



Dynamical realizations of l -conformal Newton–Hooke group

Anton Galajinsky*, Ivan Masterov

Laboratory of Mathematical Physics, Tomsk Polytechnic University, Lenin Ave. 30, 634050 Tomsk, Russian Federation

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ABSTRACT

The method of nonlinear realizations and the technique previously developed in [A. Galajinsky, I. Masterov, Nucl. Phys. B 866 (2013) 212, arXiv:1208.1403] are used to construct a dynamical system without higher derivative terms, which holds invariant under the l -conformal Newton–Hooke group. A configuration space of the model involves coordinates, which parametrize a particle moving in d spatial dimensions and a conformal mode, which gives rise to an effective external field. The dynamical system describes a generalized multi-dimensional oscillator, which undergoes accelerated/decelerated motion in an ellipse in accord with evolution of the conformal mode. Higher derivative formulations are discussed as well. It is demonstrated that the multi-dimensional Pais–Uhlenbeck oscillator enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry for a particular choice of its frequencies.

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1. Introduction

The Newton–Hooke algebra is an analogue of the Galilei algebra in the presence of a universal cosmological repulsion or attraction [1–3]. It is derived from the (anti) de Sitter algebra by the nonrelativistic contraction in the same way as the Galilei algebra is obtained from the Poincaré algebra [1]. A peculiar feature of the Newton–Hooke algebra is that the structure relation involving the generators of time and space translations yields the Galilei boost: $[H, P_i] = \pm \frac{1}{R^2} K_i$. The positive constant R is called the characteristic time.¹ The upper/lower sign above is realized in nonrelativistic spacetime with the negative/positive cosmological constant $\Lambda = \mp \frac{1}{R^2}$. In the flat space limit $\Lambda \rightarrow 0$ the Galilei algebra is recovered.

Conformal extensions of the Galilei algebra have recently attracted considerable interest mostly due to potential uses within the context of the nonrelativistic AdS/CFT -correspondence. Such extensions are parameterized by a positive half-integer l [4,5], thus justifying the term the l -conformal Galilei algebra [5].² In general, $(2l + 1)$ vector generators enter the algebra, which along with spatial translations and Galilei boosts involve accelerations. In constructing dynamical realizations, the generators in the algebra are

linked to constants of the motion. Because the number of functionally independent constants of the motion needed to integrate a differential equation correlates with its order, dynamical realizations of the l -conformal Galilei algebra in general involve higher derivative terms (see e.g. [6–9]). Dynamical realizations without higher derivatives have been constructed recently in [10,11] within the method of nonlinear realizations. In particular, it was demonstrated in [11] that, while the accelerations are involved in the formal algebraic structure behind the generalized oscillator equations in [10,11], they prove to be functionally dependent.

Although the l -conformal extension of the Newton–Hooke algebra is known for a long time [12,13],³ its dynamical realizations remain almost completely unexplored (for related earlier studies see [14–19]). In this work, we continue the analysis initiated in [11] and construct dynamical realizations of the l -conformal Newton–Hooke group in terms of second order differential equations. Higher derivative formulations are considered as well. In particular, it is shown that the multi-dimensional Pais–Uhlenbeck oscillator enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry for a particular choice of its frequencies.

The work is organized as follows. In Section 2 the method of nonlinear realizations and the technique previously developed in [11] are used to construct a dynamical realization without higher derivatives for the l -conformal Newton–Hooke group with $\Lambda < 0$. Similar analysis for $\Lambda > 0$ is presented in Section 3. Higher derivative formulations are discussed in Section 4. In particular, it is

* Corresponding author.

E-mail addresses: galajin@tpu.ru (A. Galajinsky), masterov@tpu.ru (I. Masterov).

¹ Being multiplied with the speed of light, it yields the radius of the parent (anti) de Sitter space.

² In modern literature the reciprocal $N = \frac{1}{l}$ is called the rational dynamical exponent. The corresponding algebra is referred to as the conformal Galilei algebra with rational dynamical exponent or N -Galilean conformal algebra (see e.g. [6,7]). In this work we use the terminology originally introduced in [5].

³ Note that the flat space limit of the l -conformal Newton–Hooke algebra in [12] does not yield the l -conformal Galilei algebra. This shortcoming was overcome in [13].

shown that a variant of the multi-dimensional Pais–Uhlenbeck oscillator is invariant under the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry. In Section 5 we summarize our results and discuss possible further developments.

