conversely, if the linearized hyperbolic problem is uniformly stable and the vector bundle $\mathbb{E}^-(\check{H}(\underline{u},\check{\zeta},\rho))$ *has a continuous extension to* $\overline{S}^d_+ \times [0,\rho_0]$*, then the reduced uniform spectral stability condition is satisfied and the linearized problem* (2.17) *satisfies the uniform low-frequency stability condition.*

Proof. The uniform estimate (2.75) implies that

$$|u| \leqslant C\left|\Gamma_{\mathrm{red}}(\zeta)u\right|$$

for $u \in \mathbb{E}^-(\check{H}(\underline{u},\check{\zeta},\rho))$, $\check{\zeta} \in \overline{S}^d_+$ and $\rho > 0$ small. If $\check{\gamma} > 0$, every term is continuous up to $\rho = 0$ and the estimate above implies (2.64), that is

$$|u| \leqslant C\left|\Gamma_{\mathrm{red}}(0)u\right|$$

for $u \in \mathbb{E}^-(H_0(\underline{u},\check{\zeta}))$, $\check{\zeta} \in S^d_+$. This implies that the hyperbolic problem in uniformly stable.

If $\mathbb{E}^-(\check{H}(\underline{u},\check{\zeta},\rho))$ has a continuous extension to $\overline{S}^d_+ \times [0,\rho_0]$, the reduced Evans function is has a continuous extension to $\overline{S}^d_+ \times [0,\rho_0]$. The hyperbolic uniform stability and (2.81) imply that

$$\check{D}(\check{\zeta},\rho) \geqslant c > 0$$

for $\check{\zeta} \in S^d_+$ and $\rho = 0$. By continuity, this extends first to $\check{\zeta} \in \overline{S}^d_+$ and next to $\rho \in [0,\rho_1]$ for some $\rho_1 > 0$. □

Remark 2.31. It is proved in [22] that when the eigenvalues of the hyperbolic symbol $\overline{A}(u,\xi)$ have constant multiplicity, and more generally when there is a smooth K-family of symmetrizers (see the definition below), then the vector bundle $\mathbb{E}^-(\check{H}(\underline{u},\check{\zeta},\rho))$ has a continuous extension to $\rho = 0$. The main concern of this paper is to construct K-families for systems with variable multiplicity. This is possible under suitable assumptions, and therefore the two theorems above extend a result of F. Rousset [24]. However, we will also show that the bundle $\mathbb{E}$ does not always admit a continuous extension, with the result that the hyperbolic problem can be uniformly stable while the viscous problem is strongly unstable in the low-frequency regime. This seems to be a new phenomenon.

Assuming transversality of w, Theorem 2.28 implies that the uniform spectral stability for low frequency is equivalent to the spectral stability for the reduced problem. There is an analogue for maximal estimates. The *maximal estimates for the reduced problem* (2.79) read

$$(\check{\gamma}+\rho)^{\frac{1}{2}}\|\check{u}_H\|_{L^2} + \left|\check{u}_H(0)\right| \leqslant C\big((\check{\gamma}+\rho)^{-\frac{1}{2}}\|\check{f}_H\|_{L^2} + |h|\big) \tag{2.82}$$

with C independent of $\check{\zeta} \in \overline{S}^d_+$ and $\rho \in\,]0,\rho_0]$. Note that for $\rho = 0$ and $\check{\gamma} > 0$, this is the maximal estimate for the inviscid problem. Scaling back to the original variables, this estimate is equivalent to

$$\big(\gamma + |\zeta|^2\big)^{\frac{1}{2}}\|u_H\|_{L^2} + \left|u_H(0)\right| \leqslant C\big(\big(\gamma+|\zeta|^2\big)^{-\frac{1}{2}}\|f_H\|_{L^2} + |h|\big) \tag{2.83}$$

for the solutions of (2.73).

Theorem 2.32. *Suppose that the profile w is transversal. Then the maximal estimates* (2.53) *are valid for low frequencies if and only if the maximal estimates* (2.82) *for the reduced problem hold true.*

Proof. By Lemma 2.12 $P(\underline{u},\zeta)$ has no purely imaginary eigenvalues. Thus, using symmetrizers (see e.g. [20] and Section 3 below), there holds

$$\|u_P^+\|_{L^2} + |u_P^+(0)| \lesssim \|f_P^+\|_{L^2}, \tag{2.84}$$

$$\|u_P^-\|_{L^2} \lesssim \|f_P^-\|_{L^2} + |u_P^-(0)|, \tag{2.85}$$

where $\pm$ denotes the smooth projections on the spaces $\mathbb{E}^{\pm}(P(\underline{u},\zeta))$.

The splitting (2.71) implies that the boundary condition (2.68) reads

$$\pi_H g = \Gamma_{\mathrm{red}} u_H(0) + \pi_H \Gamma_P u_P^+(0),$$
$$\pi_P g = \Gamma_P u_P^-(0) + \pi_P \Gamma_H u_H(0) + \pi_H \Gamma_P u_P^+(0).$$

Moreover Γ_P is invertible on $\mathbb{E}^-(P)$, hence $|\Gamma_P u_P^-(0)| \approx |u_P^-(0)|$ and

$$|\Gamma_{\mathrm{red}} u_H(0)| \lesssim |\pi_H g| + |u_P^+(0)|,$$
$$|u_P^-(0)| \lesssim |\pi_P g| + |u_H(0)| + |u_P^+(0)|.$$

Suppose that the estimate (2.83) is satisfied. Then,

$$\varphi\|u_H\|_{L^2} + |u_H(0)| \lesssim \varphi^{-1}\|f_H\|_{L^2} + |\pi_H g| + |u_P^+(0)|.$$

With (2.84), this implies that

$$\varphi\|u_H\|_{L^2} + \|u_P^-\|_{L^2} + |u_H(0)| + |u_P^-(0)| \lesssim \varphi^{-1}\|f_H\|_{L^2} + \|f_P^-\|_{L^2} + |g| + |u_P^+(0)|.$$

Thus, with (2.84), we obtain that

$$\varphi\|u_H\|_{L^2} + \|u_P\|_{L^2} + |u_H(0)| + |u_P(0)| \lesssim \varphi^{-1}\|f_H\|_{L^2} + \|f_P\|_{L^2} + |g|.$$

Because $V(\underline{u},0)$ has the special form (2.30), $\widetilde{U} = V(u_H,u_P) = (\widetilde{U}^1,\widetilde{U}^2)$ satisfies

$$\widetilde{U}^1 = O(1)u_H + O(1)u_P, \qquad \widetilde{U}^2 = O(|\zeta|)u_H + O(1)u_P.$$

Therefore, the solutions of (2.25) satisfy

$$\varphi\|\widetilde{U}^1\|_{L^2} + \|\widetilde{U}^2\|_{L^2} + |\widetilde{U}(0)| \lesssim \varphi^{-1}\|\widetilde{F}\|_{L^2} + |g|$$

that is the maximal estimate (2.53).

Conversely, assume that the maximal estimate (2.53) is satisfied. Suppose that u_H is a solution of (2.66). By transversality, Γ_P is surjective from $\mathbb{E}^-(P,\zeta)$ to its image $\mathbb{F}_P(\zeta)$ and there exists there is $u_P(0)$ in $\mathbb{E}^-(P,\zeta)$ such that

$$\Gamma_P u_P(0) = -\pi_P \Gamma_H u_H(0) \in \mathbb{F}_P(\zeta). \tag{2.86}$$

Consider $u_P = e^{zP} u_P(0)$ which is well defined and rapidly decaying at infinity since $u_P(0) \in \mathbb{E}^-(P, \zeta)$. It is a solution of (2.67) with $f_P = 0$. Then $\widetilde{U} = V(u_H, u_P)$ is a solution of (2.25) with $\widetilde{F} = V(f_H, 0)$. Thus $(u_H, u_P) = V^{-1}\widetilde{U}$ and there holds

$$\|u_H\|_{L^2} \lesssim \|\widetilde{U}\|_{L^2}, \qquad \bigl|u_H(0)\bigr| \lesssim \bigl|\widetilde{U}(0)\bigr|, \qquad \|\widetilde{F}\|_{L^2} \lesssim \|f_H\|_{L^2}$$

and, by (2.86), $\widetilde{\Gamma}\widetilde{U}(0) = \Gamma_H u_H(0) + \Gamma_P u_P(0) = \Gamma_{\mathrm{red}} u_H(0)$. Thus the estimate (2.53) immediately implies (2.83). □

3. Low frequency analysis: The main results

This section is mainly devoted to the study of the reduced equation (2.79), which is a non-singular perturbation of the inviscid problem (2.33). Our goal is to perform an analysis without assuming constant multiplicity of eigenvalues, thus allowing for examples such as MHD. The inviscid case is considered in [21], and we want to extend the results to small viscous perturbations.

3.1. Symmetrizers

Consider the constant–coefficient linear first order system (2.25). For clarity, we drop the tildes and reserve the notation $u, U, \ldots$ for the unknowns and call $p \in \mathcal{U}$ the parameter called $\underline{u}$ in this equation, which now reads

$$\partial_z U = G(p,\zeta)U + F, \qquad \Gamma(p,\zeta)U(0) = g. \tag{3.1}$$

To prove energy estimates for the solutions of this equation, the main step is to construct *symmetrizers*. They are self adjoint matrices $\Sigma(p,\zeta)$ such that

$$\mathrm{Re}\bigl(\Sigma(p,\zeta)G(p,\zeta)\bigr) > 0. \tag{3.2}$$

The symmetrizer is adapted to the boundary conditions and provides maximal estimates for the traces when

$$\Sigma(p,\zeta) > 0 \quad \text{on } \ker\Gamma(p,\zeta). \tag{3.3}$$

The construction of such symmetrizers is in two steps: first, one constructs a family of symmetrizers Σ^κ, which is independent of the boundary conditions; second one uses the uniform Lopatinski or Evans condition, to prove that if κ is large enough then the symmetrizer is adapted to the boundary condition.

More precisely, one considers a splitting

$$\mathbb{C}^{N+N'} = \mathbb{E}^-(p,\zeta) \oplus \mathbb{E}^+(p,\zeta) \tag{3.4}$$

where $\mathbb{E}^-(p,\zeta)$ is the negative invariant space of $G(p,\zeta)$ as above while $\mathbb{E}^+(p,\zeta)$ can be chosen arbitrarily so that the splitting (3.4) holds. Denoting by $\Pi^\pm(p,\zeta)$ the projectors associate to this splitting, the family of symmetrizers Σ^κ is searched so that

$$\Sigma^\kappa \geqslant m(\kappa)\big(\Pi^+\big)^*\Pi^+ - \big(\Pi^-\big)^*\Pi^- \tag{3.5}$$

where $m(\kappa) \to +\infty$ as $\kappa \to +\infty$.

Since $\mathbb{E}^- = \ker \Pi^+$, the stability condition (2.44) which reads

$$\ker \Gamma(p,\zeta) \cap \mathbb{E}^-(p,\zeta) = \{0\} \tag{3.6}$$

is also equivalent to an estimate

$$\big|\Pi^- u\big|^2 \leqslant C\big(|\Gamma u|^2 + \big|\Pi^+ u\big|^2\big). \tag{3.7}$$

Therefore, if the family Σ^κ satisfies (3.5), then for κ large enough, there holds

$$\Sigma^\kappa \geqslant c\,\mathrm{Id} - C'\Gamma^*\Gamma, \quad c > 0 \tag{3.8}$$

and therefore Σ^κ is adapted to the boundary condition Γ.

If $\operatorname{Re}\Sigma^\kappa G \geqslant \delta_\kappa \,\mathrm{Id}$, then multiplying the equation by Σ^κ and integrating by parts yields the estimate

$$\delta_\kappa \|U\|_{L^2}^2 + c\big|U(0)\big|^2 \leqslant \frac{1}{\delta_\kappa}\|F\|_{L^2}^2 + C'|g|^2. \tag{3.9}$$

This is the sketch of the general argument. To obtain usable estimates, uniform versions of (3.5), (3.7) are needed as well as more precise versions of (3.2) (see below). Note that in this approach, the construction of symmetrizers is completely independent of the boundary conditions, and in particular of the validity of the stability conditions. In this paper, we concentrate on the construction of families of symmetrizers which satisfy (3.5). They are called K-families in [21].

3.2. Main results

The construction of symmetrizers for middle frequencies, is performed in [20]. By Lemma 2.12, the matrix $G(p,\zeta)$ has no eigenvalues on the imaginary axis when $\zeta \in \overline{\mathbb{R}}^{d+1} \setminus \{0\}$. Therefore,

Lemma 3.1. *For all $\underline{\zeta} \in \overline{\mathbb{R}}^{d+1} \setminus \{0\}$, there is a neighborhood of $(\underline{p},\underline{\zeta})$ in $\mathcal{U} \times \mathbb{R}^{d+1}$ such that for (p,ζ) in this neighborhood there is a smooth splitting*

$$\mathbb{C}^{N'} = \mathbb{E}^-(p,\zeta) \oplus \mathbb{E}^+(p,\zeta), \tag{3.10}$$

where $\mathbb{E}^\pm(p,\zeta)$ denote the invariant space of $G(p,\zeta)$ associated to the spectrum in $\{\pm \operatorname{Re}\mu > 0\}$. Denoting by $\Pi^\pm(p,\zeta)$ the smooth spectral projectors associate to this splitting, *there is a smooth family* $\Sigma^\kappa(p,\zeta)$ *of self adjoint matrices such that for all* (p,ζ) *in the given neighborhood and all* $\kappa \geqslant 1$:

$$\begin{aligned} &\text{(i)} \quad \operatorname{Re}\Sigma^\kappa G > 0, \\ &\text{(ii)} \quad \operatorname{Re}\Sigma^\kappa \geqslant \kappa\big(\Pi^+\big)^*\Pi^+ - \big(\Pi^-\big)^*\Pi^-. \end{aligned} \tag{3.11}$$

Corollary 3.2. *If the weak spectral stability condition is satisfied, then for all* $\underline{\zeta} \in \overline{\mathbb{R}}^{d+1} \setminus \{0\}$, *there are a constant* C *and a neighborhood of* $(\underline{p},\underline{\zeta})$ *in* $\mathcal{U} \times \mathbb{R}^{d+1}$ *such that for* (p,ζ) *in this neighborhood the solutions of* (3.1) *satisfy*

$$\|U\|_{L^2} + \big|U(0)\big| \leqslant C\big(\|F\|_{L^2} + |g|\big). \tag{3.12}$$

We now concentrate on low frequencies. By Lemma 2.12, the matrix $G(p,\zeta)$ is locally smoothly conjugated to a block diagonal matrix (2.28) with diagonal blocks with $H(p,\zeta)$ of dimension $N \times N$ and $P(p,\zeta)$ of dimension $N' \times N'$. The system (3.1) is therefore equivalent to Eqs. (2.66), (2.67) coupled by the boundary conditions (2.68).

In the block diagonal reduction (2.28), we construct symmetrizers

$$\Sigma^\kappa = \begin{pmatrix} \Sigma_H^\kappa & 0 \\ 0 & \Sigma_P^\kappa \end{pmatrix} \tag{3.13}$$

such that the properties (3.2) and (3.5) are satisfied for each block independently.

The construction of symmetrizers for the elliptic block P is standard and identical to the construction for middle frequencies, since $P(\underline{p},0)$ has no eigenvalues on the imaginary axis. Denote by $\mathbb{E}_P^\pm(p,\zeta)$ the subspaces of $\mathbb{C}^{N'}$, invariant for $P(p,\zeta)$, associated to the spectrum in $\{\pm \operatorname{Re}\mu > 0\}$. Thus, for (p,ζ) in a neighborhood of $(\underline{p},0)$, there is a smooth splitting

$$\mathbb{C}^{N'} = \mathbb{E}_P^- \oplus \mathbb{E}_P^+. \tag{3.14}$$

Denote by $\Pi_P^\pm(p,\zeta)$ the smooth spectral projectors associate to this splitting.

Proposition 3.3. *There is a smooth family of self adjoint matrices* Σ_P^κ *on a neighborhood of* $(\underline{p},0)$ *such that*

$$\begin{aligned} &\text{(i)} \quad \operatorname{Re}\Sigma_P^\kappa P > 0, \\ &\text{(ii)} \quad \operatorname{Re}\Sigma_P^\kappa \geqslant \kappa\big(\Pi_P^+\big)^*\Pi_P^+ - \big(\Pi_P^-\big)^*\Pi_P^-. \end{aligned} \tag{3.15}$$

This implies the estimates (2.84), (2.85) which where used in the previous section.

To analyze H, we use polar coordinates for $\zeta = \rho\check{\zeta}$ as in (2.77) so that

$$H(p,\zeta) = \rho\check{H}(p,\check{\zeta},\rho), \quad \check{H}(p,\check{\zeta},\rho) = H_0(p,\check{\zeta}) + O(\rho). \tag{3.16}$$

By Lemma 2.12, for $\zeta \in \overline{\mathbb{R}}_+^{d+1} \setminus \{0\}$, $\check{H}$ has no eigenvalue on the imaginary axis, hence the number N^- of eigenvalues of $\check{H}$ in $\{\operatorname{Re}\mu < 0\}$ is constant.

We fix a point $\check{\underline{\zeta}} \in \overline{S}^d_+$, that is $\check{\underline{\zeta}} = (\check{\underline{\tau}}, \check{\underline{\eta}}, \check{\underline{\gamma}})$ in the unit sphere with $\check{\underline{\gamma}} \geqslant 0$. The goal is to construct smooth symmetrizers for $\check{H}$, for $(p, \check{\zeta}, \rho)$ close to $(\underline{p}, \check{\underline{\zeta}}, 0)$. For convenience we introduce the following terminology.

Definition 3.4. A smooth symmetrizer for $\check{H}$ on a neighborhood ω of $(\underline{p}, \check{\underline{\zeta}}, 0)$ is a smooth self adjoint matrix $\check{\Sigma}^H(p, \check{\zeta}, \rho)$ such that

$$\operatorname{Re} \check{\Sigma}^H \check{H} = \sum V_k^* \Sigma_k V_k, \tag{3.17}$$

where the V_k and Σ_k are smooth matrices on ω of appropriate dimension so that the products make sense, satisfying

(i) $\sum V_k^* V_k$ is definite positive,
(ii) either Σ_k is definite positive or $\Sigma_k = \gamma \Sigma_{k,1} + \rho \Sigma_{k,2}$ with $\Sigma_{k,1}$ and $\Sigma_{k,2}$ definite positive.

Definition 3.5. A family of smooth symmetrizers Σ^κ on neighborhoods ω^κ of $(\underline{p}, \check{\underline{\zeta}}, 0)$ is called a K-family of symmetrizers for $\check{H}$ if there are a decomposition

$$\mathbb{C}^N = \underline{\mathbb{E}}_H^- \oplus \underline{\mathbb{E}}_H^+ \tag{3.18}$$

with $\dim \underline{\mathbb{E}}^- = N_-$ and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$ such that for all κ

$$\Sigma^\kappa(\underline{p}, \check{\underline{\zeta}}, 0) \geqslant m(\kappa) \Pi_+^* \Pi_+ - \Pi_-^* \Pi_-, \tag{3.19}$$

where $\Pi_\pm$ are the projectors associated to the splitting (3.18).

Remark 3.6. Recall from [22] that if there is K-family of symmetrizers, then $\underline{\mathbb{E}}^-$ is the limit of the negative spaces $\mathbb{E}^-(p, \check{\zeta}, \rho)$ as $(p, \check{\zeta}, \rho)$ tends to $(\underline{p}, \check{\underline{\zeta}}, 0)$ with $\rho > 0$. Thus $\underline{\mathbb{E}}^-$ is uniquely determined. On the other hand, $\underline{\mathbb{E}}^+$ is arbitrary, provided that the splitting (3.18) holds: if (3.19) holds for some choice of $\underline{\mathbb{E}}^+$, then it also holds for another choice for a multiple of Σ^κ with some other function $m(\kappa)$.

We can now state the main result of this paper, which extends [21].

Theorem 3.7. *Suppose that the assumptions of Section* 2.1 *are satisfied. Assume further that one of the following two condition is satisfied*:

(i) *all the real characteristic roots* $(\underline{p}, \tau, \xi)$ *with* $|\xi| = 1$ *satisfy the condition* (BS) *of Definition* 4.9.
(ii) *the system is symmetric dissipative in the sense of Definition* 2.5 *and the real characteristic roots* $(\underline{p}, \tau, \xi)$ *with* $|\xi| = 1$ *are either totally nonglancing in the sense of Definition* 4.3 *or satisfy the condition* (BS) *of Definition* 4.9.

Then, for all $\check{\underline{\zeta}} \in \overline{S}^d_+$, *there exists K-families of smooth symmetrizers for* $\check{H}(p, \zeta, \rho)$ *near* $(\underline{p}, \check{\underline{\zeta}}, 0)$.

The condition (BS) ensures that a suitable generalized block structure condition is satisfied. From a technical point of view, this condition makes the construction of symmetrizers given in [20] work. For hyperbolic problems, the block structure condition was introduced by A. Majda and S. Osher [16] as the technical condition which allows to construct Kreiss symmetrizers (see [12]). In [21], it is shown that the block structure condition is satisfied if and only if the system is smoothly diagonalizable. In the viscous case, things are more subtle and the generalized block structure condition is discussed in details in Section 3. We just point out here the following example.

Theorem 3.8. *If* $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ *is a semi-simple characteristic root of constant multiplicity, then the condition* (BS) *of Definition* 4.9 *is satisfied at that point.*

Together with Theorem 3.7, this implies Theorem 1.1. Finally, we quote that the existence of K-families implies the validity of the maximal estimates when the boundary conditions satisfy the uniform spectral stability conditions.

Theorem 3.9. *Suppose that there exists a K-families of symmetrizers for* $\check{H}$ *near* $(\underline{p}, \underline{\check{\zeta}}, 0)$ *and suppose that the boundary conditions are such that the uniform spectral stability condition is satisfied for low frequencies. Then the uniform stability estimates* (2.53) *are satisfied.*

Similarly, if the reduced boundary conditions satisfy the reduced uniform stability condition then the uniform estimates (2.82) *and* (2.83) *hold true.*

3.3. Block reductions

The advantage of the notion of K-families is that it is independent of the boundary conditions. Therefore, their construction depend only on an analysis of $\check{H}$. In particular, we can use spectral block decompositions of $\check{H}$.

Fix $\underline{\check{\zeta}} \in \overline{S}^d_+$. Consider the *distinct* eigenvalues $\underline{\mu}_k$ of $H_0(\underline{p}, \underline{\check{\zeta}})$. For $(p, \check{\zeta}, \rho)$ in a neighborhood of $(\underline{p}, \underline{\check{\zeta}}, 0)$, there is a smooth block reduction

$$V^{-1} \check{H} V = \operatorname{diag}(\check{H}_k) \tag{3.20}$$

where the H_k have their spectrum in small discs centered at $\underline{\mu}_k$ that are pairwise disjoints. Equivalently, there is a smooth decomposition

$$\mathbb{C}^N = \bigoplus_k \mathbb{E}_k(p, \check{\zeta}, \rho) \tag{3.21}$$

in invariant spaces for $\check{H}(p, \check{\zeta}, \rho)$ and $\check{H}_k$ is the restriction of $\check{H}$ to $\mathbb{E}_k$. We denote by N_k the dimension of $\mathbb{E}_k$, that is the size of $\check{H}_k$.

The K-families of symmetrizers are constructed for each block $\check{H}_k$ separately. If Σ_k^κ is a K-family for $\check{H}_k$, it is clear that $\Sigma^\kappa = V^* \operatorname{diag}(\Sigma_k^\kappa) V$ has the form (3.17) and is a K-family for $\check{H}$.

When the mode is *elliptic*, that is when $\operatorname{Re} \underline{\mu}_k \neq 0$, the construction of symmetrizers is easy (see e.g. [1,12,20]).

Proposition 3.10. *Suppose that $\underline{\mu}_k$ is an eigenvalue of $H_0(\underline{p}, \underline{\check{\zeta}})$ with $\operatorname{Re} \underline{\mu}_k \neq 0$. Then is a smooth family of self adjoint matrices Σ_k^κ on a neighborhood of $(\underline{p}, \underline{\check{\zeta}}, 0)$ such that*

$$\begin{aligned} &\text{(i)} \quad \operatorname{Re}\big(\Sigma_k^\kappa \check{H}_k\big) > 0, \\ &\text{(ii)} \quad \operatorname{Re} \Sigma_k^\kappa \geqslant \kappa \operatorname{Id} \quad \textit{if } \operatorname{Re} \underline{\mu}_k > 0, \\ &\qquad\ \ \operatorname{Re} \Sigma_k^\kappa \geqslant -\operatorname{Id} \quad \textit{if } \operatorname{Re} \underline{\mu}_k < 0. \end{aligned} \tag{3.22}$$

Therefore we now restrict our attention to a nonelliptic mode:

$$\underline{\mu}_k = i\underline{\check{\xi}}_d, \quad \underline{\check{\xi}}_d \in \mathbb{R}. \tag{3.23}$$

By definition of H_0, this implies that $-\underline{\check{\tau}} + i\underline{\check{\gamma}}$ is an eigenvalue $\underline{\lambda}$ of $A(\underline{p}, \underline{\check{\xi}})$ with $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$. In particular, by hyperbolicity, this can only happen when $\underline{\check{\gamma}} = 0$. By Lemma 2.12, $\check{H}_k$ has no eigenvalues on the imaginary axis when $\rho > 0$, thus the number of eigenvalues in $\{\operatorname{Re} \mu < 0\}$ is constant. We call it N_k^-. The next definition reformulates Definitions 3.4 and 3.5 for nonelliptic blocks $\check{H}_k$.