2. Dynamical realization of l -conformal Newton–Hooke algebra with $\Lambda < 0$

The l -conformal Newton–Hooke algebra involves the generator of time translations H , the generator of dilatations D , the generator of special conformal transformations K , the set of vector generators $C_i^{(n)}$ with $n = 0, \dots, 2l$, $i = 1, \dots, d$, and the generators of spatial rotations M_{ij} . The structure relations for the algebra read [13]

$$\begin{aligned} [H, D] &= i \left(H \mp \frac{2}{R^2} K \right), \\ [H, C_i^{(n)}] &= i \left(n C_i^{(n-1)} \pm \frac{(n-2l)}{R^2} C_i^{(n+1)} \right), \\ [H, K] &= 2iD, \quad [D, K] = iK, \\ [D, C_i^{(n)}] &= i(n-l)C_i^{(n)}, \quad [K, C_i^{(n)}] = i(n-2l)C_i^{(n+1)}, \\ [M_{ij}, C_k^{(n)}] &= i\delta_{jk}C_i^{(n)} - i\delta_{ik}C_j^{(n)}, \\ [M_{ij}, M_{kl}] &= i(\delta_{jk}M_{il} + \delta_{il}M_{jk} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik}). \end{aligned} \tag{1}$$

In the first and second lines the upper/lower sign corresponds to a negative/positive cosmological constant. Below we construct a dynamical realization of the l -conformal Newton–Hooke algebra in terms of second order differential equations for the case of a negative cosmological constant. Positive cosmological constant is discussed in Section 3.

In order to construct a dynamical realization, we resort to the method of nonlinear realizations [20–22]. The first ingredient is the coset space arising from the quotient by a subgroup of rotations⁴

$$G = e^{itH} e^{izK} e^{iuD} e^{ix_i^{(n)} C_i^{(n)}}, \tag{2}$$

which is parametrized by the real coordinates t, u, z and $x_i^{(n)}$. Left multiplication by the group element $g = e^{iaH} e^{ibK} e^{icD} e^{ix_i^{(n)} C_i^{(n)}} \times e^{\frac{1}{2}\omega_{ij}M_{ij}}$ determines the action of the group on the coset. For practical applications the infinitesimal form is enough. In particular, for the case at hand one finds the following generators of the infinitesimal transformations⁵

$$\begin{aligned} \tilde{H} &= \frac{\partial}{\partial t}, \\ \tilde{D} &= \frac{R}{2} \sin \frac{2t}{R} \frac{\partial}{\partial t} + \cos \frac{2t}{R} \frac{\partial}{\partial u} - \left(\frac{1}{R} \sin \frac{2t}{R} + z \cos \frac{2t}{R} \right) \frac{\partial}{\partial z}, \\ \tilde{K} &= \frac{R^2}{2} \left(1 - \cos \frac{2t}{R} \right) \frac{\partial}{\partial t} + R \sin \frac{2t}{R} \frac{\partial}{\partial u} \\ &\quad + \left(\cos \frac{2t}{R} - zR \sin \frac{2t}{R} \right) \frac{\partial}{\partial z}, \\ \tilde{C}_i^{(0)} &= \sum_{n=0}^{2l} \sum_{m=0}^n \frac{(-1)^n}{R^m} \frac{(2l)!}{m!(n-m)!(2l-n)!} \end{aligned}$$

⁴ Here and in what follows the summation over repeated indices is understood.

⁵ As usual, given a generator \tilde{T} , an infinitesimal parameter α and a coordinate y , the infinitesimal transformation reads $y' = y + \alpha \tilde{T}y$, where \tilde{T} is to be understood as a differential operator acting on y .

$$\begin{aligned} &\times \left(\cos \frac{t}{R} \right)^{2l-m} \left(\sin \frac{t}{R} \right)^m z^{n-m} e^{u(n-l)} \frac{\partial}{\partial x_i^{(n)}}, \\ \tilde{M}_{ij} &= x_i^{(n)} \frac{\partial}{\partial x_j^{(n)}} - x_j^{(n)} \frac{\partial}{\partial x_i^{(n)}}, \end{aligned} \tag{3}$$

while $\tilde{C}_i^{(n)}$ with $n > 0$ are obtained from $C_i^{(0)}$ and K by computing the successive commutators

$$\tilde{C}_i^{(n)} = \frac{(-1)^n (2l-n)!}{(2l)!} \underbrace{[\tilde{K}, [\tilde{K}, \dots [\tilde{K}, \tilde{C}_i^{(0)}] \dots]]}_{n \text{ times}}. \tag{4}$$

In obtaining (3) the standard Baker–Campbell–Hausdorff formula proves to be helpful. Then one constructs the Maurer–Cartan one-forms

$$G^{-1}dG = i(w_H H + w_D D + w_K K + w_i^{(n)} C_i^{(n)}), \tag{5}$$

where

$$\begin{aligned} w_H &= e^{-u} dt, \quad w_D = du - 2z dt, \\ w_K &= e^u (dz + z^2 dt) + \frac{2}{R^2} \sinh u dt, \\ w_i^{(n)} &= dx_i^{(n)} - (n+1)x_i^{(n+1)} w_H - (n-l)x_i^{(n)} w_D \\ &\quad - (n-2l-1)x_i^{(n-1)} \left(w_K + \frac{1}{R^2} w_H \right), \end{aligned} \tag{6}$$

which are invariant under all the transformations from the l -conformal Newton–Hooke group but for rotations with respect to which $w_i^{(n)}$ is transformed as a vector. It is assumed in (6) that $x_i^{(-1)} = x_i^{(2l+1)} = 0$.

The Maurer–Cartan one-forms are the building blocks of a dynamical realization. Setting some of them to zero one can either reduce the number of degrees of freedom via algebraic equations or obtain dynamical equations of motion, which are automatically invariant under the action of a given group [22]. In this work we choose the following constraints

$$w_D = 0, \quad \tilde{\gamma}^{-1} w_K - \tilde{\gamma} w_H = 0, \quad w_i^{(n)} = 0, \tag{7}$$

where $\tilde{\gamma}$ is an arbitrary (coupling) constant. The first two restrictions in (7) are the conventional constraints associated with the conformal subalgebra $so(2, 1)$ [23]. Taking t to be the temporal coordinate and introducing the new variable

$$\rho = e^{\frac{u}{2}} \tag{8}$$

from the first two equations in (7) one readily gets

$$z = \frac{\dot{\rho}}{\rho}, \quad \ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{R^2}, \tag{9}$$

where we abbreviated $\gamma^2 = \tilde{\gamma}^2 + \frac{1}{R^2}$ and the dot denotes the derivative with respect to time. From (9) one concludes that z is not independent and can be discarded, while ρ describes the conformal particle in one dimension in the harmonic trap. This degree of freedom is conveniently described by the effective action functional

$$S_\rho = \int dt \left(\dot{\rho}^2 - \frac{\gamma^2}{\rho^2} - \frac{\rho^2}{R^2} \right). \tag{10}$$

In particular, applying the Noether theorem to the infinitesimal conformal symmetry transformations, which can be deduced from (3), one gets the conserved charges

$$\begin{aligned}\mathcal{H} &= \dot{\rho}^2 + \frac{\gamma^2}{\rho^2} + \frac{\rho^2}{R^2}, \\ \mathcal{D} &= \rho \dot{\rho} \cos \frac{2t}{R} - \frac{1}{2} \mathcal{H} R \sin \frac{2t}{R} + \frac{1}{R} \rho^2 \sin \frac{2t}{R}, \\ \mathcal{K} &= \rho^2 \cos \frac{2t}{R} - \rho \dot{\rho} R \sin \frac{2t}{R} + \frac{1}{2} \mathcal{H} R^2 \left(1 - \cos \frac{2t}{R}\right),\end{aligned}\quad (11)$$

which allow one to determine the dynamics of ρ by purely algebraic means

$$\rho(t) = \sqrt{\frac{(\mathcal{D} R \sin \frac{t}{R} + \mathcal{K} \cos \frac{t}{R})^2 + (\gamma R \sin \frac{t}{R})^2}{\mathcal{K}}}. \quad (12)$$

Note that the conserved charges (11) are functionally dependent

$$\mathcal{H}\mathcal{K} - \mathcal{D}^2 - \frac{\mathcal{K}^2}{R^2} = \gamma^2. \quad (13)$$

From the latter relation it also follows that \mathcal{K} cannot be zero or negative, which is essential in Eq. (12) above.