Definition 3.11. A smooth symmetrizer for a nonelliptic block $\check{H}_k$ on a neighborhood ω of $(\underline{p}, \underline{\check{\zeta}}, 0)$ is a smooth self adjoint matrix $\Sigma(p, \check{\zeta}, \rho)$ such that for all $(p, \check{\zeta}, \rho) \in \omega$:

$$\operatorname{Re} \Sigma \check{H}_k = \check{\gamma} \Sigma_1 + \rho \Sigma_2, \tag{3.24}$$

with $\Sigma_1(\underline{p}, \underline{\check{\zeta}}, 0)$ and $\Sigma_2(\underline{p}, \underline{\check{\zeta}}, 0)$ definite positive.

A family of smooth symmetrizers Σ_k^κ on neighborhoods ω^κ of $(\underline{p}, \underline{\check{\zeta}}, 0)$ is called a K-family of symmetrizers for $\check{H}_k$ if there are a decomposition

$$\mathbb{E}_k(\underline{p}, \underline{\check{\zeta}}, 0) = \underline{\mathbb{E}}_k^- \oplus \underline{\mathbb{E}}_k^+ \tag{3.25}$$

with $\dim \underline{\mathbb{E}}_k^-$ equal to N_k^- and $m(\kappa) \to +\infty$ as $\kappa \to +\infty$ such that for all κ

$$\Sigma_k^\kappa(\underline{p}, \underline{\check{\zeta}}, 0) \geqslant m(\kappa)\big(\Pi_k^+\big)^* \Pi_k^+ - \big(\Pi_k^-\big)^* \Pi_k^-, \tag{3.26}$$

where $\Pi_k^\pm$ are the projectors associated to the splitting (3.25).

Given $\underline{\check{\zeta}} = (\underline{\check{\tau}}, \underline{\check{\eta}}, 0) \in \overline{S}_+^d$ and a nonelliptic mode $\underline{\mu}_k = i\underline{\check{\xi}}_d$, $-\underline{\check{\tau}}$, is an eigenvalue of $A(\underline{p}, \underline{\check{\xi}})$ with $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$. Therefore, Theorem 3.7, is an immediate corollary of Proposition 3.10 and the following two theorems.

Theorem 3.12. *Suppose that the system is symmetric dissipative in the sense of Definition* 2.5*; suppose in addition that $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is a totally incoming or outgoing characteristic root in the sense of Definition* 4.3*. Then there are K-families of symmetrizers for the associated block $\check{H}_k$, with $\underline{\mathbb{E}}_k^- = \{0\}$ in the outgoing case and $\underline{\mathbb{E}}_k^- = \mathbb{C}^{N_k}$ in the incoming case.*

Theorem 3.13. *If $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ is a characteristic root which satisfies the generalized block structure condition of Definition 4.22. Then there are K-families of symmetrizers for the associated block $\check{H}_k$.*

4. The generalized block structure condition

4.1. Hyperbolic multiple roots

We first recall from [21] several notations and definitions concerning the characteristic roots of the hyperbolic part L. For simplicity, we suppose, as we may, that the coefficient of ∂_t is $A_0 = \mathrm{Id}$, so that, with notations (2.3), $L = \overline{L}$. The characteristic determinant is denoted by

$$\Delta(p, \tau, \xi) := \det\bigl(\tau\, \mathrm{Id} + A(p, \xi)\bigr). \tag{4.1}$$

Definition 4.1. Consider a root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of $\Delta(\underline{p}, \underline{\tau}, \underline{\xi}) = 0$, of algebraic multiplicity m in τ.

(i) $(\underline{p}, \underline{\tau}, \underline{\xi})$ is algebraically regular, if on a neighborhood ω of $(\underline{p}, \underline{\xi})$ there are m smooth real functions $\lambda_j(p, \xi)$, analytic in ξ, such that $\lambda_j(\underline{p}, \underline{\xi}) = -\underline{\tau}$ and for $(p, \xi) \in \omega$:

$$\Delta(p, \tau, \xi) = e(p, \tau, \xi) \prod_{j=1}^{m} \bigl(\tau + \lambda_j(p, \xi)\bigr) \tag{4.2}$$

where e is a polynomial in τ with smooth coefficients such that $e(\underline{p}, \underline{\tau}, \underline{\xi}) \neq 0$.

(ii) $(\underline{p}, \underline{\tau}, \underline{\xi})$ is geometrically regular if in addition there are m smooth functions $e_j(p, \xi)$ on ω with values in $\mathbb{C}^N$, analytic in ξ, such that

$$A(p, \xi) e_j(p, \xi) = \lambda_j(p, \xi) e_j(p, \xi), \tag{4.3}$$

and the $e_1, \ldots, e_m$ are linearly independent.

(iii) $(\underline{p}, \underline{\tau}, \underline{\xi})$ is semi-simple with constant multiplicity if all the λ_j's are equal.

Case (iii) occurs when $\lambda(p, \xi)$ is a continuous semi-simple eigenvalue of $A(p, \xi)$ with constant multiplicity near $(\underline{p}, \underline{\xi})$, such $\underline{\tau} + \lambda(\underline{p}, \underline{\xi}) = 0$. This implies that λ is smooth and analytic in ξ as well as the eigenspace $\ker(A - \lambda)$. In this case, one can choose for $\{e_j\}$ any smooth basis of this eigenspace.

If all the roots at $(\underline{p}, \underline{\xi})$ are geometrically regular, then, locally near $(\underline{p}, \underline{\xi})$, $A(p, \xi)$ is smoothly diagonalizable, meaning that it has a smooth basis of eigenvectors.

Example 4.2. For the inviscid MHD, the multiple eigenvalues are algebraically regular, but some are not geometrically regular (see [21] and Section 8 below).

The second notion which plays an important role in the analysis of hyperbolic boundary value problems is the notion of *glancing modes*. Recall from [21] the following definition. If $\underline{\tau}$ is a root of multiplicity m of the polynomial $\Delta(\underline{p}, \cdot, \underline{\xi})$, then by hyperbolicity, the Taylor expansion of Δ at $(\underline{p}, \underline{\tau}, \underline{\xi})$ at the order $m - 1$ vanishes so that

$$\Delta(\underline{p}, \underline{\tau}+\tau, \underline{\xi}+\xi) = \underline{\Delta}_m(\tau,\xi) + O\big(|\tau,\xi|^{m+1}\big) \tag{4.4}$$

and $\underline{\Delta}_m$ is homogeneous of degree m. Moreover, $\underline{\Delta}_m$ is hyperbolic in the time direction. Indeed, any direction of hyperbolicity for $\Delta(\underline{p},\cdot)$ is a direction of hyperbolicity for $\underline{\Delta}_m$. Denote by $\underline{\Gamma}_+$ the open convex cone of hyperbolic directions for $\underline{\Delta}_m$ which contains dt.

Definition 4.3. The root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of Δ, of multiplicity m, is said nonglancing when the boundary is noncharacteristic for $\underline{\Delta}_m$.

It is totally incoming (respectively outgoing) when the inward (respectively outward) conormal to the boundary belongs to $\underline{\Gamma}_+$. It is totally nonglancing if is either totally incoming or totally outgoing.

Example 4.4. This definition agrees with the usual one for simple roots, given by $\tau + \lambda(p,\xi) = 0$. In this case $\partial_t + \nabla_\xi \lambda \cdot \partial_x$ is the Hamiltonian transport field for the propagation of singularities or oscillations and the glancing condition $\partial_{\xi_d}\lambda = 0$ precisely means that the field is tangent to the boundary. More generally, if the root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of Δ is algebraically regular, then, with notations as in (4.2)

$$\underline{\Delta}_m(\tau,\xi) = e(\underline{p}, \underline{\tau}, \underline{\xi}) \prod_{j=1}^{m} \big(\tau + \xi \cdot \nabla_\xi \lambda_j(\underline{p}, \underline{\xi})\big). \tag{4.5}$$

The mode is nonglancing if none of the tangential speed $\partial_{\xi_d}\lambda_j(\underline{p}, \underline{\xi})$ vanish. It is totally incoming (respectively outgoing) if they all are positive (respectively negative). In particular, in the constant multiplicity case, all the λ_j are equal and they are all glancing, incoming or outgoing at the same time.

In the study of boundary value problems, the dichotomy incoming vs outgoing plays a crucial role: for instance, for transport equations one boundary condition is needed in the first case and none in the second. Using symmetrizers to prove energy estimates, they are constructed in opposite ways. The general Kreiss construction also reflects this dichotomy. Introduce the following definition:

Definition 4.5. Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is an algebraically regular root of Δ. With notations as in (4.2), denote by ν_j the order of $\underline{\xi}_d$ is a root of order of the equation $\underline{\tau} + \lambda_j(\underline{p}, \underline{\xi}_1, \ldots, \underline{\xi}_{d-1}, \cdot) = 0$, that is the positive integer such that

$$\partial^a_{\xi_d}\lambda_j(\underline{p}, \underline{\xi}) = 0 \quad \text{for } a < \nu_j \quad \text{and} \quad \beta_j := \frac{1}{\nu_j!}\partial^{\nu_j}_{\xi_d}\lambda_j(\underline{p}, \underline{\xi}) \neq 0. \tag{4.6}$$

We say that λ_j is of type I when either ν_j is even or ν_j is odd and $\beta_j > 0$. It is of type O when ν_j is odd and $\beta_j < 0$.

We denote by J_O (respectively J_I) the set of indices j of the corresponding type.

Remark 4.6. When $(\underline{p}, \check{\underline{\tau}}, \check{\underline{\xi}})$ is nonglancing, then the all the ν_j are equal to 1, and being of type I (respectively type O) means to be incoming (respectively outgoing). They are all of the same type exactly when the mode is totally nonglancing.

Remark 4.7. The details of the construction of Kreiss' symmetrizers depend strongly on being of type I or O, see [1,12,19] and Section 5. There are no reason other than technical why even roots are of type I rather than O.

4.2. The decoupling condition

The spectral properties of $A(\xi)$ are modified by the perturbation B. In particular, since the construction of symmetrizers depends deeply on the property of being incoming/outgoing, it is very important that the perturbation respects the decoupling between the different type of modes.

Definition 4.8. Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is a geometrically regular root of Δ of order m. Consider a basis $\{e_j\}$ as in (4.3) and dual left eigenvectors $\underline{\ell}_j$ such that

$$\underline{\ell}_j\big(\underline{\tau}\,\mathrm{Id} + A(\underline{p}, \underline{\xi})\big) = 0, \qquad \underline{\ell}_j \cdot e_{j'}(\underline{p}, \underline{\xi}) = \delta_{j,j'}. \tag{4.7}$$

Consider the and the $m \times m$ matrix with entries

$$B^{\sharp}_{j,j'} = \underline{\ell}_j B(\underline{p}, \underline{\xi}) e_{j'}(\underline{p}, \underline{\xi}). \tag{4.8}$$

(i) We say that the decoupling condition is satisfied if

$$B^{\sharp}_{j,j'} = 0 \quad \text{when } (j, j') \in (J_O \times J_I) \cup (J_I \times J_O) \tag{4.9}$$

where J_O and J_I are introduced in Definition 4.5.

(ii) We say that the basis $\{e_j\}$ is adapted to B if

$$\mathrm{Re}\, B^{\sharp} > 0. \tag{4.10}$$

Definition 4.9. We say that the root $(\underline{p}, \underline{\tau}, \underline{\xi})$ of Δ satisfies the condition (BS) if it is geometrically regular root, satisfies the decoupling condition (4.9) and there is an eigenbasis basis $\{e_j\}$ adapted to B.

We give several examples and counterexamples. The next result rephrases Theorem 3.8.

Proposition 4.10 *(Constant multiplicity). Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is a semi-simple characteristic root with constant multiplicity of Δ. Then the condition* (BS) *is satisfied.*

Proof. For semi-simple characteristic root λ with constant multiplicity either J_O or J_I is empty so that the decoupling condition (4.9) is trivially satisfied. The perturbation argument for the spectrum of $i\underline{A} + \rho\underline{B}$, used in [20] for fully parabolic viscosity, applies to the general case and the Assumption (H4) for small frequencies implies that the spectrum of $B^{\sharp}$ is located in $\{\mathrm{Re}\, z > 0\}$. Thus there is a basis $\{\underline{e}_j\}$ in $\ker(A(\underline{p}, \underline{\xi}) + \underline{\tau}\,\mathrm{Id})$ such that $\mathrm{Re}\, B^{\sharp}$ is definite positive. Next, since any smooth basis $\{e_j\}$ in $\ker(A - \lambda)$ satisfies (4.3), one can choose it such that $e_j(\underline{p}, \underline{\xi}) = \underline{e}_j$. □

Proposition 4.11 *(Artificial viscosity). Suppose that $(\underline{p}, \underline{\tau}, \underline{\xi})$ is geometrically regular for $iA + B$ in the sense that there are m smooth functions $\lambda_j(p, \xi, \rho)$ and m linearly independent smooth* *vectors $e_j(p,\xi,\rho)$ on a neighborhood of $(\underline{p},\underline{\xi},\rho)$, analytic in ξ, such that $\lambda_j(\underline{p},\underline{\xi},0)=-\underline{\tau}$ for all j and*

$$\big(iA(p,\xi)+\rho B(p,\xi)\big)e_j(p,\xi,\rho)=i\lambda_j(p,\xi,\rho)e_j(p,\xi,\rho). \tag{4.11}$$

Then, the decoupling condition is satisfied and the basis $\{e_j|_{\rho=0}\}$ is adapted to B.

Proof. Alternately, differentiating (4.3) with respect to ρ and multiplying on the left by $\underline{\ell}_{j'}$, implies that $B^\sharp_{j',j}=0$ when $j\neq j'$. Moreover, (H1) implies that $B^\sharp_{j,j}>0$. □

For example, if $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular for A in the sense of Definition 4.1 and if $B=\Delta_x\,\mathrm{Id}$ is an artificial viscosity, then $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular for $iA+B$. However, this condition is too restrictive for applications, in particular when A and B do not commute.

Example 4.12. If the root is totally nonglancing, then the decoupling condition is trivially satisfied since either J_I or J_O is empty. This applies to fast shocks in MHD.

Counterexample 4.13. Slow shocks in MHD do not satisfy the decoupling condition, see Section 8.

The decoupling condition is crucial in the construction of symmetrizers. The second condition (4.10) is more technical. One could expect that with the positivity Assumption (H1), one could always find an adapted basis. This is not clear, except for multiplicity 2 or symmetric systems.

Proposition 4.14. *Suppose that $(\underline{p},\underline{\tau},\underline{\xi})$ is geometrically regular of multiplicity m. Assume that either $m=2$ or that the system is symmetric dissipative in the sense of Definition* 2.5. *Then, there is a basis $\{e_j\}$ adapted to B.*

If in addition all the eigenvalues λ_j are of the same type O or I, then the condition (BS) *is satisfied.*

The proof is given Section 6.

4.3. The hyperbolic block structure condition

We turn back to the construction of symmetrizers for nonelliptic blocks $\check{H}_k$ in the splitting (3.20). The construction of K-families is performed in [20] provided that $\check{H}_k$ can be put in a suitable normal form. This is the so-called *block structure condition*. We first review this condition in the hyperbolic case, and next extend it to the hyperbolic–parabolic case.

Consider $\underline{p}$ and a frequency $\underline{\check{\zeta}}=(\underline{\check{\tau}},\underline{\check{\eta}},0)\neq 0$ and a purely imaginary eigenvalue (3.23) $\underline{\mu}_k=i\underline{\check{\xi}}_d$ of $H_0(\underline{p},\underline{\check{\zeta}})$. Let $\underline{\check{\xi}}=(\underline{\check{\eta}},\underline{\check{\xi}}_d)$. Then $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$ is a root of Δ. We consider the block $\check{H}_k$ associated to $\underline{\mu}_k$ and denote by $\mathbb{E}_k$ the corresponding invariant space of $\check{H}$. We use the notations $\check{H}_{k,0}(p,\check{\zeta})=\check{H}_k(p,\check{\zeta},0)$ and $\mathbb{E}_{k,0}(p,\check{\zeta})=\mathbb{E}_k(p,\check{\zeta},0)$.

Definition 4.15. $\check{H}_{k,0}$ has the block structure property near $(\underline{p},\underline{\check{\zeta}})$ if there exists a smooth invertible matrix $V_{k,0}$ on a neighborhood of that point such that $V_{k,0}^{-1}\check{H}_{k,0}V_{k,0}$ is block diagonal,

$$V_{k,0}^{-1}\check{H}_{k,0}V_{k,0} = \begin{bmatrix} Q_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & Q_{m'} \end{bmatrix}, \tag{4.12}$$

with diagonal blocks Q_j of size $\nu_j \times \nu_j$ such that:

$Q_j(p, \check{\zeta})$ has purely imaginary coefficients when $\check{\gamma} = 0$,

$$Q_j(\underline{p}, \check{\zeta}) = \underline{\mu}_k \,\mathrm{Id} + i \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & \cdots & 0 \end{bmatrix}, \tag{4.13}$$

and the real part of the lower left-hand corner of $\partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\zeta})$, denoted by $q_j^\flat$, does not vanish.

When $\nu_j = 1$, $Q_j(p, \check{\zeta})$ is a scalar. In this case, (4.13) has to be understood as $Q_j(\underline{p}, \check{\zeta}) = \underline{\mu}_k$, with no Jordan's block. The lower left-hand corner of the matrix is Q_j itself and the condition reads $q_j^\flat := \partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\check{\zeta}}) \neq 0$.

Proposition 4.16. *(See [21].) If the root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is geometrically regular in the sense of Definition* 4.1*, the corresponding block $\check{H}_{k,0}$ satisfies the block structure condition.*

Conversely, if $\check{H}_{k,0}$ satisfies the block structure condition with matrices V that are real analytic in $\check{\zeta}$, then the root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is geometrically regular.

Remark 4.17. There is a slight discrepancy here between the necessary and the sufficient condition, due to analyticity conditions. Definition 4.1 requires analyticity in $\check{\xi}$. This is used in the proof of sufficiency. In addition, it implies that the block structure condition holds with matrices V that are real analytic in $\check{\zeta}$. Thus, there is an "if and only if" theorem. However, for the construction of symmetrizers, analyticity of V_k is not needed, this is why we do not insist on it in the definition above. In addition, note that *for fixed* p, the existence of C^∞ eigenvalues and eigenvectors for A, implies that these eigenvalues are real analytic in ξ and that one can choose analytic eigenvectors (see e.g. [17,25]). The question is to control the domain of analyticity as p varies. In applications, for this problem, proving analyticity is not harder than proving the C^∞ smoothness.

To prepare the hyperbolic–parabolic analysis, we have to review the proof of Proposition 4.16 (see [18,21]). In particular, we reformulate the conditions of Definition 4.15 in a more intrinsic way. The choice of a smooth matrix $V_{k,0}$ is equivalent to the choice of a smooth basis of $\mathbb{E}_{k,0}$, denoted by $\{\varphi_{j,a}(p, \check{\zeta})\}_{1 \leqslant j \leqslant m', 1 \leqslant a \leqslant \nu_j}$. The property (4.13) reads

$$\big(H_0(\underline{p}, \underline{\check{\zeta}}) - \underline{\mu}_k\big)\varphi_{j,1}(\underline{p}, \underline{\check{\zeta}}) = 0, \tag{4.14}$$

$$\big(H_0(\underline{p}, \underline{\check{\zeta}}) - \underline{\mu}_k\big)\varphi_{j,a}(\underline{p}, \underline{\check{\zeta}}) = i\varphi_{j,a-1}(\underline{p}, \underline{\check{\zeta}}), \quad 2 \leqslant a \leqslant \nu_j. \tag{4.15}$$

With (3.21), there is a unique smooth dual basis $\psi_{j,a}(p, \check{\zeta})$ such that

$$\psi_{j,a} \cdot \mathbb{E}'_{k,0} = 0,$$
$$\psi_{j,a} \cdot \varphi_{j',a'} = \delta_{j,j'}\delta_{a,a'}. \tag{4.16}$$

Here, $\mathbb{E}'_{k,0}$ denotes the invariant space of $H_0(p, \check{\zeta})$ such that $\mathbb{C}^N = \mathbb{E}_{k,0} \oplus \mathbb{E}'_{k,0}$. It is the sum of invariant subspaces associated to eigenvalues $\underline{\mu}_{k'} \neq \underline{\mu}_k$.

In the basis $\varphi_{j,a}$, the entries of the matrix $V_{k,0}^{-1}\check{H}_{k,0}V_{k,0}$ are $\psi_{j,a}H_0\varphi_{j',a'}$. The diagonal block structure means that

$$\psi_{j,a}H_0\varphi_{j',a'} = 0 \quad \text{when } j \neq j'. \tag{4.17}$$

The other conditions read:

$$\operatorname{Re}(\psi_{j,a}H_0\varphi_{j,a'}) = 0 \quad \text{when } \check{\gamma} = 0, \tag{4.18}$$
$$\operatorname{Re}\partial_{\check{\gamma}}(\psi_{j,\nu_j}H_0\varphi_{j,1})(\underline{p}, \underline{\check{\zeta}}) \neq 0. \tag{4.19}$$

We first show how to compute this quantity in terms of A only.

Lemma 4.18. *Suppose that $\check{H}_{k,0}$ has a block diagonal decomposition* (4.12) *in a smooth basis $\varphi_{j,a}$ of $\mathbb{E}_k(p, \check{\zeta}, 0)$ which satisfies* (4.14), (4.15). *Let $\psi_{j,a}$ denote a dual basis satisfying* (4.16). *The lower left-hand corner entry of $\partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\check{\zeta}})$ is equal to the lower left-hand corner entry of $-i\partial_{\check{\tau}} Q_j(\underline{p}, \underline{\check{\zeta}})$ and equal to*

$$\underline{q}_j = -\psi_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}})A_d^{-1}(\underline{p})\varphi_{j,1}(\underline{p}, \underline{\check{\zeta}}). \tag{4.20}$$

Proof. Let $\underline{H}_0 = H_0(\underline{p}, \underline{\check{\zeta}})$. Then $\underline{H}_0 - \underline{\mu}_k$ is invertible on $\mathbb{E}'_{k,0}(\underline{p}, \underline{\check{\zeta}})$. With (4.14), (4.15), this implies that

$$\operatorname{range}(\underline{H}_0 - \underline{\mu}_k \operatorname{Id}) = \{\psi_{1,\nu_1}(\underline{p}, \underline{\check{\zeta}}), \ldots, \psi_{m',\nu_{m'}}(\underline{p}, \underline{\check{\zeta}})\}^{\perp}, \tag{4.21}$$
$$\ker(\underline{H}_0 - \underline{\mu}_k \operatorname{Id}) = \{\varphi_{1,1}(\underline{p}, \underline{\check{\zeta}}), \ldots, \varphi_{m',1}(\underline{p}, \underline{\check{\zeta}})\}. \tag{4.22}$$

In particular,

$$(\underline{H}_0 - \underline{\mu}_k \operatorname{Id})\varphi_{j,1} = 0 \quad \text{and} \quad \psi_{j,\nu_j}(\underline{H}_0 - \underline{\mu}_k \operatorname{Id}) = 0. \tag{4.23}$$

The entry in consideration is

$$q_j(p, \check{\zeta}) = \psi_{j,\nu_j}H_0\varphi_{j,1} = \psi_{j,\nu_j}(H_0 - \underline{\mu}_k \operatorname{Id})\varphi_{j,1} + \underline{\mu}_k\delta_{\nu_j,1}.$$

Therefore, differentiating in $\check{\gamma}$ and $\check{\tau}$ and using (2.27), implies that

$$\partial_{\check{\gamma}} q_j(\underline{p}, \underline{\check{\zeta}}) = -i\partial_{\check{\tau}} q_j(\underline{p}, \underline{\check{\zeta}}) = \underline{q}_j \tag{4.24}$$

is given by (4.20). □

We now discuss how much flexibility there is in the choice of the basis $\varphi_{j,a}$. Recall that we are considering a purely imaginary eigenvalue $\underline{\mu}_k = i\underline{\xi}_d$ of $H_0(\underline{p}, \underline{\check{\zeta}})$, so that $-\underline{\check{\tau}}$ is an eigenvalue $\underline{\lambda}$ of $A(\underline{p}, \underline{\check{\xi}})$ with $\underline{\check{\xi}} = (\underline{\check{\eta}}, \underline{\check{\xi}}_d)$.

Lemma 4.19. *Suppose that $\check{H}_{k,0}$ has the block structure property near $(\underline{p}, \underline{\check{\zeta}})$ in a smooth basis $\varphi_{j,a}$ and denote by $\psi_{j,a}$ the dual basis* (4.16)*. Then,*

(i) *$\underline{\lambda}$ is a semi-simple eigenvalue of $A(\underline{p}, \underline{\check{\xi}})$ with multiplicity m equal to the number m' of blocks Q_j,*

(ii) *on a neighborhood of $(\underline{p}, \underline{\check{\xi}})$, there are m smooth eigenvalues $\lambda_j(p, \check{\xi})$ of $A(p, \check{\xi})$ and m smooth linearly independent eigenvectors $e_j(p, \check{\xi})$, such that*

$$\lambda_j(\underline{p}, \underline{\check{\xi}}) = \underline{\lambda}, \tag{4.25}$$

$$A(p, \check{\xi})e_j(p, \check{\xi}) = \lambda_j(p, \check{\xi})e_j(p, \check{\xi}), \tag{4.26}$$

$$e_j(\underline{p}, \underline{\check{\xi}}) = \varphi_{j,1}(\underline{p}, \underline{\check{\zeta}}), \tag{4.27}$$

(iii) *the order of $\underline{\check{\xi}}_d$ as a root of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\eta}}, \cdot) = 0$ is equal to ν_j,*

(iv) *denoting by $\{\underline{\ell}_j\}$ the left eigenvector dual basis of $\{e_j\}$ as in* (4.7)*, there holds*

$$\underline{\ell}_j A_d(\underline{p}) = \beta_j \psi_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}), \tag{4.28}$$

with $\beta_j := \frac{1}{\nu_j!}\partial_{\xi_d}^{\nu_j}\lambda_j(\underline{p}, \underline{\check{\xi}})$ as in (4.6)*,*

(v) *the lower left-hand corner entry of $\partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\check{\zeta}})$ is*

$$\underline{q}_j = -1/\beta_j \in \mathbb{R}. \tag{4.29}$$

Proof. (a) Define $\tilde{\varphi}_{j,\nu_j} = \varphi_{j,\nu_j}$ and for $a < \nu_j$

$$\tilde{\varphi}_{j,a}(p, \zeta) = -i\big(H_0(p, \zeta) - \underline{\mu}_k\big)\tilde{\varphi}_{j,a+1}. \tag{4.30}$$

By (4.15), there holds

$$\tilde{\varphi}_{j,a}(\underline{p}, \underline{\check{\zeta}}) = \varphi_{j,a}(\underline{p}, \underline{\check{\zeta}}). \tag{4.31}$$

Moreover, in the new basis $\tilde{\varphi}_{j,a}$, the matrix of Q_j has the form

$$Q_j = i\underline{\check{\xi}}_d \,\mathrm{Id} + i\begin{pmatrix} * & 1 & \dots & 0 \\ \vdots & 0 & \ddots & 0 \\ * & 0 & \dots & 1 \\ * & 0 & \dots & 0 \end{pmatrix}. \tag{4.32}$$

Thanks to (4.31), the dual basis $\{\tilde{\psi}_{j,a}\}$ associated to $\{\tilde{\varphi}_{j,a}\}$ also satisfies $\tilde{\psi}_{j,a}(\underline{p}, \underline{\check{\zeta}}) = \psi_{j,a}(\underline{p}, \underline{\check{\zeta}})$. This implies that the lower left-hand corner of $\partial_{\check{\gamma}} Q_j(\underline{p}, \underline{\check{\zeta}})$ is unchanged in the new basis.

(b) Consider the determinant

$$\Delta_j(p,\check{\zeta},\check{\xi}_d) = \det\bigl(\xi_d \operatorname{Id} + i Q_j(p,\check{\zeta})\bigr).$$

It is independent of the basis $\{\psi_{j,a}\}$ or $\{\tilde{\psi}_{j,a}\}$. Thus, it is real when $\check{\gamma} = 0$ and vanishes at $(\underline{p}, \check{\underline{\zeta}}, \check{\underline{\xi}}_d)$. Moreover, (4.13) implies that

$$\partial_{\check{\tau}} \Delta_j(\underline{p}, \check{\underline{\zeta}}, \underline{\xi}_d) = -\underline{q}_j.$$

As a byproduct, using also (4.24) this shows that

$$\underline{q}_j \in \mathbb{R} \quad \text{thus} \quad \underline{q}_j = \operatorname{Re} \underline{q}_j = q_j^\flat \neq 0. \tag{4.33}$$

In particular, the implicit function theorem implies that there is a smooth function $\lambda_j(p,\check{\xi})$, in a real neighborhood of $(\underline{p}, \check{\underline{\xi}})$, such that $\lambda_j(\underline{p}, \check{\underline{\xi}}) = -\underline{\check{\tau}}$ and for $\check{\zeta} = (\check{\tau}, \check{\eta}, 0)$:

$$\Delta_j(p,\check{\zeta},\check{\xi}_d) = \alpha_j(p,\check{\zeta},\check{\xi}_d)\bigl(\check{\tau} + \lambda_j(p,\check{\xi})\bigr) \tag{4.34}$$

with $\alpha_j(\underline{p}, \check{\underline{\zeta}}, \check{\underline{\xi}}_d) \neq 0$.

(c) Consider next the eigenvector equation

$$\bigl(\check{\xi}_d \operatorname{Id} + i Q_j(p,\check{\zeta})\bigr) e_j = 0. \tag{4.35}$$

By (4.32), in the basis $\{\tilde{\psi}_{j,a}\}$, the $\nu_j - 1$ first equation determine the last $\nu_j - 1$ components of e_j

$$(e_j)_a = (\check{\xi}_d - \check{\underline{\xi}}_d)^{a-1} (e_j)_1, \quad a \geqslant 2. \tag{4.36}$$

Substituting these values, the last equation is a scalar equation equivalent to $\Delta_j = 0$. Introduce

$$\zeta_j(p,\check{\eta},\check{\xi}) = \bigl(-\lambda_j(p,\check{\xi}), \check{\eta}, 0\bigr),$$

and

$$e_j(p,\check{\xi}) = \tilde{\varphi}_{j,1}(p,\check{\zeta}) + \sum_{a=2}^{\nu_j} (\check{\xi}_d - \check{\underline{\xi}}_d)^{j-1} \tilde{\varphi}_{j,a}(p,\check{\zeta}). \tag{4.37}$$

This vector is smooth and satisfies (4.35), thus

$$\bigl(A(p,\check{\xi}) - \lambda_j(p,\check{\xi}) \operatorname{Id}\bigr) e_j(p,\check{\xi}) = A_d(p)\bigl(i H_0(p,\check{\zeta}_j) + \check{\xi}_d \operatorname{Id}\bigr) e_j(p,\check{\xi}) = 0.$$

Moreover, the $e_j(\underline{p}, \check{\underline{\xi}}) = \varphi_{j,1}(\underline{p}, \check{\underline{\zeta}})$ are linearly independent.

(d) By (4.34), for $\check{\zeta} = (\check{\tau}, \check{\eta}, 0)$, there holds

$$\det\bigl(\check{\tau}\,\mathrm{Id} + A(p, \check{\xi})\bigr) = \det(A_d)\det\bigl(iH_0(p, \check{\zeta}) + \check{\xi}_d\,\mathrm{Id}\bigr) \\ = \alpha(p, \check{\tau}, \check{\xi}) \prod_{j=1}^{m'} \bigl(\check{\tau} + \lambda_j(p, \check{\xi})\bigr)$$

where $\alpha(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}}) \neq 0$ and m' is the number of blocks Q_j. This shows that $-\underline{\check{\tau}}$ is an eigenvalue of algebraic order m' of $A(\underline{p}, \underline{\check{\xi}})$. By step (c), the geometric multiplicity is at least m', implying that $-\underline{\check{\tau}}$ is semi-simple of order m'.

Moreover, by (4.14), there holds

$$\Delta_j(\underline{p}, \underline{\check{\zeta}}, \check{\xi}_d) = (\check{\xi}_d - \underline{\check{\xi}}_d)^{\nu_j},$$

showing that $\underline{\check{\xi}}_d$ is a root of multiplicity ν_j of Δ_j, thus of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\eta}}, \check{\xi}) = 0$.

(e) Let $\underline{\ell}_j$ satisfy (4.7). Thus

$$\mathrm{Range}\bigl(\check{H}_0(\underline{p}, \underline{\check{\zeta}}) - \underline{\mu}_k\,\mathrm{Id}\bigr) = A_d^{-1}(\underline{p})\,\mathrm{Range}\bigl(\underline{\check{\tau}}\,\mathrm{Id} + A(\underline{p}, \underline{\check{\xi}})\bigr) \\ = A_d^{-1}(\underline{p})\{\underline{\ell}_1, \ldots, \underline{\ell}_m\}^{\perp}.$$

Comparing with (4.21), this implies that

$$\mathrm{span}\bigl\{\psi_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}),\ 1 \leqslant j \leqslant m\bigr\} = \mathrm{span}\{\underline{\ell}_j,\ 1 \leqslant j \leqslant m\}. \tag{4.38}$$

For $a \in \{1, \ldots, \nu_j\}$, introduce

$$\underline{e}_{j,a} = \frac{1}{(a-1)!}\partial_{\xi_d}^{a-1} e_j(\underline{p}, \underline{\check{\xi}}). \tag{4.39}$$

Because $\underline{\check{\xi}}_d$ is a root of order ν_j of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\eta}}, \check{\xi}) = 0$, the definition (4.37) implies that

$$\underline{e}_{j,a} = \tilde{\varphi}_{j,a}(\underline{p}, \underline{\check{\zeta}}) = \varphi_{j,a}(\underline{p}, \underline{\check{\zeta}}) \quad \text{for } 1 \leqslant a \leqslant \nu_j.$$

In particular, (4.16) implies that

$$\psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}}) \cdot \underline{e}_{j,\nu_j} = \psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}}) \cdot \varphi_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}) = \delta_{j,j'}. \tag{4.40}$$

Differentiating the equation

$$\bigl(A(\underline{p}, \check{\xi}) - \lambda_j(\underline{p}, \check{\xi})\bigr)e_j(\underline{p}, \check{\xi}) = 0 \tag{4.41}$$

with respect to $\check{\xi}_d$ and at order ν_j yields

$$\bigl(\underline{\check{\tau}}\,\mathrm{Id} + A(\underline{p}, \underline{\check{\xi}})\bigr)\partial_{\xi_j}^{\nu_j} e_j(\underline{p}, \underline{\check{\xi}}) = -\nu_j A_d(\underline{p})\partial_{\xi_d}^{\nu_j - 1} e_j(\underline{p}, \underline{\check{\xi}}) + \partial_{\xi_j}^{\nu_j}\lambda_j(\underline{p}, \underline{\check{\xi}})e_j(\underline{p}, \underline{\check{\xi}}).$$

Multiplying on the left by $\underline{\ell}_{j'}$ annihilates the left-hand side, implying

$$\underline{\ell}_{j'} A_d(\underline{p}) e_{j,\nu_j}(\underline{p}, \underline{\check{\zeta}}) = \beta_j \underline{\ell}_{j'} \cdot e_j(\underline{p}, \underline{\check{\xi}}) = \beta_j \delta_{j',j}.$$

By (4.38), the $\underline{\ell}_j A_d$ and ψ_{j,ν_j} span the same space. Therefore, comparing with (4.40) implies that $\underline{\ell}_{j'} A_d(\underline{p}) = \beta_j \psi_{j',\nu_{j'}}(\underline{p}, \underline{\check{\zeta}})$.

(f) By (4.20) and (4.28), we have

$$-\beta_j \underline{q}_j = \underline{\ell}_j \varphi_{j,1}(\underline{p}, \underline{\check{\zeta}}) = \underline{\ell}_j e_j(\underline{p}, \underline{\check{\xi}}) = 1.$$

The proof of the lemma is complete. □

Remark 4.20. This lemma is a variation on the necessary part in Proposition 4.16 (see [21]), with useful additional remarks. It shows that the block structure condition is closely related to a smooth diagonalization of A. Conversely, if one starts from a smooth basis e_j and a root of $\underline{\check{\tau}} + \lambda_j(\underline{p}, \underline{\check{\xi}})$ with (4.6), one constructs a basis $\varphi_{j,a}$ such that $\varphi_{j,a}(\underline{p}, \underline{\check{\zeta}})$ is given by (4.39), using an holomorphic extension of e_j to complex values of $\check{\xi}_d$ (see [21]). Lemma 4.19 implies that the change of bases which preserve the block structure form are linked to change of bases which preserve the smooth diagonalization of A.

The construction of K-families of symmetrizers for the blocks Q_j is performed in [12,14, 19]. The sign of β_j and the parity of ν_j play an important role. Hyperbolicity implies that H_0 and thus the $\check{H}_k$ and Q_j have no purely imaginary eigenvalues when $\check{\gamma} > 0$. Denote by $\mathbb{E}^-_{Q_j}$ the invariant space of Q_j associated to the spectrum in $\{\operatorname{Re}\mu < 0\}$ since the definition of the limiting space $\underline{\mathbb{E}}^-_{Q_j}$. Recall that the limit space at $(\underline{p}, \underline{\check{\zeta}})$ is

$$\underline{\mathbb{E}}^-_{Q_j} = \mathbb{C}^{\nu'_j} \times \{0\}^{\nu_j - \nu'_j} \tag{4.42}$$

with

$$\nu'_j = \begin{cases} \nu_j/2 & \text{when } \nu_j \text{ is even}, \\ (\nu_j+1)/2 & \text{when } \nu_j \text{ is odd and } \beta_j > 0, \\ (\nu_j-1)/2 & \text{when } \nu_j \text{ is odd and } \beta_j < 0. \end{cases} \tag{4.43}$$

Remark 4.21. As a corollary, we have the following characterization of the sets J_O and J_I:

$$\begin{cases} j \in J_I & \text{if } \nu_j \text{ is even or } \nu_j \text{ is odd and } q^\flat_j < 0, \\ j \in J_O & \text{if } \nu_j \text{ is odd and } q^\flat_j > 0. \end{cases} \tag{4.44}$$

4.4. The hyperbolic–parabolic case

We still consider a block $\check{H}_k$ associated to a purely imaginary eigenvalue (3.23). In the next section, we show that the following technical conditions are the natural one for the construction of Kreiss symmetrizers.

Definition 4.22. $\check{H}_k$ has the generalized block structure property near $(\underline{p}, \check{\underline{\zeta}}, 0)$ if there exists a smooth invertible matrix V_k on a neighborhood of that point such that

$$V_k^{-1} \check{H}_k V_k = \begin{pmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_m \end{pmatrix} + \rho \begin{pmatrix} \widetilde{B}_{1,1} & \cdots & \widetilde{B}_{1,m} \\ \vdots & \ddots & \vdots \\ \widetilde{B}_{m,1} & \cdots & \widetilde{B}_{m,m} \end{pmatrix} \tag{4.45}$$

where the $Q_j(p, \check{\zeta})$ satisfy the properties of Definition 4.15. Moreover, the $m \times m$ matrix $B^\flat$ with entries $B^\flat_{j,j'}$ equal to the lower left-hand corner of $\widetilde{B}_{j,j'}(\underline{p}, \check{\underline{\zeta}}, 0)$ satisfies

$$B^\flat_{j,j'} = 0 \quad \text{when } (j, j') \in (J_O \times J_I) \cup (J_I \times J_O) \tag{4.46}$$

where J_O and J_I are defined by (4.44) and there is a real diagonal matrix $D^\flat$, with entries $d^\flat_j$ such that

$$d^\flat_j q^\flat_j > 0, \qquad \operatorname{Re} D^\flat B^\flat > 0. \tag{4.47}$$

We show that these conditions are related to the condition (BS) of Definition 4.9 formulated on the original system. We need first a more detailed form of the block reduction H in (2.28). Introduce the following notations:

$$B_{**}(p, \zeta) := \sum_{j=1,k}^{d-1} \eta_j \eta_k B_{j,k}(p), \tag{4.48}$$

$$B_{*d}(p, \zeta) := \sum_{j=1}^{d-1} \eta_j \big(B_{j,d}(p) + B_{d,j}(p)\big). \tag{4.49}$$

Lemma 4.23. *One can choose the matrix V in* (2.28) *such that there holds*

$$H(p, \zeta) = H_0(p, \zeta) - H_1(p, \zeta) + O\big(|\zeta|^3\big) \tag{4.50}$$

where

$$H_1 = A_d^{-1}\big(B_{*,*} - i B_{*,d} H_0 - B_{d,d} H_0^2\big). \tag{4.51}$$

Proof. Direct computations show that the kernel of $G(p, 0)$ is $\mathbb{C}^N \times \{0\}$ and, using that A_d is invertible, that $\ker G(p, 0) \cap \operatorname{range} G(p, 0) = \{0\}$. This shows that 0 is a semi-simple eigenvalue of $G(p, 0)$.

If μ is a purely imaginary eigenvalue of $G(p, 0)$, then 0 is an eigenvalue of $iA(p, \xi) + B(p, \xi)$ with $\xi = (0, -i\mu)$. By Assumption (H1) this requires that $\xi = 0$, thus $\mu = 0$. This shows that the nonvanishing eigenvalues of $G(p, 0)$ are not on the imaginary axis.

This implies that there is a smooth matrix $V(p, \zeta)$ on a neighborhood of $(\underline{p}, 0)$ such that (2.28) holds with $H(p, 0) = 0$ and $P(p, 0)$ invertible with no eigenvalue on the imaginary axis.

The image of the first N columns of V is the invariant space of G, and H is the restriction of G to that space. At $\zeta = 0$ this space is $\ker G$, and performing a smooth change of basis in $\mathbb{C}^N$, we can always assume that the first N columns of V are of the form

$$V_I(p,\zeta) = \begin{pmatrix} \mathrm{Id}_{N\times N} \\ W(p,\zeta) \end{pmatrix} \tag{4.52}$$

with W of size $N' \times N$ vanishing at $\zeta = 0$. This implies (2.30).

By (2.28) $GV_I = V_I H$, hence $MV_I = G_d V_I H$ and

$$\mathcal{M} = -\mathcal{A}H + \overline{B}_d W H, \quad W = JH.$$

Therefore,

$$\mathcal{M} = -\mathcal{A}H + \overline{B}_d J H^2 = -\mathcal{A}H + B_{d,d}H^2. \tag{4.53}$$

Taking the first order term at $\zeta = 0$ shows that the first order term in H_0 in H satisfies

$$(i\tau + \gamma)\,\mathrm{Id} + \sum_{j=1}^{d-1} i\eta_j A_j = -A_d(p)H_0$$

and hence is given by (2.27). The second order term H_1 in H satisfies

$$B_{*,*} = -A_d H_1 + iB_{*,d}H_0 + B_{d,d}JH_0^2$$

implying (4.50) and (4.51). □

Parallel to Lemma 4.18, we can now state:

Lemma 4.24. *Suppose that the matrix of $\check{H}_k$ is given by the right-hand side of* (4.45) *in a smooth basis $\varphi_{j,a}$ of $\mathbb{E}_k(p,\check{\zeta},\rho)$ which satisfies* (4.14) *and* (4.15) *for $\rho = 0$. Let $\{\underline{\ell}_j\}$ denote the dual basis of $\{e_j = \varphi_{j,1}\}$ satisfying* (4.7). *The entries of $B^\flat$ are*

$$B^\flat_{j,j'} = -\frac{1}{\beta_j}\underline{\ell}_j B(\underline{p},\check{\underline{\xi}})\varphi_{j',1}(\underline{p},\check{\underline{\zeta}},0). \tag{4.54}$$

Proof. In the block reduction (4.45), the lower left-hand corner entry of the (j,j')-block is

$$h_{j,j'} = \psi_{j,\nu_j}\check{H}\varphi_{j',1} = \psi_{j,\nu_j}(\check{H} - \underline{\mu}_k)\varphi_{j',1} + \underline{\mu}_k\delta_{j,j'}.$$

Differentiating in ρ and using the relations (4.23) yields

$$-B^\flat_{j,j'} = \partial_\rho h_{j,j'}(\underline{p},\check{\underline{\zeta}},0) = -\underline{\psi}_{j,\nu_j}\widetilde{B}(\underline{p},\check{\underline{\zeta}})\underline{\varphi}_{j,1},$$

where $\underline{\psi}_{j,\nu_j}$ and $\underline{\varphi}_{j,1}$ stand for the evaluation at $(\underline{p},\check{\underline{\zeta}},0)$ of the corresponding function. Using the explicit form of $\widetilde{B}$ and the relations

$$\underline{H}_0\underline{\varphi}_{j,1} = i\underline{\check{\xi}}_d\underline{\varphi}_{j,1}, \qquad \underline{\psi}_{j,\nu_j}\underline{H}_0 = i\underline{\check{\xi}}_d\underline{\psi}_{j,\nu_j}$$

we obtain

$$\begin{aligned}\underline{\psi}_{j,\nu_j}\widetilde{B}(\underline{p},\underline{\check{\zeta}})\underline{\varphi}_{j,1} &= \underline{\psi}_{j,\nu_j}A_d^{-1}\big(B_{*,*}(\underline{p},\underline{\check{\eta}}) + \underline{\check{\xi}}_d B_{*;d}(\underline{p},\underline{\check{\eta}}) + \underline{\check{\xi}}_d^2 B_{d,d}(\underline{p})\big)\underline{\varphi}_{j,1}\\ &= \underline{\psi}_{j,\nu_j}B(\underline{p},\underline{\check{\xi}})\underline{\varphi}_{j,1}.\end{aligned}$$

With (4.28), this implies (4.54). □

Theorem 4.25. *If* $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$ *is a geometrically regular characteristic root of* Δ *which satisfies the condition* (BS) *of Definition* 4.9*. Then the associated block* $\check{H}_k$ *satisfies the generalized block structure condition.*

Proof. Since $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$ is geometrically regular, the hyperbolic part $\check{H}_{k,0}$ satisfies the block structure condition. Moreover, if e_j is a basis analytic in ξ, there is a basis $\varphi_{j,a}$ such that $\varphi_{j,a}(\underline{p},\underline{\check{\zeta}}) = e_j(\underline{p},\underline{\check{\xi}})$ (see Remark 4.20 or [21]). By Lemma 4.24, (4.9) is equivalent to (4.46).

If once can choose the base $\{e_j\}$ such that (4.10) holds, then choose $d_j^\flat = -\beta_j$ and by (4.29) and (4.54) there holds $d_j^\flat q_j^\flat = 1$ so that $DB^\flat = B^\sharp$ satisfies (4.47). □

Remark 4.26. Conversely, if the generalized block structure condition holds with matrices V_k which are real analytic in $\check{\zeta}$, then, by Proposition 4.16 $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$ is geometrically regular. By (4.54), (4.46) is equivalent to the decoupling condition (4.9). Moreover, (4.47) implies that there is a diagonal matrix with positive entries $d_j^\sharp = d_j^\flat/q_j^\flat$ such that $\operatorname{Re} D^\sharp B^\sharp > 0$. Consider the diagonal matrix $C = (D^\sharp)^{-1/2} = \operatorname{diag}(c_j)$ and the new basis $\tilde{e}_j = c_j e_j$. The new dual basis is $\widetilde{\ell}_j = c_j^{-1}\underline{c}_j$ and the new matrix $\widetilde{B}^\sharp$ is $C^{-1}B^\sharp C = CD^\sharp B^\sharp C$ and therefore $\operatorname{Re}\widetilde{B}^\sharp = C\operatorname{Re}(D^\sharp B^\sharp)C$ is definite positive.

5. Symmetrizers

In this section, we prove Theorems 3.12 and 3.13. We are given a frequencies $\underline{\check{\zeta}} = (\underline{\check{\tau}},\underline{\check{\eta}},0)$ and a purely imaginary eigenvalue $\mu_k = -i\underline{\check{\xi}}_d$ of $H_0(\underline{p},\underline{\check{\zeta}})$, so that $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$, with $\underline{\check{\xi}} = (\underline{\check{\eta}},\underline{\check{\xi}}_d)$ is a root of the characteristic determinant Δ, of multiplicity m. Our goal is to construct K-families of symmetrizers for the block $\check{H}_k(p,\check{\zeta},\rho)$ associated to μ_k.

5.1. Proof of Theorem 3.13

We assume here that $(\underline{p},\underline{\check{\tau}},\underline{\check{\xi}})$ is geometrically regular and satisfies the condition (BS).

We follow closely [20] (Lemma 4.11 and Appendix A therein. See also [19]) where the constant multiplicity case is studied. In this case, all the blocks Q_j are equal and thus have the same dimensions ν, but more importantly, all the eigenvalues are of the same type O or I. So we review the main steps of the construction and indicate where the proof of [19,20] has to be modified.

In the block reduction (4.45) of $\check{H}_k$, we choose the symmetrizers Σ_k^κ to be block diagonal:

$$\Sigma_k^\kappa = \begin{pmatrix} S_1^\kappa & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_m^\kappa \end{pmatrix},$$

$$S_j^\kappa(p, \check{\zeta}, \rho) = E_j^\kappa + \widetilde{E}_j^\kappa(p, \check{\zeta}) - i\gamma F_j^\kappa - i\rho \widetilde{F}_j^\kappa, \tag{5.1}$$

where E_j^κ and $\widetilde{E}_j^\kappa$ are real symmetric matrices, and F_j^κ and $\widetilde{F}_j^\kappa$ are real and skew symmetric. Moreover, E_j^κ, F_j^κ and $\widetilde{F}_j^\kappa$ are constant, $\widetilde{E}_j^\kappa$ depends only on $(p, \check{\tau}, \check{\eta})$ and the E_j^κ have the special form

$$E_j^\kappa = \begin{bmatrix} 0 & \cdots & \cdots & 0 & e_{j,1}^\kappa \\ \vdots & & & \iddots & e_{j,2}^\kappa \\ \vdots & & \iddots & \iddots & \\ 0 & \iddots & \iddots & & \\ e_{j,1}^\kappa & e_{j,2}^\kappa & & & e_{j,\nu_j}^\kappa \end{bmatrix},$$

and $\widetilde{E}_j^\kappa(\underline{p}, \underline{\check{\zeta}}) = 0$.