Now let us turn to the rightmost constraint in (7) which determines $x_i^{(n)}$. Taking into account the first two relations in (7) and Eqs. (6) and (8), one readily finds

$$\rho^2 \dot{x}_i^{(n)} = (n+1)x_i^{(n+1)} + (n-2l-1)\gamma^2 x_i^{(n-1)}. \quad (14)$$

Beautifully enough, this system of ordinary differential equations is analogous to that, which appeared recently in [11] in constructing dynamical realizations for the l -conformal Galilei algebra. There are two essential differences, however. As compared to flat spacetime, the effective frequency of oscillations is increased due to the contribution proportional to the cosmological constant

$$\tilde{\gamma}^2 \rightarrow \gamma^2 = \tilde{\gamma}^2 + \frac{1}{R^2} \quad (15)$$

and the external field provided by the conformal mode $\rho_G(t)$ [11]

$$\rho_G(t) = \sqrt{\frac{(\mathcal{D} + t\mathcal{H})^2 + \tilde{\gamma}^2}{\mathcal{H}}}, \quad (16)$$

is promoted to (12).

The system (14) can be treated in two different ways. Taken literally, it is equivalent to one dynamical equation, which involves higher derivative terms [7,9,11]. On the other hand, it can be rewritten as a collection of subsystems of differential equations [11], each of which is invariant under the l -conformal Newton–Hooke group and involves at most two derivatives. In order to construct such invariant subsystems, it proves sufficient to rewrite (14) in the matrix form

$$\rho^2 \dot{x}_i^{(n)} = x_i^{(m)} A^{mn} \quad (17)$$

and find eigenvalues and eigenvectors of A^{mn} . As was demonstrated in [11], the eigenvalues appear in complex conjugate pairs $\pm ip\gamma$, where $p = 0, 2, 4, \dots, 2l$ for an integer l and $p = 1, 3, \dots, 2l$ for a half-integer l . The corresponding eigenvectors go in pairs as well, which we denote by $v_{(n)}^p$ and $\bar{v}_{(n)}^p$. The desired dynamical realization is constructed by analogy with [11]

$$\ddot{\rho} = \frac{\gamma^2}{\rho^3} - \frac{\rho}{R^2}, \quad \rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} X_i^p \right) + (p\gamma)^2 X_i^p = 0, \quad (18)$$

where the dynamical field X_i^p is built from the eigenvectors of A^{mn} and the original variables $x_i^{(n)}$ in the following way:

$$X_i^p = x_i^{(n)} (v_{(n)}^p + \bar{v}_{(n)}^p). \quad (19)$$

It turns out that the option $p = 0$ leads to a first order differential equation [11], which should be discarded on physical grounds. On the space of fields $\rho(t)$ and $X_i^p(t)$ the l -conformal Newton–Hooke group acts as follows:

$$\rho'(t') = \rho(t) + \delta\rho, \quad X_i^p(t') = X_i^p(t) + \delta X_i^p, \quad (20)$$

where $\delta\rho$ and δX_i are inherited from the coset transformations with the generators (3).

Thus, for any value of p from the range $1, \dots, 2l$ one can construct a pair of the second order differential equations (18), which provide a dynamical realization of the l -conformal Newton–Hooke group. Focusing on integer l , one can notice that the value $(2\gamma)^2$ appears for any l , the equation involving $(4\gamma)^2$ is common for all $l > 1$, the frequency $(6\gamma)^2$ is shared by all $l > 2$ etc. A similar observation can be made for a half-integer l as well. The reason why one can realize different l -conformal Newton–Hooke groups in one and the same equation is that all the vector generators $C_i^{(n)}$ with $n > 1$ prove to be functionally dependent (see Ref. [11] for more details). Although $C_i^{(n)}$ with $n > 1$ are involved in the formal algebraic structure leading to (18), they prove to be irrelevant for actual solving thereof.