The block structure condition implies that

$$\check{H}_k = \operatorname{diag}(Q_j|_{\gamma=0}) + \gamma \operatorname{diag}(\partial_\gamma Q_j|_{\gamma=0}) + \rho \widetilde{B}|_{\rho=0} O(\gamma^2 + \rho^2). \tag{5.2}$$

Σ_k^κ is a symmetrizer for $\check{H}_k$, on a neighborhood (depending on κ) of $(\underline{p}, \underline{\check{\zeta}}, 0)$, if

$$\operatorname{Re}\big(\big(E_j^\kappa + \widetilde{E}_j^\kappa\big) Q_j|_{\gamma=0}\big) = 0, \tag{5.3}$$

$$\operatorname{Re}\big(E_j^\kappa \partial_\gamma Q_j(\underline{p}, \underline{\check{\zeta}}, 0) - i F_j^\kappa Q_j(\underline{p}, \underline{\check{\zeta}})\big) > 0, \tag{5.4}$$

$$\operatorname{Re}\big(\operatorname{diag}\big(E_j^\kappa\big) \widetilde{B}(\underline{p}, \underline{\check{\zeta}}, 0) - i \operatorname{diag}\big(\widetilde{F}_j^\kappa Q_j(\underline{p}, \underline{\check{\zeta}})\big)\big) > 0. \tag{5.5}$$

Moreover, the condition (3.26) reads

$$\big(E_j^\kappa w, w\big) \geqslant C_1\big(\kappa \big|\Pi_j^+ w\big|^2 - \big|\Pi_j^- w\big|^2\big), \tag{5.6}$$

where $\Pi_j^\pm$ is the projection onto $\underline{\mathbb{E}}_j^\pm$ in the decomposition $\mathbb{C}^{\nu_j} = \underline{\mathbb{E}}_j^- \oplus \underline{\mathbb{E}}_j^+$, where

$$\underline{\mathbb{E}}_j^- = \mathbb{C}^{\nu_j'} \times \{0\}^{\nu_j - \nu_j'}, \qquad \underline{\mathbb{E}}_j^+ = \{0\}^{\nu_j'} \times \mathbb{C}^{\nu_j - \nu_j'}, \tag{5.7}$$

with ν_j' given by (4.43).

Before starting the construction, we note that Ralston's lemma [23] (see also [19,20]) implies that one can perform an additional change of basis $\mathrm{Id} + \rho \widetilde{V}$ such that the matrices $\widetilde{B}_{j,j'}$ in (4.45) are of the form

$$\widetilde{B}_{j,j'}(\underline{p},\check{\underline{\zeta}}) = \begin{pmatrix} * & 0 & \dots & 0 \\ \vdots & 0 & \dots & 0 \\ B^\flat_{j,j'} & 0 & \dots & 0 \end{pmatrix}. \tag{5.8}$$

This does not affect the previous choices, made at $\rho = 0$. Next, we introduce some notations. A vector $w \in \mathbb{C}^{N_k} = \bigoplus \mathbb{C}^{\nu_j}$, is broken into m blocks $w_j \in \mathbb{C}^{\nu_j}$, with components denoted by $w_{j,a}$. We now proceed to the construction of the symmetrizers.

(a) One first choose the E^κ_j such that (5.6) holds and

$$\operatorname{Re}\big(E^\kappa_j \partial_\gamma Q_j(\underline{p},\check{\underline{\zeta}})w_j, w_j\big) \geqslant 2|w_{j,1}|^2 - C_\kappa\big|w'_j\big|^2, \tag{5.9}$$

$$\operatorname{Re}\big(\operatorname{diag}\big(E^\kappa_j\big)\widetilde{B}(\underline{p},\check{\underline{\zeta}},0)w_j, w_j\big) \geqslant 2|w_{*,1}|^2 - C'_\kappa\big|w'_*\big|^2, \tag{5.10}$$

with $w_{j,1}$ denoting the first component of $w_j \in \mathbb{C}^{\nu_j}$ and $w'_j \in \mathbb{C}^{\nu_j-1}$ denotes the other components and $w_{*,1} \in \mathbb{C}^m$ is the collection of the first components $w_{j,1}$ while w'_* denotes the remaining components.

Note that

$$\operatorname{Re}\big(E^\kappa_j \partial_\gamma Q_j(\underline{p},\check{\underline{\zeta}})w, w\big) = e^\kappa_{j,1}q^\flat_j|w_{j,1}|^2 + O\big(|w_j|\big|w'_j\big|\big),$$
$$\operatorname{Re}\big(\operatorname{diag}\big(E^\kappa_j\big)\widetilde{B}(\underline{p},\check{\underline{\zeta}},0)w_j, w_j\big) = \operatorname{Re}\big(E^\flat B^\flat w_{*,1}, w_{*,1}\big) + O\big(\big|w'_*\big||w|\big),$$

where $E^\flat$ is the $m \times m$ diagonal matrix with entries $e^\kappa_{j,1}$. Moreover, the decoupling condition (4.9) implies that $B^\flat$ has a block diagonal structure: ordering the base $\{e_j\}$ according to the type I or O, with obvious notations there holds:

$$B^\flat = \begin{pmatrix} B^\flat_I & 0 \\ 0 & B^\flat_O \end{pmatrix}. \tag{5.11}$$

Similarly we note $E^\flat = \operatorname{diag}(E^\flat_I, E^\flat_O)$ and (5.9) and (5.10) are satisfied if

$$e^\kappa_{j,1}q^\flat_j \geqslant 3, \qquad \operatorname{Re} E^\flat_I B^\flat_I \geqslant 3\,\mathrm{Id}, \qquad \operatorname{Re} E^\flat_O B^\flat_O \geqslant 3\,\mathrm{Id}. \tag{5.12}$$

On the other hand, to satisfy (5.6), one chooses the $e^\kappa_{j,a}$ inductively, starting from $a = 1$, but this choice depends on the type of the eigenvalue. Remember also from (4.29) that $\beta_j = -1/q^\flat_j$. According to [1,12] or Lemma 8.4.2 in [19], the $e^\kappa_{j,a}$ are chosen as follows.

(1) If λ_j is of type I, then $e^\kappa_{j,1} = e_{j,1}$ is taken $O(1)$, independent of κ, and the $e^\kappa_{j,a}$ for $a \geqslant 2$ are chosen successively and depend on κ. In particular, when ν_j is even, $e^\kappa_{j,2} \geqslant c\kappa$. When ν_j is odd and $\beta_j > 0$, then $q^\flat_j < 0$ and $e_{j,1} < 0$; when $\nu_j \geqslant 3$, then $e^\kappa_{j,3} \geqslant c\kappa$.

(2) If λ_j is of type O, that is ν_j odd and $q^\flat_j > 0$, one chooses $e^\kappa_{j,1} \geqslant c\kappa$ and the other $e^\kappa_{j,a}$ are chosen inductively.

By assumption, there is a diagonal real matrix $D^\flat = \operatorname{diag}(D^\flat_I, D^\flat_O)$ such that

$$d^\flat_j q^\flat_j > 0, \qquad \operatorname{Re} D^\flat_I B^\flat_I > 0, \qquad \operatorname{Re} D^\flat_O B^\flat_O > 0.$$

Therefore, there is a positive constant c such that if we choose $e^{\kappa}_{j,1} = cd^{\flat}_j$ when λ_j is of type I and $e^{\kappa}_{j,1} = c\kappa d^{\flat}_j$ when λ_j is of type I, the condition (5.12) is satisfied. Next, according to [1,12,19], we can choose the $e^{\kappa}_{j,a}$ for $a \geqslant 2$ such that the inequality (5.6) is also satisfied.

Remark 5.1. The construction above shows that the conditions of Definition 4.22 are more or less necessary for the construction of K-families of symmetrizers. First, the different magnitude in κ of $e^{\kappa}_{j,1}$ for different types forces the decoupling (5.11), that it condition (4.9). Second, a spectral condition on $B^{\sharp}$ is not sufficient in general to insure the existence of a diagonal matrix $E^{\flat}$ such that (5.12) holds. This indicates that condition (4.47) is also necessary for the construction above.

(b) Once the matrices E_j are chosen, the construction goes on as in [19,20]. We omit the details. By (4.13), $\mathrm{Re}(E_j Q_j(\underline{p}, \underline{\check{\zeta}})) = 0$. Next, using the implicit function theorem and the property that $\frac{1}{i} Q_k$ is real when $\check{\gamma} = 0$, the real symmetric matrix $\widetilde{E}_k(p, \check{\tau}, \check{\eta})$ is chosen so that such that $\mathrm{Re}(E_j + \widetilde{E}_j)(Q_j|_{\gamma=0}) = 0$.

Since F_j is real and skew symmetric, there holds $\mathrm{Re}\, {-i} F_j Q_j(\underline{p}, \underline{\check{\zeta}}) = \mathrm{Re}\, F_j J_j$ where J_j is the Jordan matrix in (4.13). One can choose F_j such that

$$\mathrm{Re}(F_j J_j w_j, w_j) \geqslant -|w_{j,1}|^2 + (C+1)\left|w'_j\right|^2,$$

where C is the constant in (5.9). Adding to (5.9) implies (5.4).

Similarly, $\mathrm{Re}\, {-i} \widetilde{F}_j Q_j(\underline{p}, \underline{\check{\zeta}}) = \mathrm{Re}\, \widetilde{F}_j J_j$ and one can choose $\widetilde{F}_j$ such that

$$\mathrm{Re}(F_j J_j w_j, w_j) \geqslant -|w_{j,1}|^2 + (C'+1)\left|w'_j\right|^2,$$

where C' is the constant in (5.10), implying (5.5).

5.2. Proof of Theorem 3.12

We now assume that the system is symmetric dissipative in the sense of Definition 2.5 and that the root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ is totally nonglancing. In [21], symmetrizers for $\check{H}_k(p, \check{\zeta}, 0)$ are constructed. We show that they also symmetrize $\check{H}_k(p, \check{\zeta}, \rho)$ when $\rho > 0$.

In [21], it is proved that the nonglancing condition implies that the multiplicity of μ_k as an eigenvalue of $H_0(\underline{p}, \underline{\check{\zeta}}) = \check{H}(\underline{p}, \underline{\check{\zeta}}, 0)$ is equal to m. Denote by V_k the $N \times m$ sub-matrix of V which corresponds to the block $\check{H}_k$. Therefore, for $(p, \check{\zeta}, \rho)$ close to $(\underline{p}, \underline{\zeta}, 0)$, the corresponding invariant space of $\check{H}_h$ is $\mathbb{E}_k(p, \check{\zeta}, \rho) = V_k(p, \check{\zeta}, \rho)\mathbb{C}^m$ and

$$V_k \check{H}_k = \check{H} V_k. \tag{5.13}$$

Recall that $\mathbb{E}^-_k(p, \check{\zeta}, \rho)$ is the negative space of $\check{H}_k$ for $\check{\zeta} \in \overline{S}^d_+$, $\rho \geqslant 0$ with $\check{\gamma} > 0 + \rho > 0$.

Lemma 5.2.

(ii) *If $(\underline{p}, \underline{\tau}, \underline{\xi})$ is totally incoming, then, for (p, ζ) in a neighborhood of $(\underline{p}, \underline{\zeta})$, $\mathbb{E}^-_k(p, \zeta) = \mathbb{C}^m$.*
(iii) *If $(\underline{p}, \underline{\tau}, \underline{\xi})$ is totally outgoing, then, for (p, ζ) in a neighborhood of $(\underline{p}, \underline{\zeta})$, $\mathbb{E}^-_k(p, \zeta) = \{0\}$.*

Proof. The dimension is constant for $\check{\gamma} + \rho > 0$, and the result is proved in [21] when $\rho = 0$. □

By assumption, there is a definite positive matrix $S(p)$ such that the SA_j are symmetric.

Lemma 5.3. *The symmetric matrix*

$$\Sigma_{k,0}(p,\zeta) = -V_k^*(p,\zeta,0)S(p)A_d(p)V_k(p,\zeta,0) \tag{5.14}$$

is a symmetrizer for $\check{H}_k$ on a neighborhood of $(\underline{p}, \underline{\check{\zeta}}, 0)$. More precisely, there holds

$$\operatorname{Re} \Sigma_k \check{H}_k = \gamma R_1 + \rho R_2 \tag{5.15}$$

with $\Sigma_1(\underline{p}, \underline{\check{\zeta}}, 0)$ and $\Sigma_2(\underline{p}, \underline{\check{\zeta}}, 0)$ definite positive.

In addition, $\Sigma_k(\underline{p}, \underline{\zeta}, \underline{0})$ is definite positive (respectively negative) when the mode is totally incoming (respectively outgoing).

Proof. According to (3.16), there holds

$$\check{H}(p,\check{\zeta},\rho) = H_0(p,\check{\zeta}) + \rho H'(p,\check{\zeta},\rho).$$

Using (5.13) and the definition (2.27) of H_0, one obtains the identity (5.15) with

$$R_1 = V_k^* S V_k, \tag{5.16}$$

$$R_2 = V_k^*(\operatorname{Re} S A_d H')V_k. \tag{5.17}$$

Because S is definite positive, R_1 also has this property. Next, Lemma 4.23 implies that $H'(p,\check{\zeta},0) = -H_1(p,\check{\zeta})$ with H_1 given by (4.51). Since $H_0(\underline{p}, \underline{\check{\zeta}}) = \mu_k \operatorname{Id} = -i\xi_k \operatorname{Id}$ on $\mathbb{E}_k(\underline{p}, \underline{\check{\zeta}}, 0)$, there holds

$$H'(\underline{p}, \underline{\check{\zeta}}, 0)V_k(\underline{p}, \underline{\check{\zeta}}, 0) = -A_d^{-1}(\underline{p})B(\underline{p}, \underline{\check{\xi}}).$$

Therefore, at the base point $(\underline{p}, \underline{c}z, 0)$, there holds

$$R_2(\underline{p}, \underline{c}z, 0) = V_k^*(\operatorname{Re} SB)V_k.$$

The symmetry assumption implies that SB is definite positive on the space $\mathbb{E}_k(\underline{p}, \underline{\check{\zeta}}, 0) = \ker(A(\underline{p}, \underline{\check{\xi}}) + \underline{\check{\tau}} \operatorname{Id})$, implying that R_2 is definite positive at $(\underline{p}, \underline{c}z, 0)$, hence on a neighborhood of that point.

That $\Sigma_k(\underline{p}, \underline{\zeta}, 0)$ is definite positive (respectively negative) when the mode is totally incoming (respectively outgoing) is proved in [21]. □

With Lemma 5.2, this implies that

$$\Sigma_k^\kappa = \begin{cases} \Sigma_k & \text{in the incoming case,} \\ \kappa \Sigma_k & \text{in the outgoing case,} \end{cases} \tag{5.18}$$

are K-families of symmetrizers for $\check{H}_k$.

6. Further remarks and examples

6.1. Adapted basis. Proof of Proposition 4.14

In this section, we always assume that Assumptions (H1), (H4) are satisfied. Consider a geometrically regular root $(\underline{p}, \underline{\check{\tau}}, \underline{\check{\xi}})$ of Δ. We show that there are eigenbasis $\{e_j\}$ satisfying (4.3) which are adapted to B, in the sense of Definition 4.8, either when the multiplicity is 2 or when the system is symmetric.

6.1.1. The case of multiplicity two

Projecting on the 2-dimensional invariant space of $iA(p, \check{\xi}) + \rho B(p, \check{\xi})$ associated to the eigenvalues close the $i\lambda_1$ and $i\lambda_2$, we are reduced to consider 2×2 matrices

$$i\widetilde{A}(p, \check{\xi}) + \rho \widetilde{B}(p, \check{\xi}, \rho) \quad \text{with } \widetilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{6.1}$$

Assumption (H4) implies that the spectrum $i\widetilde{A} + \rho\widetilde{B}$ is contained in $\operatorname{Re}\lambda \geqslant c\rho$, for $(p, \check{\xi})$ close to $(\underline{p}, \underline{\check{\xi}})$ and $\rho \in [0, \rho_0]$, for some $\rho_0 > 0$. Changing $\check{\xi}$ to $-\check{\xi}$ and using (H1) near $-\underline{\check{\xi}}$, we see that the spectrum $\pm i\widetilde{A} + \rho\widetilde{B}$ is contained in $\operatorname{Re}\lambda \geqslant c\rho$.

We show that, changing the base $\{e_1, e_2\}$ if necessary, one always meet condition (4.10).

Lemma 6.1. *With assumptions as above, there is a smooth change of bases preserving* (6.1)*, such that* $\operatorname{Re}\widetilde{B}(\underline{p}, \underline{\check{\xi}})$ *is definite positive.*

Proof. The constant multiplicity case $\lambda_1 = \lambda_2$ being already treated, we assume that $\lambda_1 \neq \lambda_2$ on any neighborhood of $(\underline{p}, \underline{\check{\xi}})$. In this case we are limited to consider diagonal change of basis and we prove that there exists a diagonal matrix D, such that

$$\operatorname{Re}\big(D\widetilde{B}(\underline{p}, \check{\xi}, 0)D^{-1}\big) > 0. \tag{6.2}$$

(a) Recall that there is $c > 0$ such that the spectrum $\pm i\widetilde{A} + \rho\widetilde{B}$ is contained in $\operatorname{Re}\lambda \geqslant c\rho$. We first show that for all $t \in \mathbb{R}$ the spectrum of

$$\begin{pmatrix} 0 & 0 \\ 0 & it \end{pmatrix} + \widetilde{B}(\underline{p}, \underline{\check{\xi}}, 0) - \frac{c}{4}\,\mathrm{Id} \tag{6.3}$$

is contained in $\operatorname{Re}\lambda > 0$. If not, there are t, $\rho_1 > 0$ and a neighborhood ω of $(\underline{p}, \underline{\check{\xi}})$ such that

$$\begin{pmatrix} 0 & 0 \\ 0 & it \end{pmatrix} + \widetilde{B}(\underline{p}, \check{\xi}, \rho) \tag{6.4}$$

has an eigenvalue in $\operatorname{Re}\lambda < c/2$ when $(p, \check{\xi}) \in \omega$ and $\rho \in [0, \rho_1]$. There is $(p', \check{\xi}') \in \omega$ such that $\lambda_2(p', \check{\xi}') - \lambda_1(p', \check{\xi}') = t_1 \neq 0$. Choose $\rho \in [0, \rho_1[$ such that $\rho|t| \leqslant |t_1|$. By continuity, since $\lambda_2 - \lambda_1$ vanishes at $(\underline{p}, \underline{\check{\xi}})$, there is $(p, \check{\xi}) \in \omega$ such that $\lambda_2(p, \check{\xi}) - \lambda_1(p, \check{\xi}) = \pm t\rho$. Therefore the matrix $\pm i\widetilde{A}(p, \check{\xi}) + \rho\widetilde{B}(p, \check{\xi}, \rho)$ has an eigenvalue in $\{\operatorname{Re}\lambda \leqslant \rho c/2\}$.

(b) Consider a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Its spectrum is contained in $\{\operatorname{Re}\lambda > 0\}$, if and only if

$$\operatorname{Re}(a+d) > 0 \quad \text{and} \quad \bigl|\operatorname{Re}(\sqrt{f}\,)\bigr|^2 \leqslant \operatorname{Re}(a+d)^2$$

where $f = (a-d)^2 + 4bc$. Since $|\operatorname{Re}(\sqrt{f})|^2 = \frac{1}{2}(|f| + \operatorname{Re} f)$, the second condition reads

$$|\operatorname{Im} f|^2 < 4(\operatorname{Re} a + \operatorname{Re} d)^2\bigl((\operatorname{Re} a + \operatorname{Re} d)^2 - \operatorname{Re} f\bigr),$$

or

$$\begin{aligned}&\bigl(\operatorname{Re}(a-d)\operatorname{Im}(a-d) + 2\operatorname{Im}(bc)\bigr)^2 \\ &\quad < (\operatorname{Re} a + \operatorname{Re} d)^2\bigl((\operatorname{Im} a - \operatorname{Im} d)^2 + 4\operatorname{Re} a \operatorname{Re} d - 4\operatorname{Re}(bc)\bigr)\end{aligned}$$

or

$$\begin{aligned}&\operatorname{Re} a \operatorname{Re} d\bigl((\operatorname{Im} a - \operatorname{Im} d)^2 - (\operatorname{Re} a - \operatorname{Re} d)\operatorname{Im}(bc)(\operatorname{Im} a - \operatorname{Im} d) \\ &\quad + (\operatorname{Re} a + \operatorname{Re} d)^2(\operatorname{Re} a \operatorname{Re} d - \operatorname{Re} bc) - \bigl|\operatorname{Im}(bc)\bigr|^2\bigr) > 0. \qquad (6.5)\end{aligned}$$

We apply this criterion to the matrices (6.4). In this case, when t varies in $\mathbb{R}$ the coefficient $\operatorname{Im}(a-d)$ varies from $-\infty$ to $+\infty$ while the other coefficients are fixed. Therefore, if the corresponding inequality (6.5) is satisfied for all t, then $\operatorname{Re} a + \operatorname{Re} b > 0$, $\operatorname{Re} a \operatorname{Re} d \geqslant 0$ and

$$\begin{aligned}&(\operatorname{Re} a - \operatorname{Re} d)^2\bigl|\operatorname{Im}(bc)\bigr|^2 \\ &\quad \leqslant 4\operatorname{Re} a \operatorname{Re} d\bigl((\operatorname{Re} a + \operatorname{Re} d)^2(\operatorname{Re} a \operatorname{Re} d - \operatorname{Re} bc) - \bigl|\operatorname{Im}(bc)\bigr|^2\bigr).\end{aligned}$$

Thus

$$\bigl|\operatorname{Im}(bc)\bigr|^2 \leqslant 4\operatorname{Re} a \operatorname{Re} d(\operatorname{Re} a \operatorname{Re} d - \operatorname{Re} bc)$$

and

$$|bc| + \operatorname{Re}(bc) \leqslant 2\operatorname{Re} a \operatorname{Re} d.$$

Denoting by $b_{j,k}$ the entries of $\widetilde{B}(\underline{p}, \check{\underline{\xi}}, 0)$, we see that the spectral condition of step (a) implies the following conditions:

$$\operatorname{Re} b_{11} > 0, \qquad \operatorname{Re} b_{22} > 0, \qquad |b_{12}b_{21}| + \operatorname{Re}(b_{12}b_{21}) < 2\operatorname{Re} b_{11}\operatorname{Re} b_{22}. \qquad (6.6)$$

($\underline{c}$) Similarly, we note that the condition $\operatorname{Re}\widetilde{B}(\underline{p},\check{\underline{\xi}},0)>0$ is equivalent to

$$\begin{gathered}\operatorname{Re} b_{11}>0, \qquad \operatorname{Re} b_{22}>0,\\ |b_{12}|^2+|b_{21}|^2+2\operatorname{Re}(b_{12}b_{21})<4\operatorname{Re} b_{11}\operatorname{Re} b_{22}.\end{gathered} \tag{6.7}$$

With $D=\left(\begin{smallmatrix}1&0\\0&\delta\end{smallmatrix}\right)$, the conjugation $D\widetilde{B}D^{-1}$ changes B to B' with

$$b'_{11}=b_{11}, \qquad b'_{22}=b_{22}, \qquad b'_{21}=\delta b_{21}, \qquad b'_{12}=\frac{1}{\delta}b_{12}.$$

For all $\varepsilon>0$, one can choose δ such that

$$\left|b'_{12}\right|^2+\left|b'_{21}\right|^2\leqslant|b_{12}b_{21}|+\varepsilon$$

and therefore, (6.6) implies that there is δ such that (6.7) holds for the b'_{jk}. □

6.1.2. Symmetric systems

Lemma 6.2. *Suppose that $(\underline{p},\check{\underline{\tau}},\check{\underline{\xi}})$ is geometrically regular and the system is symmetric dissipative. Then one can choose the eigenbasis $\{e_j\}$ such that* (4.9) *holds.*

Proof. Denote by S the symmetrizer. We show that one can choose the eigenbasis e_j such that

$$^t e_j(\underline{p},\check{\underline{\xi}})S(\underline{p})e_{j'}(\underline{p},\check{\underline{\xi}})=\delta_{j,j'}. \tag{6.8}$$

In this case, $\underline{\ell}_j={}^t e_j(\underline{p},\check{\underline{\xi}})S(\underline{p})$ and

$$B^\sharp_{j,j'}={}^t e_j(\underline{p},\check{\underline{\xi}})S(\underline{p})B(\underline{p},\check{\underline{\xi}})e_{j'}(\underline{p},\check{\underline{\xi}}) \tag{6.9}$$

showing that $\operatorname{Re} B^\sharp$ is the restriction of $\operatorname{Re}(S(\underline{p})B(\underline{p},\check{\underline{\xi}}))$ to the space spanned by the $e_j(\underline{p},\check{\underline{\xi}})$, and hence positive.

To prove (6.8), consider the partition of $\{1,\dots,m\}$ into subsets J_a such that j and j' belong to the same class J_a if and only if $\lambda_j=\lambda'_j$ on a neighborhood of $(\underline{p},\check{\underline{\xi}})$. Denote by $\mathbb{F}_a(p,\check{\xi})$ the space spanned by the $e_j(p,\check{\xi})$ for $j\in J_a$. Then, near $(\underline{p},\check{\underline{\xi}})$, $A(p,\check{\xi})=\tilde{\lambda}_a\,\mathrm{Id}$ on this space, where $\tilde{\lambda}_a$ is the common value of the λ_j for $j\in J_a$. Thus, locally, one can find a smooth basis of $\mathbb{F}_a$, analytic in $\check{\xi}$ and orthonormal for the scalar product $S(p)$. Collecting these bases of $\mathbb{F}_a$, (6.8) holds when j and j' belong to the same class J_a.

When j and j' do not belong to the same class J_a, there is a sequence $(p^n,\check{\xi}^n)$ converging to $(\underline{p},\check{\underline{\xi}})$ such that $\lambda_j(p^n,\check{\xi}^n)\neq\lambda_{j'}(p^n,\check{\xi}^n)$. The symmetry of $S(p^n)A(p^n,\check{\xi}^n)$ implies that

$$^t e_j\left(p^n,\check{\xi}^n\right)S\left(p^n\right)e_{j'}\left(p^n,\check{\xi}^n\right)=0.$$

Therefore, passing to the limit, we see that (6.8) is also satisfied when j and j' do not belong to the same class J_a. □

6.2. Discontinuity of the negative spaces $\mathbb{E}^-$

We show that the decoupling condition (4.9) is necessary for the continuity of $\mathbb{E}^-(p,\check\zeta,\rho)$ at $\rho=0$. Before stating the result, we make the following remark.

Lemma 6.3. *Suppose that $(\underline{p},\check{\underline{\tau}},\check{\underline{\xi}})$ is geometrically regular and nonglancing. With notations as in* (4.2)*,* (4.6)*,* (4.8) *let $\beta_j=\partial_{\xi_d}\lambda_j(\underline{p},\check{\underline{\xi}})\neq 0$. Then, there is $c>0$ such that for all $\rho>0$ and $t\in\mathbb{R}$, the spectrum of $it\,\mathrm{diag}(\beta_j)+\rho B^\sharp$ is contained in $\{\mathrm{Re}\,\mu\geqslant c\rho\}$.*

Proof. We fix $p=\underline{p}$ and forget it in the notations. For ξ close to $\check{\underline{\xi}}$, consider the invariant space of $iA(\xi)+\rho B(\xi)$ associated to eigenvalues close to $-\check{\underline{\tau}}$. In the basis $\{e_j\}$, its matrix is

$$i\,\mathrm{diag}\big(\lambda_j(\xi)\big)+\rho\widetilde{B}(\xi,\rho),\tag{6.10}$$

with $\widetilde{B}(\check{\underline{\xi}},0)=B^\sharp$. Assumption (H4) implies that the spectrum of this matrix lies in $\{\mathrm{Re}\,\mu\geqslant c\rho\}$.

Adding $\check{\underline{\tau}}\,\mathrm{Id}$, we can assume, without loss of generality, that $\lambda_j(\check{\underline{\xi}})=0$. Taking $t>0$, $\xi=\check{\underline{\xi}}\pm(0,t)$ and $\rho=t\sigma>0$ the matrix in (6.10) is

$$tM(t,\sigma)=t\big(\pm i\,\mathrm{diag}(\beta_j)+\sigma B^\sharp+O(t)\big)$$

and the spectrum of $M(t,\sigma)$ lies in $\{\mathrm{Re}\,\mu\geqslant c\sigma\}$. Letting t tend to zero, implies that the spectrum of $M(0,\sigma)$ is also contained in $\{\mathrm{Re}\,\mu\geqslant c\sigma\}$ and the lemma follows by homogeneity. □

Corollary 6.4. *If $(\underline{p},\check{\underline{\tau}},\check{\underline{\xi}})$ is geometrically regular and nonglancing, then for all $\gamma\geqslant 0$ and $\rho\geqslant 0$, with $\gamma+\rho>0$, the matrix $\mathrm{diag}(\beta_j^{-1})(\gamma\,\mathrm{Id}+\rho B^\sharp)$ has no eigenvalues on the purely imaginary axis.*

Consider $\check{\underline{\zeta}}=(\check{\underline{\tau}},\check{\underline{\eta}},0)\neq 0$ and a purely imaginary eigenvalue $\underline{\mu}_k=i\check{\underline{\xi}}_d$ of $H_0(\underline{p},\check{\underline{\zeta}})$. Let $\check{\underline{\xi}}=(\check{\underline{\eta}},\check{\underline{\xi}}_d)$. Then $(\underline{p},\check{\underline{\tau}},\check{\underline{\xi}})$ is a root of Δ. We denote by $\check{H}_k$ the block associated to $\underline{\mu}_k$ and, for $\rho>0$, we denote by $\mathbb{E}_k^-(p,\check\zeta,\rho)$ the negative invariant space of $\check{H}_k$.

Proposition 6.5. *Suppose that $(\underline{p},\check{\underline{\tau}},\check{\underline{\xi}})$ is geometrically regular and nonglancing and suppose that there exist $j\in J_I$ and $j'\in J_O$ such that*

$$B^\sharp_{j',j}\neq 0.\tag{6.11}$$

Then the negative space $\mathbb{E}_k^-(\underline{p},\check\zeta,\rho)$ has no limit as $(\check\zeta,\rho)\to(\check{\underline{\zeta}},0)$.

In particular, there are no smooth K-families of symmetrizers for $\check{H}_k$ near $(\underline{p},\check{\underline{\zeta}})$.

Proof. By Lemmas 4.18 and 4.24, the block decomposition (4.45) implies that in a suitable basis

$$\check{H}_k(\underline{p},\check{\underline{\tau}},\check{\underline{\eta}},\gamma,\rho)=-\mathrm{diag}\big(\beta_j^{-1}\big)\big(\gamma\,\mathrm{Id}+\rho B^\sharp\big)+O\big(\gamma^2+\rho^2\big).\tag{6.12}$$

Denote by $\mathbb{E}^-(\gamma,\rho)$ the negative space of $\check{H}_k(\underline{p},\check{\underline{\tau}},\check{\underline{\eta}},\gamma,\rho)$. We show that

$$\lim_{\gamma\to 0}\mathbb{E}^-(\gamma,0)\neq\lim_{\rho\to 0}\mathbb{E}^-(0,\rho), \tag{6.13}$$

which implies that $\mathbb{E}^-(\gamma,\rho)$ has no limit as $(\gamma,\rho)\to(0,0)$.

Consider first the case where $\rho=0$. Then, (6.12) implies that the first limit in (6.13) is the space $\mathbb{E}_I$ spanned by the vectors e_j of the basis such that $\beta_j>0$, that is such that $j\in J_I$.

On the other hand, Corollary 6.4 implies that $B^\flat=-\mathrm{diag}(\beta_j^{-1})B^\sharp$ has no eigenvalue on the imaginary axis. Therefore, the second limit in (6.13) is the negative space $\mathbb{E}^-_{B^\flat}$ of $B^\flat$. If it were equal to $\mathbb{E}_I$, this would mean that $\mathbb{E}_I$ is invariant by $B^\flat$, thus by $B^\sharp=-\mathrm{diag}(\beta_j)B^\flat$, which contradicts (6.11).

By [22], the existence of smooth K-families of symmetrizers implies that the limit of $\mathbb{E}^-_k$ at $(\check{\underline{\zeta}},\rho)$ exists, and is equal to the space $\underline{\mathbb{E}}^-_k$ of Definition 3.11. Therefore, (6.13) implies that there are no smooth K-families of symmetrizers. □

6.3. *Viscous instabilities*

Consider boundary conditions as in Assumption 2.9. When the negative space $\mathbb{E}^-$ is not continuous in $(\check{\zeta},\rho)$, then the Evans function is likely not continuous and one can expect that the low-frequency uniform stability condition for the viscous problem is strictly stronger than the similar condition for the inviscid problem. In particular, the inviscid problem can be strongly stable while the viscous one is strongly unstable. We illustrate here this phenomenon on an explicit example.

6.3.1. *An example*

Consider the system

$$\begin{cases}(\partial_t+\partial_y)u_1+\partial_x u_2=\varepsilon\mu\Delta u_1,\\ (\partial_t+\partial_y)u_2+\partial_x u_1=\varepsilon\nu\Delta u_2.\end{cases} \tag{6.14}$$

Taking linear combinations and changing ε, the system is equivalent to

$$(\partial_t+\partial_y)\,\mathrm{Id}+A\partial_x-\varepsilon B\Delta,\qquad A=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ B=\begin{pmatrix}1&a\\a&1\end{pmatrix}, \tag{6.15}$$

with $a=|\nu-\mu|/(\nu+\mu)\in[0,1[$. This system is symmetric and satisfy Assumptions (H1) and (H2).

The hyperbolic part is diagonal: the eigenvalues are

$$\lambda_1=\eta+\xi,\qquad \lambda_2=\eta-\xi. \tag{6.16}$$

They cross on the line $\xi=0$ and are trivially geometrically regular since the system is already in diagonal form. One of the eigenvalue is incoming, one is outgoing. The decoupling condition (4.9) is satisfied if and only if $a=0$. In the sequel, we assume that $a>0$.

6.3.2. Boundary conditions

Next, consider boundary conditions for (6.15):

$$u|_{x=0} + \varepsilon \Gamma \partial_x u|_{x=0} = 0. \tag{6.17}$$

We first compute the limiting inviscid boundary conditions, using boundary layers. The bounded solutions $u = w(x/\varepsilon)$ of (6.15) are

$$w(z) = u + e^{zB^{-1}A} h, \quad h \in \mathbb{E}^-_{B^{-1}A},\ u \in \mathbb{C}^2, \tag{6.18}$$

where $\mathbb{E}^-_{B^{-1}A}$ is the negative space of $B^{-1}A$. Therefore, u is the endpoint of a profile which satisfies the boundary condition (6.17), if and only if

$$u \in \left(\mathrm{Id} + \Gamma B^{-1} A\right)\mathbb{E}^-_{B^{-1}A}. \tag{6.19}$$

Note that given any complex number $\underline{c}$, one can choose Γ such that this boundary condition reads

$$u_1 = \underline{c} u_2. \tag{6.20}$$

6.3.3. Low frequency stability

The first order system (2.21) reads

$$\partial_z U - G(\zeta)U, \quad G(\zeta) = \begin{pmatrix} 0 & \mathrm{Id} \\ \sigma B^{-1} + \eta^2 \mathrm{Id} & B^{-1}A \end{pmatrix}, \tag{6.21}$$

with $\zeta = (\tau, \eta, \gamma)$ and $\sigma = \gamma + i(\tau + \eta)$. Perform the small frequency reduction (2.28), using the change of unknowns

$$\begin{pmatrix} u \\ \partial_z u \end{pmatrix} = V(\zeta) \begin{pmatrix} u_H \\ u_P \end{pmatrix}.$$

Then, by Lemma 4.23, there holds

$$V^{-1} G V = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix}$$

with $P(0) = B^{-1}A$ and

$$H(\zeta) = -\sigma A + \left(\sigma^2 - \eta^2\right) AB + O\left(|\zeta|^3\right). \tag{6.22}$$

Since $V(0)$ has the triangular form (2.30), we see that the boundary condition reads

$$u_H + \widetilde{\Gamma}(\zeta) u_P = 0, \qquad \widetilde{\Gamma}(0) = \Gamma + A^{-1}B. \tag{6.23}$$

The Evans condition is violated at ζ if there is $u_H \in \mathbb{E}^-_H(\zeta)$ and $u_P \in \mathbb{E}^-_P(\zeta)$ satisfying this boundary condition. The negative space of $P(\zeta)$, $\mathbb{E}^-_P(\zeta)$ is smooth in ζ and equal to $\mathbb{E}^-_{B^{-1}A}$

when $\zeta = 0$. Thus, the Evans condition is violated at ζ if and only if there is $u_H \in \mathbb{E}_H^-(\zeta)$ such that

$$\mathbb{E}_H^-(\zeta) \cap \widetilde{\Gamma}(\zeta)\mathbb{E}_P^-(\zeta) \neq \{0\}.$$

Since $A^{-1}B = (B^{-1}A)^{-1}$, there holds

$$\widetilde{\Gamma}(0)\mathbb{E}_P^-(0) = \big(\mathrm{Id} + \Gamma B^{-1}A\big)\mathbb{E}_{B^{-1}A}^-.$$

Comparing with (6.19) and (6.20), we see that for ζ small, the space $\widetilde{\Gamma}(\zeta)\mathbb{E}_P^-(\zeta)$ is generated by ${}^t(c(\zeta), 1)$ where $c(\zeta)$ is a smooth function such that $c(0) = \underline{c}$. Therefore, the Evans condition is violated at ζ if and only if

$$\begin{pmatrix} c(\zeta) \\ 1 \end{pmatrix} \in \mathbb{E}_H^-(\zeta). \tag{6.24}$$

Remark 6.6. Using the terminology of [21], the analysis above shows that the *reduced boundary condition* for the hyperbolic part $H(\zeta)$ reads

$$u_1 = c(\zeta)u_2. \tag{6.25}$$

Taking $\zeta = 0$ in this equation, we recover that (6.20) is the natural limiting boundary condition for the hyperbolic operator H_0.

Proposition 6.7. *There are choices of a and Γ, such that*

(i) *the inviscid problem* (6.15) *for $\varepsilon = 0$ with the boundary condition* (6.20) *is maximal strictly dissipative thus uniformly stable,*
(ii) *the viscous problem with boundary conditions* (6.17) *is strongly unstable for small frequencies, in the sense that the Evans functions vanishes for arbitrarily small frequencies ζ with $\gamma > 0$.*

Proof. The matrix

$$S = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}, \quad s > 0 \tag{6.26}$$

is a symmetrizer for the inviscid problem. If

$$|\underline{c}|^2 < s, \tag{6.27}$$

the boundary condition is strictly dissipative for S. This implies that the uniform Lopatinski condition is satisfied.

We consider frequencies $\zeta = \rho\check{\zeta}$ with $\check{\zeta}$ close to $(-1, 1, 0)$ where $H_0(\check{\zeta}) = 0$ has a double eigenvalue. More precisely we consider frequencies

$$\zeta = \big(-\rho + \rho^2\hat{\tau}, \rho, \rho^2\hat{\gamma}\big). \tag{6.28}$$

In this case, we see that G is a function of $\hat\sigma = \hat\gamma + i\hat\tau$ and ρ, holomorphic in $\hat\sigma$, as well as V, P, H and c. Moreover

$$H(\zeta) = -\rho^2\big(\hat\sigma A + A^{-1}B + O(\rho)\big) = \rho^2 \widehat{H}(\hat\sigma, \rho). \tag{6.29}$$

The model operator is

$$\widehat{H}(\hat\sigma, 0) = -\hat\sigma A - A^{-1}B = \begin{pmatrix} -\hat\sigma - 1 & -a \\ a & \hat\sigma + 1 \end{pmatrix}.$$

$\widehat{H}(1,0)$ has one eigenvalue with positive real part, with eigenvector ${}^t(\underline{b}, 1)$ with $\underline{b} = (2 + \sqrt{4-a^2})/a$. (Note here the importance of the assumption $a \neq 0$.) Therefore, for $\hat\sigma$ close to 1 and ρ small, the negative space of $\widehat{H}(\hat\sigma, \rho)$ is generated by ${}^t(b(\hat\sigma, \rho), 1)$ where b is smooth and holomorphic in $\hat\sigma$ and $b(1,0) = \underline{b}$. Moreover

$$\partial_{\hat\sigma} b(1,0) = \frac{1}{a}\left(1 + \frac{2}{\sqrt{4-a^2}}\right) \neq 0. \tag{6.30}$$

Comparing with (6.24), we see that the stability condition is violated at ζ given by (6.28), if and only if

$$b(\hat\sigma, \rho) = c(\zeta) = \hat c(\hat\sigma, \rho). \tag{6.31}$$

Given $a \in\,]0,1[$, we choose $\underline{c} = \underline{b}$ and Γ such that the inviscid boundary condition reads (6.20). Note that $\hat c(\hat\sigma, 0) = \underline{c}$ for all $\hat\sigma$. Thus Eq. (6.31) holds at $\hat\sigma = 1$ and $\rho = 0$. Moreover, with (6.30), the implicit function theorem shows that for $\rho > 0$ small, there is $\hat\sigma(\rho)$ close to 1 solution of (6.31), providing frequencies $\zeta(\rho) = O(\rho)$ with $\gamma(\rho) \sim \rho^2 > 0$, where the stability condition is violated. □

6.3.4. Smooth symmetrizers

We briefly discuss here the existence of smooth symmetrizers for the hyperbolic operator $\check H$ (3.16). In the present case, we deduce from (6.22) that in polar coordinates $\zeta = \rho\check\zeta$, there holds

$$\check H(\check\zeta, \rho) = -\check\sigma A + \rho\big(\check\sigma^2 - \check\eta^2\big)AB + O\big(\rho^2\big), \quad \check\sigma = \check\gamma + i(\check\tau + \check\eta). \tag{6.32}$$

Fix $\underline{\check\zeta} = (1, -1, 0)$, which corresponds to a multiple root of the hyperbolic part. Then $\underline{\check\sigma} = 0$, and near $(\underline{\check\zeta}, 0)$

$$\check H(\check\zeta, \rho) = -A\big(\check\sigma\,\mathrm{Id} + \rho\beta(\check\zeta)B\big) + O\big(\rho^2\big) \tag{6.33}$$

with $\beta(\underline{\check\zeta}) = 1$. Dropping the $\check{}$, and changing ρb to ρ, the matrix $\check H$ is a perturbation for (σ, ρ) close to $(0,0)$ of the following *canonical example*

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_x + \sigma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho\begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \quad \operatorname{Re}\sigma \geqslant 0,\ \rho \geqslant 0. \tag{6.34}$$

Note that (6.29) derives from (6.33) choosing $\check{\sigma} = \rho\hat{\sigma}$.

Denote by $\mathbb{E}^-$ the negative space of $\check{H}$ for $\operatorname{Re}\sigma + \rho > 0$. On can check directly on this example that the negative spaces have no limit as $(\sigma, \rho) \to (0, 0)$: the limits are different when $\rho = 0$ and $\sigma = 0$, since the positive spaces of A and AB are different when $a \neq 0$.

On the other hand, blowing up once more the local coordinates near $\underline{\check{\zeta}}$, that is taking polar coordinates $(\sigma, \rho) = r(\hat{\sigma}, \hat{\rho})$, it is clear from (6.33) that $\mathbb{E}^-$ is a smooth function of $(\hat{\sigma}, \hat{\rho})$.

If $\Sigma(\check{\zeta}, \rho)$ is a smooth symmetrizer for $\check{H}$, then (3.17) implies that $\underline{\Sigma} = \Sigma(\underline{\check{\zeta}}, 0)$ must be a symmetrizer for $-(\sigma A + \rho AB)$ for all σ and ρ, equivalently that $S = \underline{\Sigma} A$ is a symmetrizer for (6.34), that is

$$S = S^* \gg 0, \qquad SA = AS, \qquad \operatorname{Re}(SB) \gg 0. \tag{6.35}$$

The first two conditions are satisfied if and only if S is diagonal and positive. Multiplying it by a positive factor, it must be of the form (6.26).

The third condition holds if and only if

$$s > a^2(1+s)^2/4.$$

Denoting by $s_{\min}(a) < 1 < s_{\max}(a) < \infty$ the roots of the equation $4s = a^2(1+s^2)$, the condition reads

$$s_{\min}(a) < s < s_{\max}(a). \tag{6.36}$$

This shows that the choice of symmetrizers is much more limited in the viscous case compared to the inviscid one. In particular, when a is close to 1, (6.36) forces to choose s in a small interval around 1.

The boundary condition (6.25) is strictly dissipative for Σ, then (6.20) is strictly dissipative for $\underline{\Sigma}$. This holds if and only if $s > |\underline{c}|^2$. Therefore:

There is a smooth symmetrizer $\Sigma(\check{\zeta}, \rho)$ *for* $\check{H}$ *on a neighborhood of* $(\underline{\check{\zeta}}, 0)$, *adapted to the boundary conditions* (6.25) *only if*

$$|\underline{c}|^2 < s_{\max}(a). \tag{6.37}$$

7. The high-frequency analysis

7.1. The main high-frequency estimate

This section is devoted to an analysis of uniform maximal estimates for high-frequencies. We still assume that the assumptions of Section 2 are satisfied and we prove that the anticipated (2.58) are satisfied when the uniform spectral stability conditions are satisfied, under the following additional structural assumptions which strengthens (H3): it means first that the block L^{11} is hyperbolic with constant multiplicity with respect to time, and second that it is totally incoming our outgoing.

Assumption 7.1.

(H5) For all $u \in \mathcal{U}^*$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, the eigenvalues of $\overline{A}^{11}(u, \xi)$ are real, semi-simple and have constant multiplicities.
(H6) $L^{11}(u, \partial)$ is also hyperbolic with respect to the normal direction dx_d.

For Navier–Stokes and MHD equations and in many examples L^{11} is a transport field

$$L^{11} = \partial_t + \sum_{j=1}^{d} a_j(u)\partial_j \tag{7.1}$$

and the condition reduces to $a_d(u) \neq 0$ for $u \in \mathcal{U}^*$, that is to Assumption 2.6, which means inflow or outflow boundary conditions. The hyperbolicity condition (H6) in the normal direction is important as shown on an example below. On the other hand, the constant multiplicity condition (H5) is more technical, and could be replaced by symmetry conditions: this is briefly discussed in Remark 7.12.

We consider the linearized equation (2.21):

$$\partial_z u = \mathcal{G}(z, \zeta)u + f, \qquad \Gamma(\zeta)u(0) = g, \tag{7.2}$$

with $u = {}^t(u^1, u^2, u^3)$, $f = {}^t(f^1, f^2, f^3)$, Γ as in (2.56) and $g = {}^t(g^1, g^2, g^3)$.

Theorem 7.2. *With assumptions as indicated above, assume that the uniform spectral stability condition is satisfied for high frequencies. Then there are $\rho_1 > 0$ and C such that for all $\zeta \in \overline{\mathbb{R}}^{d+1}_+$ with $|\zeta| \geqslant \rho_1$, the solutions of* (7.2) *satisfy*

$$\begin{aligned}
&(1+\gamma)\|u^1\|_{L^2} + \Lambda\|u^2\|_{L^2} + \|u^3\|_{L^2} + (1+\gamma)^{\frac{1}{2}}|u^1(0)| + \Lambda^{\frac{1}{2}}|u^2(0)| + \Lambda^{-\frac{1}{2}}|u^3(0)| \\
&\quad \leqslant C\big(\|f^1\|_{L^2} + \|f^2\|_{L^2} + \Lambda^{-1}\|f^3\|_{L^2}\big) + C\big((1+\gamma)^{\frac{1}{2}}|g^1| + \Lambda^{\frac{1}{2}}|g^2| + \Lambda^{-\frac{1}{2}}|g^3|\big).
\end{aligned} \tag{7.3}$$

High frequencies require a particular analysis for two reasons. First, the splitting hyperbolic vs parabolic is quite different in this regime and second the conjugation operator Φ of Lemma 2.10 is not uniform for large ζ. The analysis is made in [20] for full viscosities and Dirichlet boundary conditions. For partial viscosities and shocks, that is for transmission condition, the problem is solved in [7]. The presentation below is more systematic and allows for more general boundary conditions of the form (2.10).

We now explain the general strategy of the proof. We use the notations

$$\begin{aligned}
\|u\|_{\mathrm{sc}} &= (1+\gamma)\|u^1\|_{L^2} + \Lambda\|u^2\|_{L^2} + \|u^3\|_{L^2}, \\
\|f\|'_{\mathrm{sc}} &= \|f^1\|_{L^2} + \|f^2\|_{L^2} + \Lambda^{-1}\|f^3\|_{L^2}, \\
|u(0)|_{\mathrm{sc}} &= (1+\gamma)^{\frac{1}{2}}|u^1(0)| + \Lambda^{\frac{1}{2}}|u^2(0)| + \Lambda^{-\frac{1}{2}}|u^3(0)|, \\
|g|_{\mathrm{sc}} &= (1+\gamma)^{\frac{1}{2}}|g^1| + \Lambda^{\frac{1}{2}}|g^2| + \Lambda^{-\frac{1}{2}}|g^3|.
\end{aligned} \tag{7.4}$$

(1) The main step in the proof of the theorem is to separate off the incoming and outgoing components of u. This is done using a change of variables $\hat{u} = \mathcal{V}^{-1}(z,\zeta)u$ which transforms Eq. (7.2) to

$$\partial_z \hat{u} = \widehat{\mathcal{G}}(z,\zeta)\hat{u} + \hat{f}, \qquad \widehat{\Gamma}(\zeta)\hat{u}(0) = g. \tag{7.5}$$

There are norms similar to (7.4) for $\hat{u}$ and $\hat{f}$ as well; with little risk of confusion, we use here the same notations. An important property is that:

$$\begin{aligned} \|u\|_{\mathrm{sc}} \leqslant C\|\hat{u}\|_{\mathrm{sc}}, \qquad \|\hat{f}\|'_{\mathrm{sc}} \leqslant C\|f\|'_{\mathrm{sc}}, \\ \big|u(0)\big|_{\mathrm{sc}} \leqslant C\big|\hat{u}(0)\big|_{\mathrm{sc}}, \qquad \big|\hat{u}(0)\big|_{\mathrm{sc}} \leqslant C\big|u(0)\big|_{\mathrm{sc}}, \end{aligned} \tag{7.6}$$

with C independent of ζ. Moreover, $\widehat{\Gamma}(\zeta) = \Gamma(\zeta)\mathcal{V}(0,\zeta)$ satisfies

$$\big|\widehat{\Gamma}(\zeta)\hat{u}(0)\big|_{\mathrm{sc}} \leqslant C\big|\hat{u}(0)\big|_{\mathrm{sc}}. \tag{7.7}$$

The new matrix $\widehat{\mathcal{G}}$ has the important property that

$$\widehat{\mathcal{G}} = \begin{pmatrix} \widehat{\mathcal{G}}^+ & 0 \\ 0 & \widehat{\mathcal{G}}^- \end{pmatrix} + \widehat{\mathcal{G}}' \tag{7.8}$$

with

$$\big\|\widehat{\mathcal{G}}'\hat{u}\big\|'_{\mathrm{sc}} \leqslant \varepsilon(\zeta)\|\hat{u}\|_{\mathrm{sc}} \tag{7.9}$$

where $\varepsilon(\zeta)$ tends to 0 as $|\zeta|$ tends to infinity. The block structure corresponds to a splitting $\hat{u} = (\hat{u}^+, \hat{u}^-)$ with $\hat{u}^- \in \mathbb{C}^{N_b}$ and $\hat{u}^+ \in \mathbb{C}^{N+N'-N_b}$ denoting the incoming and outgoing components, respectively.