Because the variables ρ and X_i^p are separated in (18), one can solve the differential equation for the conformal mode ρ and substitute its solution (12) to the equation for X_i^p . The latter then describes a particle in d spatial dimensions moving in an external field. The general solution of the rightmost equation in (18) is readily obtained by introducing a new variable s such that

$$\begin{aligned}\rho^2 \frac{d}{dt} &= \frac{d}{ds} \rightarrow \frac{ds}{dt} = \frac{1}{\rho^2} \\ \rightarrow s(t) &= \frac{1}{\gamma} \arctan \frac{\mathcal{D}\mathcal{K} + (\mathcal{D}^2 + \gamma^2)R \tan \frac{t}{R}}{\gamma\mathcal{K}}.\end{aligned}\quad (21)$$

This yields

$$X_i^p = \alpha_i^p \cos(p\gamma s(t)) + \beta_i^p \sin(p\gamma s(t)), \quad (22)$$

where α_i^p and β_i^p are constants of integration. We assume γ to be positive.

Let us confront a qualitative behavior of a particle described by Eq. (22) with that invariant under the l -conformal Galilei symmetry. In the latter case, the solution reads as in (22), while the subsidiary function $s(t)$ takes the form [11]

$$s(t) = \frac{1}{\tilde{\gamma}} \arctan \left(\frac{\mathcal{D} + t\mathcal{H}}{\tilde{\gamma}} \right) \rightarrow \dot{s}(t) = \frac{1}{\rho_G^2(t)} \quad (23)$$

with $\rho_G(t)$ displayed above in Eq. (16). Here \mathcal{D} and \mathcal{H} are constants of the motion and $\tilde{\gamma}$ is the coupling constant corresponding to the conventional conformal mechanics in $d = 1$ without the oscillator potential. We assume $\tilde{\gamma}$ to be positive. For definiteness, in what follows we choose $p = 2$ ($l = 1$) and stick to the case of three spatial dimensions. Making use of the rotation invariance, one can choose a coordinate system in which the motion occurs in the xy -plane, with α_i^6 in (22) being parallel to the x -axis. The orbit is an ellipse with one point corresponding to the polar angle $\phi = 2\tilde{\gamma}s = \pi$ removed. One can readily verify that, being initially at rest close to $X_i = -\alpha_i$ (as $t \rightarrow -\infty$), the particle starts moving towards $X_i = \alpha_i$ with growing angular velocity $\frac{d\phi}{dt} = 2\tilde{\gamma} \frac{ds}{dt} = \frac{2\tilde{\gamma}}{\rho_G^2(t)}$.

Given the initial data \mathcal{D} , \mathcal{H} , it arrives there at $t = -\frac{\mathcal{D}}{\mathcal{H}}$, which corresponds to the polar angle $\phi = 2\tilde{\gamma}s = 0$. Then it continues

⁶ Here and in what follows we omit the superscript $p = 2$ attached to α_i and X_i .

to move towards $\chi_i = -\alpha_i$ with decreasing angular velocity and freezes up as $t \rightarrow \infty$.

For a particle invariant under the l -conformal Newton–Hooke group the shape of the orbit is the same but a qualitative behavior is different. The range of the temporal coordinate which enters Eqs. (21) and (22) is $-\frac{\pi R}{2} < t < \frac{\pi R}{2}$. As $t \rightarrow \pm \frac{\pi R}{2}$ the angular velocity $\frac{d\phi}{dt} = 2\gamma \frac{ds}{dt} = \frac{2\gamma}{\rho^2}$ tends to the constant value $(\frac{2\gamma\mathcal{K}}{\mathcal{D}^2 + \gamma^2}) \frac{1}{R^2}$, which is proportional to the cosmological constant $\Lambda = \frac{1}{R^2}$. If $\mathcal{D} > 0$, one reveals three regimes in which the angular velocity first increases, then decreases and then increases again. Likewise, for $\mathcal{D} < 0$ two phases of decelerated motion are separated by the acceleration phase in the middle. In both the cases the three regimes are separated by two roots of $\tan \frac{2t}{R} = \frac{2\mathcal{D}\mathcal{K}R}{\mathcal{K}^2 - (\mathcal{D}^2 + \gamma^2)R^2}$. For $\mathcal{D} = 0$ there are two regimes, in which acceleration is followed by deceleration for $\gamma^2 - \frac{\mathcal{K}^2}{R^2} > 0$ and vice versa for $\gamma^2 - \frac{\mathcal{K}^2}{R^2} < 0$. If $\mathcal{D} = 0$ and $\mathcal{K} = \gamma R$ the motion is uniform. Note that, because the angular velocity is fully determined by the conformal mode, the qualitative difference in the motion of a particle along the orbit in the case of the l -conformal Galilei symmetry and its Newton–Hooke counterpart correlates with the dynamics of $\rho(t)$. For the former case, the conformal mode is scattered off the repulsive potential $\frac{\tilde{\gamma}^2}{\rho^2}$ and its motion is unbounded, while for the latter case it is confined to move in the potential well $\frac{\tilde{\gamma}^2}{\rho^2} + \frac{\rho^2}{R^2}$. One can verify that for $p > 2$ the qualitative picture is similar but a particle makes more than one revolution in the ellipse.