(2) One proves separate estimates for the incoming and outgoing components:

$$\big\|\hat{u}^+\big\|_{\mathrm{sc}} + \big|\hat{u}^+(0)\big| \leqslant C\big\|\big(\partial_z - \widehat{\mathcal{G}}^+\big)\hat{u}^+\big\|_{\mathrm{sc}}, \tag{7.10}$$

$$\big\|\hat{u}^-\big\|_{\mathrm{sc}} \leqslant C\big\|\big(\partial_z - \widehat{\mathcal{G}}^-\big)\hat{u}^-\big\|_{\mathrm{sc}} + C\big|\hat{u}^-(0)\big|, \tag{7.11}$$

with C independent of ζ. (The norms are defined, identifying $\hat{u}^- \in \mathbb{C}^{N_b}$ to $(0,\hat{u}^-) \in \mathbb{C}^N$ etc.) As a result, with (7.9), this implies that if $\hat{u}$ is a solution of (7.5), then

$$\big\|\hat{u}^+\big\|_{\mathrm{sc}} + \big|\hat{u}^+(0)\big| \leqslant C\|\hat{f}\|_{\mathrm{sc}} + \varepsilon(\zeta)\|\hat{u}\|_{\mathrm{sc}}, \tag{7.12}$$

$$\big\|\hat{u}^-\big\|_{\mathrm{sc}} \leqslant C\|\hat{f}\|_{\mathrm{sc}} + \varepsilon(\zeta)\|\hat{u}\|_{\mathrm{sc}} + C\big|\hat{u}^-(0)\big|. \tag{7.13}$$

(3) We show that the estimates above imply that if the uniform spectral stability condition is satisfied, then the solutions of (7.5) satisfy for $|\zeta|$ large enough

$$\|\hat{u}\|_{\mathrm{sc}} + \big|\hat{u}(0)\big|_{\mathrm{sc}} \leqslant C\big(\|\hat{f}\|_{\mathrm{sc}} + |g|_{\mathrm{sc}}\big) \tag{7.14}$$

implying that the solutions of (7.2) satisfy

$$\|u\|_{sc} + \big|u(0)\big|_{sc} \leqslant C\big(\|f\|_{sc} + |g|_{sc}\big) \tag{7.15}$$

that is (7.3).

• Indeed, by definition, $h \in \mathbb{E}^-(\zeta)$ if and only if there is u solution of $\partial_z u = \mathcal{G}u$ with $u(0) = h$. The corresponding $\hat{u} = \mathcal{V}^{-1}u$ satisfies by (7.13)

$$\|\hat{u}^-\|_{sc} \leqslant C\big|u^-(0)\big| + \varepsilon(\zeta)\|\hat{u}^+\|_{sc}$$

if ζ is large enough. Therefore, (7.12) implies that for ζ large and all $h \in \mathbb{E}^-(\zeta)$, $\hat{h} = \mathcal{V}^{-1}(0,\zeta)h = (\hat{h}^+, \hat{h}^-)$ satisfies

$$\big|\hat{h}^+\big|_{sc} \leqslant \varepsilon(\zeta)\big|\hat{h}^-\big|_{sc}. \tag{7.16}$$

• In addition $\hat{\mathbb{E}}^-(\zeta) := \mathcal{V}^{-1}(0,\zeta)\mathbb{E}^-(\zeta)$ has dimension equal to N_b, as the space of the $\hat{h}^-$. Therefore, (7.16) shows that for ζ large, the projection $h \mapsto h^-$ is bijective from $\hat{\mathbb{E}}^-(\zeta)$ to $\mathbb{C}^{N_b}$, with inverse uniformly bounded in the norm $|\cdot|_{sc}$.

The uniform spectral stability condition reads

$$\forall h \in \mathbb{E}^-(\zeta), \quad |h|_{sc} \leqslant C\big|\Gamma(\zeta)h\big|_{sc} \tag{7.17}$$

(see (2.59)). Using (7.6), this implies

$$\forall \hat{h} \in \widehat{\mathbb{E}}^-(\zeta), \quad |\hat{h}|_{sc} \leqslant C\big|\widehat{\Gamma}(\zeta)\hat{h}\big|_{sc}. \tag{7.18}$$

Using the isomorphism between $\widehat{\mathbb{E}}^-(\zeta)$ and $\mathbb{C}^{N_b}$, we see that for ζ large enough and $\hat{h}^- \in \mathbb{C}^{N_b}$, there is $\hat{h}^+$ such that $(\hat{h}^+, \hat{h}^-) \in \widehat{\mathbb{E}}^-(\zeta)$. Together with (7.16) and (7.7), there holds

$$\big|\hat{h}^-\big|_{sc} \leqslant |\hat{h}|_{sc} \leqslant C\big|\widehat{\Gamma}(\zeta)\hat{h}\big|_{sc} \leqslant C\big|\widehat{\Gamma}(\zeta)\big(0,\hat{h}^-\big)\big|_{sc} + \varepsilon(\zeta)\big|\hat{h}^-\big|_{sc}.$$

For ζ large, the last term can be dropped, increasing C. Finally, we conclude that for all $\hat{h} \in \mathbb{C}^N$

$$|\hat{h}|_{sc} \leqslant C\big|\widehat{\Gamma}(\zeta)\hat{h}\big|_{sc} + C\big|\hat{h}^+\big|_{sc}. \tag{7.19}$$

Applying this estimate to $\hat{u}(0)$, combining with (7.10) and (7.11) and absorbing the error term $\widehat{\mathcal{G}}'\hat{u}$ for ζ large, we immediately obtain (7.14).

The third part of the proof will not be repeated. We will focus on the reduction (7.5) and on the proof of the estimates for $\hat{u}^\pm$.

7.2. *Spectral analysis of the symbol*

Consider the linearized operator (2.20)

$$-\mathcal{B}\partial_z^2 + \mathcal{A}\partial_z + \mathcal{M}.$$

The coefficients satisfy

$$
\begin{aligned}
\mathcal{B}(z) &= B_{dd}\big(w(z)\big),\\
\mathcal{A}(z,\zeta) &= A_d\big(w(z)\big) - \sum_{j=1}^{d-1} i\eta_j (B_{jd} + B_{d,j})\big(w(z)\big) + E_d(z),\\
\mathcal{M}(z,\zeta) &= (i\tau+\gamma)A_0\big(w(z)\big) + \sum_{j=1}^{d-1} i\eta_j\big(A_j\big(w(z)\big) + E_j(z)\big)\\
&\quad + \sum_{j,k=1}^{d-1} \eta_j\eta_k B_{j,k}\big(w(z)\big) + E_0(z),
\end{aligned}
\tag{7.20}
$$

where the E_k are functions, independent of ζ, which involve derivatives of w and thus converge to 0 at an exponential rate when z tends to infinity. Moreover, we note that

$$
E_k^{11} = 0, \quad E_k^{12} = 0 \quad \text{for } k > 0. \tag{7.21}
$$

With (2.2), we also remark that $\mathcal{M}^{12}$ does not depend on τ and γ.

We start with a spectral analysis of the matrix $\mathcal{G}$ in (2.21). It is convenient to use here the notations $u = (u^1, u^2, u^3) \in \mathbb{C}^{N-N'} \times \mathbb{C}^{N'} \times \mathbb{C}^{N'}$. In the corresponding block decomposition of matrices and using the notations above, there holds

$$
\mathcal{G} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & \mathcal{G}^{13}\\ 0 & 0 & \mathrm{Id}\\ \mathcal{G}^{31} & \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix}, \tag{7.22}
$$

where

$$
\begin{aligned}
\mathcal{G}^{11} &= -\big(\mathcal{A}^{11}\big)^{-1}\mathcal{M}^{11}, & \mathcal{G}^{31} &= \big(\mathcal{B}^{22}\big)^{-1}\big(\mathcal{A}^{21}\mathcal{G}^{11} + \mathcal{M}^{21}\big),\\
\mathcal{G}^{12} &= -\big(\mathcal{A}^{11}\big)^{-1}\mathcal{M}^{12}, & \mathcal{G}^{32} &= \big(\mathcal{B}^{22}\big)^{-1}\big(\mathcal{A}^{21}\mathcal{G}^{12} + \mathcal{M}^{22}\big),\\
\mathcal{G}^{13} &= -\big(\mathcal{A}^{11}\big)^{-1}\mathcal{A}^{12}, & \mathcal{G}^{33} &= \big(\mathcal{B}^{22}\big)^{-1}\big(\mathcal{A}^{21}\mathcal{G}^{13} + \mathcal{A}^{22}\big).
\end{aligned}
$$

Note that $\mathcal{G}^{11}$, $\mathcal{G}^{12}$, $\mathcal{G}^{31}$ and $\mathcal{G}^{33}$ are first order (linear or affine in ζ), that $\mathcal{G}^{32}$ is second order (at most quadratic in ζ) and that $\mathcal{G}^{13}$ is of order zero (independent of ζ). We denote by $\mathcal{G}_{\mathrm{p}}^{ab}$ their principal part (leading order part as polynomials). We note that

$$
\mathcal{G}_{\mathrm{p}}^{ab}(z,\zeta) = G_{\mathrm{p}}^{ab}\big(w(z),\zeta\big) \quad \text{when } (a,b) \neq (3,1), \tag{7.23}
$$

with

$$
\begin{aligned}
G_{\mathrm{p}}^{11}(u,\zeta) &= -\big(A_d^{11}(u)\big)^{-1}\left((\gamma+i\tau)A_0^{11}(u) + \sum_{j=1}^{d-1} i\eta_j A_j^{11}(u)\right),\\
G_{\mathrm{p}}^{12}(u,\zeta) &= -\big(A_d^{12}(u)\big)^{-1}\sum_{j=1}^{d-1} i\eta_j A_j^{12}(u),
\end{aligned}
$$

$$G_{\mathrm{p}}^{13}(u) = -\big(A_d^{11}(u)\big)^{-1} A_d^{12}(u),$$

$$G_{\mathrm{p}}^{32}(u,\zeta) = \big(B^{22}(u)\big)^{-1} \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}^{22}(u),$$

$$G_{\mathrm{p}}^{33}(u,\zeta) = -\big(B^{22}(u)\big)^{-1} \sum_{j=1}^{d-1} i\eta_j \big(B_{j,d}^{22}(u) + B_{d,j}^{22}(u)\big).$$

The principal term of $\mathcal{G}^{3,1}$ involves derivatives of the profile w. Denoting by

$$p = \lim_{z\to+\infty} w(z) = w(\infty)$$

the end state of the profile w, we note that the end state of $\mathcal{G}_{\mathrm{p}}^{31}$ is

$$\mathcal{G}_{\mathrm{p}}^{31}(\infty,\zeta) = \big(B^{22}(p)\big)^{-1}\left((\gamma + i\tau)A_0^{21}(p) + \sum_{j=1}^{d-1} i\eta_j A_j^{21}(p) + A_d^{21}(p) G_{\mathrm{p}}^{11}(p,\zeta)\right).$$

There are similar formulas using the matrices $\overline{A}_j$ and $\overline{B}_{j,k}$ of (2.3).

The spectral analysis is easier when all the terms are reduced to first order. If $u = (u^1, u^2, u^3)$ is replaced by $\tilde{u} = \mathrm{h}_{|\zeta|} u := (u^1, u^2, |\zeta|^{-1} u^3)$, $\mathcal{G}$ is replaced by

$$\widetilde{\mathcal{G}} = \mathrm{h}_{|\zeta|}\mathcal{G}\mathrm{h}_{|\zeta|}^{-1} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & |\zeta|\mathcal{G}^{13} \\ 0 & 0 & |\zeta|\,\mathrm{Id} \\ |\zeta|^{-1}\mathcal{G}^{31} & |\zeta|^{-1}\mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix} := \begin{pmatrix} \mathcal{G}^{11} & \mathcal{P}^{12} \\ \mathcal{P}^{21} & \mathcal{P}^{22} \end{pmatrix} \tag{7.24}$$

with obvious definitions of $\mathcal{P}^{ab}$. Note that $\widetilde{\mathcal{G}}$ is or order one, while $\mathcal{P}^{21}$ is of order zero. Thus

$$\widetilde{\mathcal{G}}(z,\zeta) = \widetilde{\mathcal{G}}_{\mathrm{p}}(z,\zeta) + O(1), \quad \widetilde{\mathcal{G}}_{\mathrm{p}} = \begin{pmatrix} \widetilde{\mathcal{G}}_{\mathrm{p}}^{11} & \mathcal{P}_{\mathrm{p}}^{12} \\ 0 & \mathcal{P}_{\mathrm{p}}^{22} \end{pmatrix} = O\big(|\zeta|\big). \tag{7.25}$$

Moreover, since the coefficients in $\mathcal{G}$ converge exponentially at infinity, the remainder in (7.25) is uniform in $z \in \mathbb{R}_+$ and $|\zeta| \geqslant 1$. Moreover, the principal part of $\widetilde{\mathcal{P}}^{22}$ is if the form $\widetilde{\mathcal{P}}_{\mathrm{p}}^{22}(z,\zeta) = P_{\mathrm{p}}^{22}(w(z),\zeta)$.

Lemma 7.3.

(i) *For all $\zeta \in \overline{\mathbb{R}}_+^{d+1}$ with $\gamma > 0$ and $\eta \neq 0$ and for all and $z \geqslant 0$, $\widetilde{\mathcal{G}}_{\mathrm{p}}(z,\zeta)$ has no eigenvalues on the imaginary axis*; *moreover, the number of eigenvalues in* $\{\operatorname{Re}\mu < 0\}$ *is $N_b = N_+^1 + N'$.*

(ii) *For all compact subset of $\mathcal{U}^*$, there are $c > 0$ and $\delta > 0$ such that for all u in the given compact and all $\zeta \in \overline{\mathbb{R}}_+^{d+1}$ such that either $\gamma \leqslant \delta|\zeta|$ or $|\eta| \leqslant \delta|\zeta|$, the distance between the spectrum of $G_{\mathrm{p}}^{11}(u,\zeta)$ and the spectrum of $P_{\mathrm{p}}^{22}(u,\zeta)$ is larger than $c|\zeta|$.*

Proof. The spectrum of $\widetilde{\mathcal{G}}_{\mathrm{p}}$ is the union of the spectra of G_{p}^{11} and P_{p}^{22}. By homogeneity, it suffices to consider $\zeta \in \overline{S}_+^d$.

(a) $G_{\rm p}^{11}$ is related to L^{11} since $A_d^{11}(i\xi + G_{\rm p}^{11}(u,\zeta)) = L^{11}(u,\gamma+i\tau,i\eta,i\xi)$. By Assumption (H3), L^{11} is hyperbolic in the time direction, hence $G_{\rm p}^{11}$ has no eigenvalues on the imaginary axis when $\gamma > 0$; moreover, the boundary is noncharacteristic for L^{11} by Assumption 2.6, implying that the number of eigenvalues of $G_{\rm p}^{11}$ in $\{\operatorname{Re}\mu < 0\}$ is equal to the number of positive eigenvalues of A_d^{11}, that is N_+^1.

Next, note that

$$P_{\rm p}^{22} = \begin{pmatrix} 0 & |\zeta|\,\mathrm{Id} \\ |\zeta|^{-1}G_{\rm p}^{32} & G_{\rm p}^{33} \end{pmatrix}.$$

Thus, $i\xi$ is an eigenvalue of $P_{\rm p}^{22}$ if and only if 0 is an eigenvalue of $B^{22}(\eta,\xi)$, which is impossible by (H2) if $\eta \neq 0$. Thus, the eigenvalues of $P_{\rm p}^{22}$ are not purely imaginary when $\eta \neq 0$. Moreover, the number of eigenvalues in $\{\operatorname{Re}\mu < 0\}$ is N' (see [20]). This finishes the proof of (i).

(b) If $\eta = 0$, $G_{\rm p}^{32}$ and $G_{\rm p}^{33}$ vanish, hence the spectrum of $P_{\rm p}^{22}$ is $\{0\}$. On the other hand, 0 is not an eigenvalue of $G_{\rm p}^{11} = -(\gamma + i\tau)(A_d^{11})^{-1}A_0^{11}$ since A_d^{11} and A_0^{11} are invertible and $|\gamma + i\tau| = |\zeta| = 1$.

If $\gamma = 0$ and $\eta \neq 0$, the eigenvalues of $P_{\rm p}^{22}$ are not in $i\mathbb{R}$. On the other hand, by Assumption (H6) the eigenvalues of $G_{\rm p}^{11}$ are purely imaginary, thus $P_{\rm p}^{22}$ and $G_{\rm p}^{11}$ have no common eigenvalue. This finishes the proof of (ii). □

The analysis in a purely "elliptic" zone $\{\gamma \geqslant \delta|\zeta|$ and $|\eta| \geqslant \delta|\zeta|\}$ with $\delta > 0$, is easy, see below. The most difficult and important part is to understand the "hyperbolic–parabolic" decoupling in an arbitrarily small cone

$$\mathrm{C}_\delta = \bigl\{0 \leqslant \gamma \leqslant \delta|\zeta|\bigr\} \cup \bigl\{|\eta| \leqslant \delta|\zeta|\bigr\} \tag{7.26}$$

with δ such that property (ii) of Lemma 7.3 holds for u in a simply connected neighborhood $\mathcal{U}_0^*$ of a compact set which contains the curve $\{w(z),\ z \in [0,+\infty[\}$. There, the usual homogeneity and the parabolic homogeneity are in competition, leading to different classes of symbols. We use the following terminology: let $\zeta = (\tau,\gamma,\eta)$ and for a multi-index $\alpha = (\alpha_\tau,\alpha_\eta,\alpha_\gamma) \in \mathbb{N} \times \mathbb{N}^{d-1} \times \mathbb{N}$, set

$$|\alpha| = \alpha_\tau + |\alpha_\eta| \quad \text{and} \quad \langle\alpha\rangle = 2(\alpha_\tau + \alpha_\gamma) + |\alpha_\eta|.$$

Recall that the parabolic weight is $\Lambda = (1 + \tau^2 + \gamma^2 + |\eta|^4)^{\frac{1}{4}}$.

Definition 7.4.

(i) $\Gamma^m(\Omega)$ denotes the space of *homogeneous symbols* of order m, that is of functions $h(z,\zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$ such that there is $\theta > 0$ such that for all $\alpha \in \mathbb{N}^{d+1}$ and all $k \in \mathbb{N}$, there are constants $C_{\alpha,k}$ such that for $|\zeta| \geqslant 1$:

$$\bigl|\partial_\zeta^\alpha h\bigr| \leqslant C_{\alpha,0}|\zeta|^{m-|\alpha|}, \quad \text{if } k = 0, \tag{7.27}$$

$$\bigl|\partial_z^k\partial_\zeta^\alpha h\bigr| \leqslant C_{\alpha,k}e^{-\theta z}|\zeta|^{m-|\alpha|}, \quad \text{if } k > 0, \tag{7.28}$$

(ii) $\mathrm{P}\Gamma^m(\Omega)$ denotes the space of *parabolic symbols* of order m, that is of functions $h(z,\zeta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$ satisfying similar estimates with $|\zeta|^{m-|\alpha|}$ replaced by $\Lambda^{m-\langle\alpha\rangle}$.

We use the same notation for spaces of homogeneous or parabolic matrix symbols of any fixed dimension.

Lemma 7.5. *For all $\hat{\zeta} \in S^d \cap \mathrm{C}_\delta$, there is a conical neighborhood Ω of $\hat{\zeta}$ and there are matrices $\mathcal{W}_{\mathrm{p}}^{12} \in \Gamma^0(\Omega)$ and $\mathcal{W}_{\mathrm{p}}^{21}$, homogeneous of degree 0 in ζ for $u \in \mathcal{U}_0^*$ such that*

$$\mathcal{W}_{\mathrm{p}}^{21}\mathcal{G}_{\mathrm{p}}^{11} - \mathcal{P}_{\mathrm{p}}^{22}\mathcal{W}_{\mathrm{p}}^{21} = |\zeta|\mathcal{P}_{\mathrm{p}}^{21}, \tag{7.29}$$

$$\mathcal{G}_{\mathrm{p}}^{11}\mathcal{W}_{\mathrm{p}}^{12} - \mathcal{W}_{\mathrm{p}}^{12}\mathcal{P}_{\mathrm{p}}^{22} = -\mathcal{P}_{\mathrm{p}}^{12}. \tag{7.30}$$

Proof. By homogeneity, it is sufficient to construct $\mathcal{W}_{\mathrm{p}}^{21}$ for $|\zeta| = 1$. By Lemma 7.3, for $\zeta \in S^{d+1} \cap \mathrm{C}_\delta$ and $u \in \mathcal{U}_0^*$, the spectra of $G_{\mathrm{p}}^{11}(u,\zeta)$ and $P_{\mathrm{p}}^{22}(u,\zeta)$ do not intersect, so that the linear system of equation

$$XG_{\mathrm{p}}^{11}(u,\zeta) - P_{\mathrm{p}}^{22}(u,\zeta)X = Y$$

has a unique solution $X = \mathcal{X}(u,\zeta)Y$. Therefore $\mathcal{W}_{\mathrm{p}}^{21}(z,\zeta) = |\zeta|\mathcal{X}(w(z),\zeta)\mathcal{P}_{\mathrm{p}}^{21}(z,\zeta)$ satisfies (7.29). (Note that $\mathcal{P}^{21}$ is of degree 0.)

The construction of $\mathcal{W}_{\mathrm{p}}^{12}$ is similar, noticing that $\mathcal{P}_{\mathrm{p}}^{12}$ is of degree 1. □

In the block structure of $\mathcal{G}$, there holds

$$\mathcal{W}_{\mathrm{p}}^{21} = \begin{pmatrix} \mathcal{V}_{\mathrm{p}}^{21} \\ \mathcal{V}_{\mathrm{p}}^{31} \end{pmatrix}, \qquad \mathcal{W}_{\mathrm{p}}^{12} = \begin{pmatrix} \mathcal{V}_{\mathrm{p}}^{12} & \mathcal{V}_{\mathrm{p}}^{13} \end{pmatrix} \tag{7.31}$$

and (7.29) reads

$$\mathcal{V}_{\mathrm{p}}^{21}\mathcal{G}_{\mathrm{p}}^{11} - |\zeta|\mathcal{V}_{\mathrm{p}}^{31} = 0, \tag{7.32}$$

$$\mathcal{V}_{\mathrm{p}}^{31}\mathcal{G}_{\mathrm{p}}^{11} - |\zeta|^{-1}\mathcal{G}_{\mathrm{p}}^{32}\mathcal{V}_{\mathrm{p}}^{21} - \mathcal{G}_{\mathrm{p}}^{33}\mathcal{V}_{\mathrm{p}}^{31} = \mathcal{G}_{\mathrm{p}}^{31}. \tag{7.33}$$

Similarly,

$$\mathcal{G}_{\mathrm{p}}^{11}\mathcal{V}_{\mathrm{p}}^{12} - |\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{13}\mathcal{G}_{\mathrm{p}}^{32} = -\mathcal{G}_{\mathrm{p}}^{12}, \tag{7.34}$$

$$\mathcal{G}_{\mathrm{p}}^{11}\mathcal{V}_{\mathrm{p}}^{13} - |\zeta|\mathcal{V}_{\mathrm{p}}^{12} - \mathcal{V}_{\mathrm{p}}^{13}\mathcal{G}_{\mathrm{p}}^{33} = |\zeta|\mathcal{G}_{\mathrm{p}}^{13}. \tag{7.35}$$

For further use, we make the following remark: by (7.23), we see that $\mathcal{G}_{\mathrm{p}}^{12}$ and $\mathcal{G}_{\mathrm{p}}^{32}$ vanish when $\eta = 0$. Therefore, (7.34) implies that $\mathcal{V}^{12}$ also vanishes when $\eta = 0$ and hence

$$\mathcal{V}^{12}(z,\zeta) = O\big(|\eta|/|\zeta|\big). \tag{7.36}$$

With these notations, let

$$\mathcal{V}_I(z,\zeta) = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ |\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{21} & \mathrm{Id} & 0 \\ \mathcal{V}_{\mathrm{p}}^{31} & 0 & \mathrm{Id} \end{pmatrix}, \qquad \mathcal{V}_{II}(z,\zeta) = \begin{pmatrix} \mathrm{Id} & \mathcal{V}_{\mathrm{p}}^{12} & |\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{13} \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$$

and $\mathcal{V} = \mathcal{V}_I\mathcal{V}_{II}$. Using the conjugation $u = \mathcal{V}\hat{u}$, $f = \mathcal{V}\hat{f}$, for ζ in the cone C_δ, Eq. (7.2) is transformed to

$$\partial_z\hat{u} = \widehat{\mathcal{G}}\hat{u} + \hat{f}, \qquad \widehat{\Gamma}\hat{u}(0) = g, \tag{7.37}$$

with $\widehat{\mathcal{G}} = \mathcal{V}^{-1}\mathcal{G}\mathcal{V} - \mathcal{V}^{-1}\partial_z\mathcal{V}$ and $\widehat{\Gamma}(\zeta) = \Gamma(\zeta)\mathcal{V}(0,\zeta)$.

Lemma 7.6. *The entries of $\widehat{\mathcal{G}}$ satisfy*:

$$\widehat{\mathcal{G}}^{11} - \left(\mathcal{G}^{11} + |\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}\right) \in \Gamma^{-1},$$
$$\widehat{\mathcal{G}}^{12} \in \Gamma^0, \qquad \widehat{\mathcal{G}}^{13} \in \Gamma^{-1}, \qquad \widehat{\mathcal{G}}^{21} \in \Gamma^{-1}, \qquad \widehat{\mathcal{G}}^{31} \in \Gamma^0,$$
$$\widehat{\mathcal{G}}^{22} \in \Gamma^0, \qquad \widehat{\mathcal{G}}^{23} - \mathrm{Id} \in \Gamma^{-1},$$
$$\widehat{\mathcal{G}}^{32} - \left(\mathcal{G}^{32} - V^{31}\mathcal{G}^{12}\right) \in \Gamma^0, \qquad \widehat{\mathcal{G}}^{33} - \mathcal{G}^{33} \in \Gamma^0.$$

Proof. We first compute the entries of $\mathcal{G}_I = \mathcal{V}_I^{-1}\mathcal{G}\mathcal{V}_I$. Direct computations show that

$$\mathcal{G}_I^{11} = \mathcal{G}^{11} + |\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}, \qquad \mathcal{G}_I^{12} = \mathcal{G}^{12}, \qquad \mathcal{G}_I^{13} = \mathcal{G}^{13},$$
$$\mathcal{G}_I^{32} = \mathcal{G}^{32} - V^{31}\mathcal{G}^{12}, \qquad \mathcal{G}_I^{33} = \mathcal{G}^{33} - V^{31}\mathcal{G}^{13}.$$

Moreover,

$$\mathcal{G}_I^{21} = -|\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{21}\mathcal{G}^{11} + \mathcal{V}^{31} - |\zeta|^{-1}\mathcal{V}^{21}\left(|\zeta|^{-1}\mathcal{G}^{12}\mathcal{V}_{\mathrm{p}}^{21} + \mathcal{G}^{13}\mathcal{V}_{\mathrm{p}}^{31}\right).$$

The first two terms are of degree zero, and by (7.32), the sum of their principal terms vanishes; the third term is of degree -1 thus $\mathcal{G}_I^{21} \in \Gamma^{-1}$. Similarly, $\mathcal{G}_I^{31}$ is of degree 1 and its principal part vanishes by (7.33). Thus,

$$\mathcal{G}_I^{21} \in \Gamma^{-1}, \qquad \mathcal{G}_I^{31} \in \Gamma^0.$$

Next

$$\mathcal{G}_I^{22} = -|\zeta|^{-1}\mathcal{V}_{\mathrm{p}}^{21}\mathcal{G}^{12} \in \Gamma^0, \qquad \mathcal{G}_I^{22} - \mathrm{Id} = -|\zeta|^{-1}\mathcal{V}^{21}\mathcal{G}^{13} \in \Gamma^{-1}.$$

The computations for $\mathcal{G}_{II} = \mathcal{V}_{II}^{-1}\mathcal{G}_I\mathcal{V}_{II}$ are quite similar. This new conjugation annihilates the principal parts of $\mathcal{G}_I^{12}$ and $\mathcal{G}_I^{13}$ and contributes to remainder terms in the other entries.

Finally, direct computations show that $\mathcal{V}^{-1}\partial_z\mathcal{V}$ only contributes to remainder. □

The main idea is to consider (7.37) as a perturbation of the decoupled system

$$\partial_z \hat{u}^1 = \widehat{\mathcal{G}}^{11} \hat{u}^1 + \hat{f}_1, \tag{7.38}$$

$$\partial_z \begin{pmatrix} \hat{u}^2 \\ \hat{u}^3 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix} \begin{pmatrix} \hat{u}^2 \\ \hat{u}^3 \end{pmatrix} + \begin{pmatrix} \hat{f}^2 \\ \hat{f}^3 \end{pmatrix}. \tag{7.39}$$

Introduce then

$$\mathcal{G}' = \widehat{\mathcal{G}} - \begin{pmatrix} \widehat{\mathcal{G}}^{11} & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix}. \tag{7.40}$$

The next lemma how the estimates are transported by the change of variables $u = \mathcal{V}\hat{u}$. We use the notations (7.4) for the scaled norms.

Lemma 7.7. *There are constant C and ρ_1 such that for all ζ in the cone C_δ with $|\zeta| \geqslant \rho_1$, there holds*

$$\begin{aligned} &\big\|\mathcal{V}^{-1}\hat{u}\big\|_{\mathrm{sc}} \leqslant C\|\hat{u}\|_{\mathrm{sc}}, \qquad \|\mathcal{V}f\|'_{\mathrm{sc}} \leqslant C\|f\|'_{\mathrm{sc}}, \\ &\big|\mathcal{V}^{-1}\hat{u}(0)\big|_{\mathrm{sc}} \leqslant C\big|\hat{u}(0)\big|_{\mathrm{sc}}, \qquad \big|\mathcal{V}u(0)\big|_{\mathrm{sc}} \leqslant C\big|u(0)\big|_{\mathrm{sc}}, \end{aligned} \tag{7.41}$$

and

$$\big|\widehat{\Gamma}(\zeta)\hat{u}(0)\big|_{\mathrm{sc}} \leqslant C\big|\hat{u}(0)\big|_{\mathrm{sc}}. \tag{7.42}$$

Moreover,

$$\|\mathcal{G}'\hat{u}\|_{\mathrm{sc}} \leqslant C\Lambda^{-1}\|\hat{u}\|_{\mathrm{sc}}. \tag{7.43}$$

Proof. Direct computations, using (7.36), show that $u = \mathcal{V}\hat{u}$ satisfies

$$\begin{aligned} u^1 &= O(1)\hat{u}^1 + O\big(|\eta||\zeta|^{-1}\big)\hat{u}^2 + O\big(|\zeta|^{-1}\big)\hat{u}^3, \\ u^2 &= O\big(|\zeta|^{-1}\big)\hat{u}^1 + O(1)\hat{u}^2 + O\big(|\zeta|^{-1}\big)\hat{u}^3, \\ u^3 &= O(1)\hat{u}^1 + O(1)\hat{u}^2 + O(1)\hat{u}^3. \end{aligned}$$

This implies the first estimate in (7.41), using the inequalities

$$(1+\gamma)|\eta|/|\zeta| \lesssim \Lambda, \qquad (1+\gamma)/|\zeta| \lesssim 1, \qquad \Lambda/|\zeta| \lesssim 1.$$

The proof of the other estimates of (7.41) is similar, using in particular for the traces the inequality $(1+\gamma)^{\frac{1}{2}}|\eta|/|\zeta| \lesssim \Lambda^{\frac{1}{2}}$.

The inequality (7.42) follows from the second line of (7.41) and the estimate $|\Gamma u(0)|_{\mathrm{sc}} \leqslant |u(0)|_{\mathrm{sc}}$ which is a direct consequence of the form (2.56) of the boundary conditions.

Finally, Lemma 7.6 implies that $\hat{f} = \mathcal{G}'\hat{u}$ satisfies

$$\hat{f}^1 = O(1)\hat{u}^2 + O\big(|\zeta|^{-1}\big)\hat{u}^3,$$
$$\hat{f}^2 = O\big(|\zeta|^{-1}\big)\hat{u}^1 + O(1)\hat{u}^2 + O\big(|\zeta|^{-1}\big)\hat{u}^3,$$
$$f^3 = O(1)\hat{u}^1 + O(1)\hat{u}^2 + O(1)\hat{u}^3$$

and (7.43) follows. □

The parabolic bloc (7.39) is studied in [20]. We now focus on the hyperbolic block (7.38), recalling and extending the analysis of [7].

7.3. *Analysis of the hyperbolic block*

7.3.1. *The genuine coupling condition*

For $u \in \mathcal{U}^*$, denote by $\lambda_j(u,\xi)$ the distinct eigenvalues of $\overline{A}^{11}(u,\xi)$, which are real and have constant multiplicity ν_j by Assumption (H5). Assumption (H6) implies the following:

Lemma 7.8. *For all $u \in \mathcal{U}^*$, all $\xi \in \mathbb{R}^d$ and all j, there holds $\partial_{\xi_d}\lambda_j(u,\xi) \neq 0$, and all these derivatives have the same sign.*

Proof. If $\partial_{\xi_d}\lambda_j(u,\underline{\eta},\underline{\xi}_d) = 0$, then the equation $\tau + \lambda(\eta,\xi_d) = 0$ would have complex roots in ξ_d for some τ close to $\underline{\tau} = -\lambda_j(u,\underline{\eta},\underline{\xi}_d)$ (recall that $\overline{\lambda}_j$ is real analytic). Thus hyperbolicity in the normal direction prevents glancing. Moreover, by continuity the sign of $\partial_{\xi_d}\lambda_j(u,\eta,\xi_d)$ is constant for all $\xi_d \in \mathbb{R}$ when $\eta \neq 0$. Thus the functions $\xi_d \mapsto \lambda_j(u,\eta,\xi_d)$ are monotone and tend to infinity as ξ_d tends to $\pm\infty$. Since $\lambda_j \neq \lambda_k$ when $j \neq k$, they must be all increasing or all decreasing. This remains true for $\eta = 0$ by continuity. □

According to the terminology of Section 4, we will say that the hyperbolic block L^{11} is *incoming* (respectively *outgoing*) when the derivatives $\partial_{\xi_d}\lambda_j(u,\xi)$ are positive (respectively negative).

Corollary 7.9.

(i) *The matrix $G_{\mathrm{p}}^{11}(u,\zeta)$ has no purely imaginary eigenvalues when $\gamma > 0$. They are all lying in $\{\operatorname{Re}\mu > 0\}$ if the 11-block is outgoing and in $\{\operatorname{Re}\mu < 0\}$ if it is incoming.*
(ii) *Near points $\underline{\zeta}$ with $\underline{\gamma} = 0$, $G_{\mathrm{p}}^{11}(u,\zeta)$ has semi-simple eigenvalues $\mu_j(u,\zeta)$ of constant multiplicity ν_j, which are purely imaginary when $\gamma = 0$. Moreover, $\partial_\gamma \operatorname{Re}\mu_j > 0$ when the 11-block is outgoing and $\partial_\gamma \operatorname{Re}\mu_j < 0$ when the 11-block is incoming.*

Proof. Note that μ is an eigenvalue of $G_{\mathrm{p}}^{11}(u,\zeta)$ if and only if $-\tau + i\gamma$ is an eigenvalue of $\overline{A}^{11}(u,\eta,\xi)$ with $\xi = -i\mu$.

Consider the equations in ξ_d: $\tau + \lambda_j(u,\eta,\xi_d) = 0$. Since λ_j is strictly monotone and tends to infinity at both infinity, it always have a unique solution, $\psi_j(u,\eta,\tau)$ and $\partial_\tau\psi_j$ has the same sign as $-\partial_{\xi_d}\lambda_j$. This solution extends analytically for $\operatorname{Im}\tau$ small. This yields distinct eigenvalues $\mu_j(u,\zeta) = i\psi_j(u,\eta,\tau - i\gamma)$ of G_{p}^{11} for ζ close to the real domain. In particular $\partial_\gamma\mu_j = \partial_\tau\psi_j$ and the eigenvalues all lie in $\{\operatorname{Re}\mu > 0\}$ if the 11-block is outgoing and in $\{\operatorname{Re}\mu > 0\}$ if it is incoming.

The kernel of $G_{\mathrm{p}}^{11} - \mu_j$ is the kernel of $\overline{A}^{11} - \lambda_j$, thus has dimension equal to the multiplicity of λ_j. Since these dimensions add up to N^1, this shows that G_{p}^{11} has only semi-simple eigenvalues of constant multiplicity, which all lie in a given half space when $\gamma > 0$.

Hyperbolicity of L^{11} implies that $G_{\mathrm{p}}^{11}(u, \zeta)$ has no purely imaginary eigenvalues when $\gamma \neq 0$ and by continuity they all lie in the same half space. □

Next we need more information on the zero-th order correction of $\widehat{\mathcal{G}}^{11}$. From (7.20), (7.21) and (7.22) we see that

$$\widehat{\mathcal{G}}^{11}(z, \zeta) - \big(\mathcal{V}^{-1}\partial_z \mathcal{V}\big)^{11} = G_{\mathrm{p}}^{11}\big(w(z), \zeta\big) + \mathcal{E}(z, \zeta), \tag{7.44}$$

where $\mathcal{E} \in \Gamma^0$. Denote its principal part by $\mathcal{E}_{\mathrm{p}}$. Its limit at $z = \infty$ is

$$E_{\mathrm{p}}(p, \zeta) = |\zeta|^{-1} G_{\mathrm{p}}^{12}(p, \zeta) V_{\mathrm{p}}^{21}(p, \zeta) G_{\mathrm{p}}^{13}(p, \zeta) V_{\mathrm{p}}^{31}(p, \zeta) \tag{7.45}$$

where $p = \lim_{z \to +\infty} w(z)$ and $V_{\mathrm{p}}^{21}(p, \zeta)$, $V_{\mathrm{p}}^{31}(p, \zeta)$ denote the end points of $\mathcal{V}_{\mathrm{p}}^{21}$ and $\mathcal{V}_{\mathrm{p}}^{31}$, that is the solutions of the intertwining relations (7.32) and (7.33) with matrices $\mathcal{G}_{\mathrm{p}}^{ab}$ replaced bay their endpoint values $G_{\mathrm{p}}^{ab}(p, \zeta)$. The next result is crucial and follows from the genuine coupling condition (H4).

Proposition 7.10. *Fix $\underline{\zeta}$ with $|\underline{\zeta}| = 1$ and $\underline{\gamma} = 0$. For ζ in a neighborhood of $\underline{\zeta}$, consider a basis where $G^{11}(u, \zeta)$ has the block diagonal form* $\operatorname{diag}(\mu_j \operatorname{Id}_{\nu_j})$. *Denote by $E_{j,k}(u, \zeta)$ the corresponding blocks of E is this basis. Then, for $u \in \mathcal{U}$ the eingenvalues of the diagonal blocks* $\operatorname{Re} E_{j,j}$ *have a positive (respectively negative) real part if the* 11*-block is outgoing (respectively incoming).*

Proof. It is sufficient to prove the positivity at $\underline{\zeta}$. Suppose that $\gamma = 0$, denote by $\varphi_{j,p}$ with $p \in \{1, \dots, \nu_j\}$ a basis of eigenvectors of $G^{11}(u, \zeta)$. Fix j and set $\xi_d = -i\mu_j(u, \zeta) \in \mathbb{R}$, $\xi = (\eta, \xi_d)$. Then the $\varphi_{j,p}$ are right eigenvectors of $\overline{A}^{11}(u, \xi)$ associated to the eigenvalue $-\tau = \lambda_j(u, \xi)$.

Consider left eigenvectors $\ell_{j,p}$ of $\overline{A}^{11}(u, \xi)$, dual to the $\varphi_{j,p}$. Then, the left eigenvectors of $G_{\mathrm{p}}^{11}(u, \zeta)$ associated to μ_j are $\frac{1}{\beta_j}\ell_j \overline{A}_d^{11}$ with $\beta_j = \partial_{\xi_d}\lambda_j(u, \eta, \xi)$, see Lemma 4.19. The entries of the block $E_{j,j}$ are

$$\frac{1}{\beta_j}\ell_{j,p}\overline{A}_d^{11} E_{\mathrm{p}}(u, \zeta)\varphi_{j,p'}. \tag{7.46}$$

Computing the eigenvalues of order ε of $\overline{B}(u, \xi) + i\varepsilon\overline{A}(u, \xi)$, leads to consider the matrix

$$i\overline{A}^{11} + \varepsilon\overline{A}^{12}\big(\overline{B}^{22}\big)^{-1}\overline{A}^{21}. \tag{7.47}$$

The genuine coupling condition (H4) implies that for $u \in \mathcal{U}$, its spectrum lies in $\operatorname{Re}\mu > c\varepsilon$ for ε small, and this implies that the matrix $F_{j,j}$ with entries

$$\ell_{j,p}\overline{A}^{12}\big(\overline{B}^{22}\big)^{-1}\overline{A}^{21}\varphi_{j,p'} \tag{7.48}$$

has its eigenvalues in the right half plane $\{\operatorname{Re}\mu > 0\}$.

Because $G_{\mathrm{p}}^{11}\varphi_{j,p'} = i\xi_d\varphi_{j,p'}$, the relation (7.32) implies

$$V_{\mathrm{p}}^{31}\varphi_{j,p'} = |\zeta|^{-1}V_{\mathrm{p}}^{21}G_{\mathrm{p}}^{11}\varphi_{j,p'} = i\xi_d|\zeta|^{-1}V_{\mathrm{p}}^{21}\varphi_{j,p'},$$

and, using the expressions of the matrices $G^{a,b}$ yields

$$\big(|\zeta|^{-1}G_{\mathrm{p}}^{12}V_{\mathrm{p}}^{21} + G_{\mathrm{p}}^{13}V_{\mathrm{p}}^{31}\big)\varphi_{j,p'} = -i|\zeta|^{-1}\big(\overline{A}_d^{11}\big)^{-1}\overline{A}^{12}(\eta,\xi)V_{\mathrm{p}}^{21}\varphi_{j,p'}$$

and

$$\big(|\zeta|^{-1}G_{\mathrm{p}}^{32}V_{\mathrm{p}}^{21} + G_{\mathrm{p}}^{33}V_{\mathrm{p}}^{31} - V_{\mathrm{p}}^{31}G_{\mathrm{p}}^{11}\big)\varphi_{j,p'} = |\zeta|^{-1}\big(\overline{B}_{dd}^{22}\big)^{-1}\overline{B}_{22}(\eta,\xi)V_{\mathrm{p}}^{21}\varphi_{j,p'}.$$

By (7.33) this is equal to

$$-G_{\mathrm{p}}^{31}\varphi_{j,p'} = -i\big(\overline{B}_{dd}^{22}\big)^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}.$$

Thus

$$|\zeta|^{-1}V_{\mathrm{p}}^{21}\varphi_{j,p'} = -i\big(\overline{B}_{22}(\eta,\xi)\big)^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}$$

and

$$E_{\mathrm{p}}\varphi_{j,p'} = -\big(\overline{A}_d^{11}\big)^{-1}\overline{A}^{12}(\eta,\xi)\big(\overline{B}_{22}(\eta,\xi)\big)^{-1}\overline{A}^{21}(\eta,\xi)\varphi_{j,p'}.$$

Multiplying on the left by $\ell_j\overline{A}_d^{11}$, this shows that the coefficients in (7.46) and (7.48) only differ by the factor $-1/\beta_j$, and the proposition follows. □

7.3.2. Estimates

We are now in position to prove maximal estimates for the solutions of Eq. (7.38).

Proposition 7.11. *There are constants C and $\rho_1 \geqslant 1$ such that for all ζ in the cone C_δ with $|\zeta| \geqslant \rho_1$ and all $\hat{u}^1$ and $\hat{f}^1$ in $L^2(\mathbb{R}_+)$ satisfying* (7.38), *there holds*

$$\begin{aligned}(1+\gamma)\|\hat{u}^1\|_{L^2} + (1+\gamma)^{\frac{1}{2}}\big|\hat{u}^{1+}(0)\big| \\ \leqslant C\big(\|\hat{f}^1\|_{L^2} + (1+\gamma)^{\frac{1}{2}}\big|\hat{u}^{1-}(0)\big|\big)\end{aligned}\tag{7.49}$$

where $\hat{u}^{1+} = \hat{u}^1$ and $\hat{u}^{1-} = 0$ if the 11-block is outgoing and $\hat{u}^{1+} = 0$ and $\hat{u}^{1-} = \hat{u}^1$ if it is incoming.

Proof. (a) Fix $\underline{\zeta} \in \overline{S}_+^{d+1}$. We prove the estimate for ζ in a conical neighborhood of $\underline{\zeta}$. Suppose first that $\underline{\gamma} = 0$ (the most difficult case). By Corollary 7.9 there is a matrix $\mathcal{V}^{11}(z,\zeta)$ homogeneous of degree 0 such that $(\mathcal{V}^{11})^{-1}\mathcal{G}_{\mathrm{p}}^{11}\mathcal{V}^{11} = \mathrm{diag}(\mu_j(w(z),\zeta))\,\mathrm{Id}_{\nu_j}$. Setting $\hat{u}^1 = \mathcal{V}^{11}u^1$ transforms the equation to

$$\partial_z u^1 = \big(\mathrm{diag}\big(\mu_j\big(w(z),\zeta\big)\big)\,\mathrm{Id}_{\nu_j} + \widetilde{\mathcal{E}},\big)u^1 + f^1\tag{7.50}$$

with $\widetilde{\mathcal{E}} = \mathcal{E} - (\mathcal{V}^{11})^{-1}\partial_z \mathcal{V}^{11} \in \Gamma^0$, whose principal part $\widetilde{\mathcal{E}}_{\mathrm{p}}$ has the same end point $E_{\mathrm{p}}(p, \zeta)$ as $\mathcal{E}_{\mathrm{p}}$.

As usual, since the μ_j are pairwise distinct, there is a new change $u^1 = (\mathrm{Id} + \mathcal{V}_{-1})\tilde{u}^1$ with $\mathcal{V}^{11}_{-1} \in \Gamma^{-1}$, such that the resulting system has the same form with the additional property that the zero-th order part is also block diagonal, so that $\widetilde{\mathcal{E}}_{\mathrm{p}} = \mathrm{diag}(\mathcal{E}_{j,j})$ and the end points of the blocks $\mathcal{E}_{j,j}$ are $E_{j,j}$ introduced in Proposition 7.10.

The term $(\widetilde{\mathcal{E}} - \widetilde{\mathcal{E}}_{\mathrm{p}})u$ is $O(|\zeta|^{-1}|u|)$, is incorporated to f^1 and finally absorbed from the right to the left of the inequality by choosing $|\zeta|$ large enough. This reduces the proof to the case where the equation reads

$$\partial_z \hat{u}^1 = \mu_j\big(w(z), \zeta\big)\hat{u}^1 + E_{j,j}(\zeta)\hat{u}^1 + F_{j,j}(z, \zeta)\hat{u}^1 + \hat{f}^1 \tag{7.51}$$

with $|F_{j,j}| \leqslant C_0 e^{-\theta z}$.

Consider the outgoing case. Then, Corollary 7.9 implies that there is a constant $c > 0$ such that $\mathrm{Re}\, \mu_j(u, \zeta) \geqslant c\gamma$. Moreover, Proposition 7.10 implies that the eigenvalues of $E_{j,j}$ have a positive real part. Thus, there is a positive definite (constant) matrix $S(\zeta) \geqslant \mathrm{Id}$ such that $\mathrm{Re}\, SE_{j,j}$ is definite positive, say $\mathrm{Re}\, SE_{j,j} \geqslant \mathrm{Id}$. Introduce $a = C_0|S| \int_0^z e^{-\theta s} ds$ such that $\partial_z a \geqslant |SF_{j,j}|$ and a is bounded in L^∞ uniformly with respect to ζ. Therefore, multiplying the equation by $e^{2a(z)} S$ and taking the L^2 scalar product with $\hat{u}^1$ implies that

$$(1 + c\gamma)\big\|e^a \hat{u}^1\big\|^2_{L^2} + \big|\hat{u}^1(0)\big|^2 \leqslant C\big\|e^a \hat{u}^1\big\|_{L^2}\big\|e^a \hat{f}^1\big\|_{L^2}$$

which implies (7.49). The proof in the incoming case is similar.

(b) Suppose next that $\underline{\gamma} = 0$. Consider again the outgoing case. Then, the eigenvalues of G^{11}_{p} satisfy $\mathrm{Re}\, \mu \geqslant c|\zeta|$ in a conical neighborhood of $\underline{\zeta}$. This is the classical "elliptic" case. There is a symmetric definite positive matrix $S(u, \zeta) \in \Gamma^0$ such that $\mathrm{Re}\, SG^{11} \geqslant c|\zeta|\,\mathrm{Id}$ and usual integrations by parts imply that

$$c|\zeta|\big\|\hat{u}^1\big\|^2_{L^2} + \big|\hat{u}^1(0)\big|^2 \leqslant C\big\|\hat{u}^1\big\|_{L^2}\big\|\hat{f}^1\big\|_{L^2} + C_1\big\|\hat{u}^1\big\|^2_{L^2}$$

where C_1 involve estimates of the zero-th order terms, which include $\partial_z S(w(z), \zeta)$. This term is eliminated choosing $|\zeta|$ large enough. The proof in the incoming case is similar. □

Remark 7.12. The proof above contains two ingredients. First, the 11-block is totally incoming or totally outgoing, in analogy with the terminology of Section 4. Thus the decoupling incoming/outgoing is trivial. More generally, this could be replaced by a decoupling condition in the spirit of Section 4. For instance, for shocks, such a decoupling is immediate in [7] corresponding to equations on each side of the front. Next, we construct symmetrizers for the incoming and outgoing components. There we use the genuine coupling condition. If the eigenvalues are not of constant multiplicity one can introduce adapted bases or use symmetry also in the spirit of Section 4.

7.3.3. About Assumption (H6)

We show on an example that hyperbolicity in the normal direction is crucial in the proof of estimates of the form (7.49). Suppose that the L^{11}- block reads

$$\begin{cases} \partial_t u - \partial_y u + \partial_x v, \\ \partial_t v + \partial_y v + \partial_x u. \end{cases} \tag{7.52}$$

Then, on the Fourier side, the 11 equation will be of the form

$$\begin{cases} \big(i(\tau - \eta) + \gamma\big)u + \partial_z v + a(z)u = f, \\ \big(i(\tau + \eta) + \gamma\big)v + \partial_x u + a(z)v = g, \end{cases} \tag{7.53}$$

and the only information we have from the genuine coupling condition is that a is positive at $z = +\infty$. Suppose that $a(z_0) < 0$ for some $z_0 > 0$. Then glancing waves for (7.52) will propagate parallel to the boundary and thus may remain in a region where a is negative and thus may never be damped. This is illustrated by choosing $\tau = \eta$, large, $\gamma = -a(z_0) >$ and

$$u_\tau(z) = \chi\big(\tau^{\frac{1}{3}}(z - z_0)\big), \qquad v_\tau(z) = \frac{-\partial_z u_\tau}{2i\tau + \gamma + a}$$

with $\chi \in C_0^\infty(\mathbb{R})$. Then (7.53) is satisfied with $f = (a(z) - a(z_0))u_\tau + \partial_z v_\tau$ and $g = 0$. Moreover, $\|f\|_{L^2} = O(\tau^{-\frac{1}{3}})\|u\|_{L^2}$ and $u(0) = v(0) = 0$, showing that no estimate of the form (7.49) can be valid.

7.4. Proof of Theorem 7.2

7.4.1. In the cone C_δ

We consider now Eq. (7.39) and briefly recall the results from [20]. It is natural to rescale the problem using the parabolic weights: with $v^2 = \hat{u}^2$ and $v^3 = \Lambda^{-1}\hat{u}^3$ and $g^2 = \hat{f}^2$ and $g^3 = \Lambda^{-1}\hat{f}^3$the system reads

$$\partial_z \begin{pmatrix} v^2 \\ v^3 \end{pmatrix} = \mathcal{G}_P \begin{pmatrix} v^2 \\ v^3 \end{pmatrix} + \begin{pmatrix} g^2 \\ g^3 \end{pmatrix} \tag{7.54}$$

with

$$\mathcal{G}_P = \begin{pmatrix} 0 & \Lambda\,\mathrm{Id} \\ \Lambda^{-1}\mathcal{G}^{32} & \mathcal{G}^{31} \end{pmatrix} \in \mathrm{P}\Gamma^1$$

of quasi-homogeneous degree one and principal part $G_P(w(z), \zeta)$ with

$$G_P(u,\zeta) = \begin{pmatrix} 0 & \Lambda\,\mathrm{Id} \\ \Lambda^{-1}((i\tau+\gamma)(\overline{B}^{22})^{-1} + G_{\mathrm{p}}^{32}(u,\eta)) & G_{\mathrm{p}}^{31}(u,\eta) \end{pmatrix}. \tag{7.55}$$

Lemma 7.13. *(See [20].) There is $c > 0$ such that the spectrum of G_P lies in $\{|\mathrm{Re}\,\mu| \geqslant c\Lambda\}$, with N' eigenvalues, counted with their multiplicity, of positive real part. There is a smooth change of variables $\mathcal{W} \in \mathrm{P}\Gamma^0$ such that*

$$\mathcal{W}^{-1}\mathcal{G}_P\mathcal{W} = \begin{pmatrix} \mathcal{P}_+ & 0 \\ 0 & \mathcal{P}_- \end{pmatrix}$$

with $\mathcal{P}_\pm \in \mathrm{P}\Gamma^1$ having their eigenvalues satisfying $\pm\mathrm{Re}\,\mu \geqslant c\Lambda$.

Introduce

$$\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} v^2 \\ v^3 \end{pmatrix}.$$

Corollary 7.14. *(See [20].) There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geqslant \rho_1$, there holds*

$$\Lambda \| v^+ \|_{L^2} + \Lambda^{\frac{1}{2}} |v^+(0)| \leqslant C \| (\partial_z - \mathcal{P}^+) v^+ \|_{L^2},$$
$$\Lambda \| v^- \|_{L^2} \leqslant C \| (\partial_z - \mathcal{P}^-) v^- \|_{L^2} + C \Lambda^{\frac{1}{2}} |v^-(0)|.$$

Scaling back, introduce

$$\begin{pmatrix} \hat u^{2,+} \\ \hat u^{3,+} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \Lambda \end{pmatrix} \mathcal{W} \begin{pmatrix} v^+ \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \hat u^{2,-} \\ \hat u^{3,-} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \Lambda \end{pmatrix} \mathcal{W} \begin{pmatrix} 0 \\ v^- \end{pmatrix}. \tag{7.56}$$

Because, $\mathcal{W}^{-1} \partial_z \mathcal{W}$ is uniformly bounded, the corollary implies the following estimate:

Proposition 7.15. *There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geqslant \rho_1$, there holds*

$$\begin{aligned} &\Lambda \| u^{2,+} \|_{L^2} + \| u^{3,+} \|_{L^2} + \Lambda^{\frac{1}{2}} |u^{2,+}(0)| + \Lambda^{-\frac{1}{2}} |u^{3,+}(0)| \\ &\quad \leqslant C \| \hat f^2 \|_{L^2} + C \Lambda^{-1} \| \hat f^3 \|_{L^2} + \| \hat u^2 \|_{L^2} + C \Lambda^{-1} \| \hat u^3 \|_{L^2}, \\ &\Lambda \| u^{2,-} \|_{L^2} + \| u^{3,-} \|_{L^2} \leqslant C \Lambda^{\frac{1}{2}} |u^{2,-}(0)| + C \Lambda^{-\frac{1}{2}} |u^{3,-}(0)| \\ &\qquad\qquad + C \| \hat f^2 \|_{L^2} + C \Lambda^{-1} \| \hat f^3 \|_{L^2} + \| \hat u^2 \|_{L^2} + C \Lambda^{-1} \| \hat u^3 \|_{L^2}. \end{aligned}$$

Finally, with $\hat u^{1,\pm}$ as in Proposition 7.11, introduce

$$\hat u^\pm = {}^t (\hat u^{1,\pm}, \hat u^{2,\pm}, \hat u^{2,\pm}). \tag{7.57}$$

Adding up the various estimates and using (7.43), one obtains the following estimates.

Proposition 7.16. *There are C and ρ_1 such that for all $\zeta \in \mathrm{C}_\delta$ with $|\zeta| \geqslant \rho_1$ and all $\hat u \in H^1(\overline{\mathbb{R}}_+)$:*

$$\| \hat u^+ \|_{\mathrm{sc}} + |\hat u^+(0)| \leqslant C \| (\partial_z - \mathcal{G}) \hat u \|_{\mathrm{sc}} + \Lambda^{-1} \| \hat u \|_{\mathrm{sc}}, \tag{7.58}$$

$$\| \hat u^- \|_{\mathrm{sc}} \leqslant C \| (\partial_z - \mathcal{G}) \hat u \|_{\mathrm{sc}} + \Lambda^{-1} \| \hat u \|_{\mathrm{sc}} + C |\hat u^-(0)|. \tag{7.59}$$

As indicated at the end of Section 7.1, these estimates imply the maximal estimates of Theorem 7.2 provided that the boundary conditions are uniformly spectral stable.

7.4.2. Analysis in the central zone

We now consider the remaining cone where

$$\zeta \in \mathbb{R}^{d+1}, \qquad \gamma \geqslant \delta|\zeta| \quad \text{and} \quad |\eta| \geqslant \delta|\zeta|. \tag{7.60}$$

We consider the rescaled $\widetilde{\mathcal{G}}$ matrix (7.25), for the rescaled unknowns $\tilde{u} = \mathrm{h}_{|\zeta|}u := (u^1, u^2, |\zeta|^{-1}u^3)$, $\tilde{f} = \mathrm{h}_{|\zeta|}f := (f^1, f^2, |\zeta|^{-1}f^3)$. We note that in the region under consideration we now have $(1+\gamma) \approx \Lambda \approx |\zeta|$, so that the rescaled norms (7.4) are equivalent to

$$\begin{aligned} \|u\|_{\mathrm{sc}} &\approx |\zeta| \|\tilde{u}\|_{L^2}, \\ \big|u(0)\big|_{\mathrm{sc}} &\approx |\zeta|^{\frac{1}{2}} \big|\tilde{u}(0)\big|, \\ \|f\|'_{\mathrm{sc}} &\approx \|\tilde{f}\|_{L^2}. \end{aligned} \tag{7.61}$$

By Lemma 7.3, there is a smooth matrix $\mathcal{V} \in \Gamma^0$ such that

$$\mathcal{V}^{-1}(z\zeta)\mathcal{G}_{\mathrm{p}}(z,\zeta)\mathcal{V}(z,\zeta) = \begin{pmatrix} \mathcal{G}_{\mathrm{p}}^+ & 0 \\ 0 & \mathcal{G}_{\mathrm{p}}^- \end{pmatrix} := \mathcal{G}_{\mathrm{p}}^{\mathrm{diag}}$$

where the spectrum of $\mathcal{G}_{\mathrm{p}}^{\pm} \in \Gamma^1$ is contained in $\{\pm \operatorname{Re}\mu \geqslant c|\zeta|\}$. We use the notations

$$\hat{u} := \mathcal{V}\tilde{u} = \begin{pmatrix} \hat{u}^+ \\ \hat{u}^- \end{pmatrix}. \tag{7.62}$$

$\tilde{u}^+$ has dimension $N + N' - N_b$ and u^- has dimension N_b. The equation for $\hat{u}$ reads

$$\partial_z \hat{u} = \widehat{\mathcal{G}}\hat{u} + \hat{f}, \tag{7.63}$$

with $\widehat{\mathcal{G}} = \mathcal{G}^{\mathrm{diag}} + O(1)$. The ellipticity of $\mathcal{G}^{\mathrm{diag}}$ immediately implies the following estimates.

Proposition 7.17. *There are constants C and ρ_1 such that for all ζ satisfying* (7.60) *and $|\zeta| \geqslant \rho_1$ and all $\tilde{u} \in H^1(\mathbb{R}_+)$ satisfying* (7.63)*, there holds*

$$|\zeta| \big\|\hat{u}^+\big\|_{L^2} + |\zeta|^{\frac{1}{2}} \big|u^+(0)\big| \leqslant C\|\hat{f}\|_{L^2} + C\|\hat{u}\|_{L^2}, \tag{7.64}$$

$$|\zeta| \big\|\hat{u}^-\big\|_{L^2} \leqslant C\|\hat{f}\|_{L^2} + C\|\hat{u}\|_{L^2} + C|\zeta|^{\frac{1}{2}} \big|\hat{u}^-(0)\big|^2. \tag{7.65}$$

Thanks to (7.61), this is the exact analogue of Proposition 7.16 and these estimates imply the maximal estimates of Theorem 7.2 provided that the boundary conditions are uniformly spectral stable, as explained in Section 7.1.

8. Application to magnetohydrodynamics

We now apply our results to the equations of isentropic magnetohydrodynamics (MHD), for which the inviscid case was treated in [21]. The full (nonisentropic) inviscid equations have been treated in [13], and have essentially the same symbolic structure as the isentropic inviscid equations.

8.1. The equations

The equations of isentropic magnetohydrodynamics (MHD) appear in basic form as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}\big(\rho u^t u\big) + \nabla p + H \times \operatorname{curl} H = \varepsilon \nu \Delta u, \\ \partial_t H + \operatorname{curl}(H \times u) = \varepsilon \mu \Delta H, \end{cases} \tag{8.1}$$

$$\operatorname{div} H = 0, \tag{8.2}$$

where $\rho \in \mathbb{R}$ represents density, $u \in \mathbb{R}^3$ fluid velocity, $p = p(\rho) \in \mathbb{R}$ pressure, and $H \in \mathbb{R}^3$ magnetic field. When $H \equiv 0$, (8.1) reduces to the equations of isentropic fluid dynamics. We assume that ν and μ are positive.

Equations (8.1) may be put in conservative form using identity

$$H \times \operatorname{curl} H = (1/2) \operatorname{div}\big(|H|^2 I - 2H^t H\big)^{\mathrm{tr}} + H \operatorname{div} H \tag{8.3}$$

together with constraint (8.2) to express the second equation as

$$\partial_t(\rho u) + \operatorname{div}\big(\rho u^t u\big) + \nabla p + (1/2) \operatorname{div}\big(|H|^2 I - 2H^t H\big)^{\mathrm{tr}} = \varepsilon \nu \Delta u. \tag{8.4}$$

They may be put in symmetrizable (but no longer conservative) form by a further change, using identity

$$\operatorname{curl}(H \times u) = (\operatorname{div} u)H + (u \cdot \nabla)H - (\operatorname{div} H)u - (H \cdot \nabla)u \tag{8.5}$$

together with constraint (8.2) to express the third equation as

$$\partial_t H + (\operatorname{div} u)H + (u \cdot \nabla)H - (H \cdot \nabla)u = \mu \varepsilon \Delta H. \tag{8.6}$$

Forgetting the constraint equation, we get a 7×7 symmetric system.

Neglecting zero-th order terms, the linearized equations of (8.1) about (ρ, u, H) are

$$\begin{cases} D_t \dot{\rho} + \rho \operatorname{div} \dot{u}, \\ \rho D_t \dot{u} + c^2 \nabla \dot{\rho} + H \times \operatorname{curl} \dot{H} - \varepsilon \nu \Delta \dot{u}, \\ D_t \dot{H} + (\operatorname{div} \dot{u}) H - H \cdot \nabla \dot{u} - \varepsilon \mu \Delta \dot{H} \end{cases} \tag{8.7}$$

with $D_t = \partial_t + u \cdot \nabla$ and $c^2 = dp/d\rho$ which we assume to be positive. This system is hyperbolic symmetric, with symmetrizer $S = \text{block-diag}(c^2, \rho \,\mathrm{Id}, \mathrm{Id})$. It enters the general framework of linearized equations studied in this paper, with parameters (ρ, u, H).

8.2. Eigenvalues and eigenvectors

Eigenvalues and eigenvectors of the symbol solve

$$\begin{cases} \tilde{\tau}\dot{\sigma} + \rho(\xi \cdot \dot{u}) = 0, \\ \tilde{\tau}\dot{u} + c^2\dot{\sigma}\xi + v \times (\xi \times \dot{v}) = i\nu|\xi^2|\dot{u}/\rho, \\ \tilde{\tau}\dot{v} + (\xi \cdot \dot{u})v - (v \cdot \xi)\dot{u} = i\mu|\xi|^2\dot{v}, \end{cases} \tag{8.8}$$

with

$$\tilde{\tau} = \tau + u \cdot \xi, \qquad v = H\sqrt{\rho}, \qquad \dot{\sigma} = \dot{\rho}/\rho, \qquad \dot{v} = \dot{H}/\sqrt{\rho}. \tag{8.9}$$

The structure condition (2.2) is satisfied with $N' = 6$. The kernel of $B(\xi)$ is generated by ${}^{\mathrm{tr}}(1, 0, \ldots, 0)$ which is never an eigenvector of $A(\xi)$ when $\xi \neq 0$. Thus Assumptions (H1′), (H1) and (H2) are satisfied.

Consider next the inviscid problem. The seven eigenvalues of $A(\rho, u, H, \xi)$ are (see e.g. [21]):

$$\begin{cases} \lambda_0 = u \cdot \xi, \\ \lambda_{\pm s} = \lambda_0 \pm c_s|\xi|, \\ \lambda_{\pm 2} = \lambda_0 \pm v \cdot \xi, \\ \lambda_{\pm f} = \lambda_0 \pm c_f|\xi|, \end{cases} \tag{8.10}$$

with

$$c_f^2 := \frac{1}{2}\Big(c^2 + |v|^2 + \sqrt{\big(c^2 - |v|^2\big)^2 + 4b^2c^2}\Big),$$
$$c_s^2 := \frac{1}{2}\Big(c^2 + |v|^2 - \sqrt{\big(c^2 - |v|^2\big)^2 + 4b^2c^2}\Big),$$
$$c^2 = p'(\rho) > 0, \quad v = H/\sqrt{\rho}, \quad b = |\hat{\xi} \times v|, \quad \hat{\xi} = \xi/|\xi|.$$

The first eigenvalue corresponds to the transport of the constraint. It can be decoupled from the system: there is a smooth one-dimensional subspace, $\mathbb{E}_0$ such that $A(\xi) = \lambda_0$ on this space and $\mathbb{E}_0^\perp$ is stable for $A(\xi)$. The other eigenvalues are in general simple.

Lemma 8.1. *(See [21].) Assume that* $0 < |v|^2 \neq c^2$. *Consider* $\xi \in \mathbb{R}^3 \setminus \{0\}$.

(i) *When* $\xi \cdot v \neq 0$ *and* $\xi \times H \neq 0$, *the eigenvalues are simple.*

(ii) *On the manifold* $\xi \times v = 0$, λ_0 *is simple. When* $|v|^2 < c^2$ (*respectively* $|v|^2 > c^2$), $\lambda_{\pm f}$ (*respectively* $\lambda_{\pm s}$) *are simple, the other eigenvalues* $\lambda_{\pm 2} = \lambda_{\pm s}$ (*respectively* $\lambda_{\pm 2} = \lambda_{\pm f}$) *are double, algebraically regular but not geometrically regular. Moreover,*

$$\lambda_{\pm 2} - \lambda_{\pm s} = O\big(|\xi \times v|^2\big) \quad \big(\textit{respectively } \lambda_{\pm 2} - \lambda_{\pm f} = O\big(|\xi \times v|^2\big)\big). \tag{8.11}$$

(iii) *On the manifold* $\xi \cdot v = 0$ *the eigenvalues* $\lambda_{\pm f}$ *are simple and the multiple eigenvalue* $\lambda_0 = \lambda_{\pm s} = \lambda_{\pm 2}$ *is geometrically regular. More precisely, there are smooth* $\lambda_{\pm 1}$ *such that* $\{\lambda_s, \lambda_{-s}\} = \{\lambda_1, \lambda_{-1}\}$. *Moreover,*

$$\lambda_{\pm 1} = u \cdot \xi \pm \delta v \cdot \xi + O\big((v \cdot \xi)^2\big), \quad \delta = \frac{c}{\sqrt{c^2 + h^2}}. \tag{8.12}$$

One can choose smooth eigenvectors $e_0, e_{\pm 1}, e_{\pm 2}$ *such that, on the manifold* $\xi \cdot v = 0$,

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ \hat{\xi} \end{pmatrix}, \qquad e_{\pm 1} = \frac{\delta}{\sqrt{2}|v|} \begin{pmatrix} -|v|^2/c^2 \\ \mp v/\delta \\ v \end{pmatrix}, \qquad e_{\pm 2} = \frac{1}{\sqrt{2}|v|} \begin{pmatrix} 0 \\ \mp w \\ w \end{pmatrix}, \tag{8.13}$$

with $w = \hat{\xi} \times v$.

8.3. Glancing and viscous coupling

The boundary $\{x_3 = 0\}$ is noncharacteristic for the hyperbolic part if and only if

$$u_3 \notin \big\{0, \pm v_3, \pm c_s(n), \pm c_f(n)\big\} \tag{8.14}$$

where $c_s(n)$ and $c_f(s)$ are the slow and fast speed computed in the normal direction $n = (0, 0, 1)$.

Lemma 8.2. *Assume that* $0 < |v| \neq c$.

(i) *On the manifold* $\xi \times v = 0$, *the multiple eigenvalues are nonglancing if and only if* $u_3 \neq \pm v_3$. *In this case, they are totally nonglancing.*
(ii) *On the manifold* $\xi \cdot v = 0$, *the multiple eigenvalues are nonglancing if and only if* $u_3 \neq 0$, $u_3 \neq \pm v_3$ *and* $u_3 \neq \pm \delta v_3$. *They are totally nonglancing when* $|u_3| > |v_3|$.

Proof. By (8.11), on $\xi \times v = 0$, with $j = s$ when $|v| < c$ and $j = f$ when $|v| > c$, there holds

$$\partial_{\xi_3} \lambda_{\pm j} = \partial_{\xi_3} \lambda_{\pm 2} = u_3 \pm v_3.$$

This implies (i).

In addition, $\partial_{\xi_3} \lambda_0 = u_3$, $\partial_{\xi_3} \lambda_{\pm 2} = u_3 \pm v_3$, and by (8.12) $\partial_{\xi_3} \lambda_{\pm 1} = u_3 \pm \delta v_3$ on the manifold $\xi \dot{v} = 0$. This implies (ii). □

Next we study the viscous coupling of vectors e_j at geometrically regular modes. In the variables $(\dot{\rho}/\rho, \dot{u}, \dot{v})$, the system (8.7) is symmetric, with symmetrizer $S = \operatorname{diag}(c^2, \mathrm{Id}, \mathrm{Id})$, and the viscosity matrix is $B(\xi) = |\xi|^2 \operatorname{diag}(0, \nu \,\mathrm{Id}/\rho, \mu \,\mathrm{Id})$. The basis (8.13) is orthonormal for S. Therefore, according to the general rule (6.9), the matrix $B^\sharp$ is symmetric with nondiagonal entries

$$\begin{gathered} B^\sharp_{0,\pm 1} = B^\sharp_{0,\pm 2} = B^\sharp_{\pm 1,\pm 2} = 0, \\ B^\sharp_{1,-1} = \frac{\delta^2 \mu}{2} - \frac{\nu}{2\rho}, \qquad B^\sharp_{2,-2} = \frac{\mu}{2} - \frac{\nu}{2\rho}. \end{gathered} \tag{8.15}$$

When $|u_3| < |v_3|$, then one of the eigenvalue $\lambda_{\pm 2}$ is incoming and the other one outgoing (depending on the sign of v_3). Therefore, if $\mu - \nu/\rho \neq 0$, the coupling coefficient $B^\sharp_{2,-2}$ does not vanish. Summing up, we have proved:

Lemma 8.3. *If* $|u_3| < |v_3|$, *and* $\nu \neq \rho\mu$, *then the decoupling condition* (4.9) *is not satisfied at modes where* $\xi \cdot v = 0$.

Remark 8.4. The decoupling of the mode λ_0 from the other ones reflects that the constraint (8.2) is propagated by the viscous equation as well. The other partial decoupling observed above depend on the particular choice of the viscosity matrices and disappear for general B.

8.4. Shocks

Consider an inviscid planar shock. We suppose that the front is $x_3 = \sigma t$ and denote by (ρ^-, u^-, H^-) and (ρ^+, u^+, H^+) the states on the left and on the right, respectively. All the analysis of the preceding section is valid, if we change u_3 to $u_3 - \sigma$.

The jump conditions are deduced from the conservative form of the equations:

$$\begin{cases} \big[\rho(u_3-\sigma)\big] = 0, \\ \big[\rho u(u_3-\sigma)\big] + r_3\Big[p + \dfrac{1}{2}|H|^2\Big] - [H_3 H] = 0, \\ \big[(u_3-\sigma)H\big] - [H_3 u] = 0, \\ [H_3] = 0, \end{cases} \tag{8.16}$$

where $r_3 = {}^t(0,0,1)$. The last jump condition comes from the constraint equation (8.2). Apparently this system of 8 scalar equations is too large. However, projecting the third equation in the normal direction yields $\sigma[H_3] = 0$ which is implied by the last equation. This shows that (8.16) is made of 7 independent equations, as expected. Denoting by u_{tg} and H_{tg} the tangential part of u and H, that is their orthogonal projection on $r_3^\perp$, (8.16) is equivalent to

$$\begin{cases} \big[\rho(u_3-\sigma)\big] = 0, \\ \big[\rho u(u_3-\sigma)\big] + r_3\Big[p + \dfrac{1}{2}|H|^2\Big] - [H_3 H] = 0, \\ \big[(u_3-\sigma)H_{\mathrm{tg}}\big] - [H_3 u_{\mathrm{tg}}] = 0, \quad [H_3] = 0. \end{cases} \tag{8.17}$$

8.4.1. Fast Lax' shocks

Consider an extreme shock. Changing x to $-x$ if necessary, the Lax condition read:

$$\begin{gathered} u_3^- + |v_3^-| < \sigma < u_3^- + c_f^-, \\ u_3^+ + c_f^+ < \sigma. \end{gathered} \tag{8.18}$$

In particular, this implies that the front is not characteristic on both side, and that the nonglancing conditions in Lemma 8.2 are also satisfied on both side and the multiple modes are totally nonglancing. Therefore:

Proposition 8.5. *For extreme Lax shocks, the assumptions of Theorem* 1.1 *are satisfied.*

8.4.2. Slow Lax' shocks

Consider a shock associated to one of the middle eigenvalue $\lambda_{\pm s}$. Changing x to $-x$ if necessary, the Lax condition read:

$$\begin{aligned} u_3^- - c_s^- < \sigma < u_3^- + c_s^-, \\ u_3^+ + c_s^+ < \sigma < u_3^+ + \left|v_3^+\right|. \end{aligned} \tag{8.19}$$

On both side we have $|u_3 - \sigma| < |v_3|$, therefore

Proposition 8.6. *For slow Lax shocks, the decoupling condition is never satisfied.*

8.5. The $H \to 0$ limit

When $H = 0$, the system (8.1) reduces to isentropic Euler's equations and (8.17) to the corresponding Rankine Hugoniot condition.

When $H = 0$, the eigenvalues are

$$\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2} = u \cdot \xi, \qquad \lambda_{\pm 3} = \lambda_0 \pm c|\xi|. \tag{8.20}$$

In particular $\partial_{\xi_3}\lambda_0 = u_3$. Moreover, at $\underline{U} = (\rho, u, 0)$, the tangent characteristic polynomial $\underline{\Delta}$ in (4.4) is $(\tau + u \cdot \xi)^5$. Therefore, if $u_3 \neq 0$, the eigenvalue λ_0 is totally nonglancing.

Lemma 8.7. *Consider a state $\underline{U} = (\underline{\rho}, \underline{u}, 0)$. Suppose that*

$$\underline{u}_3 \notin \{-\underline{c}, 0, +\underline{c}\}. \tag{8.21}$$

Then, for U in a neighborhood of $\underline{U}$, the boundary $x_3 = 0$ is noncharacteristic for the hyperbolic linearized equation and the eigenvalues $\lambda_{\pm 3}$ are simple. Moreover, for all $\xi \neq 0$, the multiple eigenvalue λ_0 is totally nonglancing at $\underline{U}$.

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