3. Dynamical realization of l -conformal Newton–Hooke algebra for $\Lambda > 0$

For a positive cosmological constant the construction of a dynamical realization proceeds along similar lines. In particular, choosing the same constraints as in (7), one gets the following equation of motion for the conformal mode:

$$\ddot{\rho} = \left(\tilde{\gamma}^2 - \frac{1}{R^2} \right) \frac{1}{\rho^3} + \frac{\rho}{R^2}. \quad (24)$$

As compared to the previous case, the full potential $(\tilde{\gamma}^2 - \frac{1}{R^2}) \frac{1}{\rho^2} - \frac{\rho^2}{R^2}$ is unbounded below because of the flip of sign in the oscillator contribution. This leads to the well known difficulties in treating $\rho(t)$ as a stable dynamical system. Yet, regarding $\rho(t)$ as a fixed background field, one can construct a reasonable dynamical equation for $x_i^{(n)}(t)$, which is subject to the rightmost constraint in (7).

The analysis depends on whether $(\tilde{\gamma}^2 - \frac{1}{R^2})$ is positive, negative or zero. If the constant is positive, the general solution of (24) reads

$$\rho(t) = \sqrt{\frac{(\mathcal{D}R \sinh \frac{t}{R} + \mathcal{K} \cosh \frac{t}{R})^2 + (\gamma R \sinh \frac{t}{R})^2}{\mathcal{K}}}, \quad (25)$$

where we abbreviated $\gamma^2 = (\tilde{\gamma}^2 - \frac{1}{R^2})$. Applying the method in [11], one then derives the dynamical equation for the vector variables

$$\rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} \chi_i^p \right) + (p\gamma)^2 \chi_i^p = 0, \quad (26)$$

which is the same as the rightmost equation in (18) but for the explicit form of the background field $\rho(t)$. As above, the fields χ_i^p are constructed from the eigenvectors of the matrix (17) in accord with (19) and the range of p is $p = 2, 4, \dots, 2l$ for an integer l and $p = 1, 3, \dots, 2l$ for a half-integer l . Given p , the pair (24) and (26) thus provides a dynamical realization of the l -conformal Newton–Hooke algebra for the case of a positive cosmological constant.

The general solution of Eq. (26) has the form (22), in which the subsidiary function $s(t)$ now reads

$$s(t) = \frac{1}{\gamma} \arctan \frac{\mathcal{D}\mathcal{K} + (\mathcal{D}^2 + \gamma^2)R \tanh \frac{t}{R}}{\gamma\mathcal{K}}, \quad \dot{s}(t) = \frac{1}{\rho^2}. \quad (27)$$

For example, choosing $p = 2$ and $d = 3$, one reveals accelerated motion of a particle in the ellipse, which turns into decelerated motion at $\tanh \frac{2t}{R} = -\frac{2\mathcal{D}\mathcal{K}R}{\mathcal{K}^2 + (\mathcal{D}^2 + \gamma^2)R^2}$. This again correlates with the evolution of $\rho(t)$.

If $(\tilde{\gamma}^2 - \frac{1}{R^2})$ is negative, one abbreviates $(\tilde{\gamma}^2 - \frac{1}{R^2}) = -\gamma^2$ and obtains the dynamical equations and their solutions by the formal substitution $\gamma \rightarrow i\gamma$ in the equations above. However, because in this case $\rho(t)$ may fall into the center, the velocity of χ_i^p may grow unbounded and the solution should be discarded as unphysical.

Finally, if $(\tilde{\gamma}^2 - \frac{1}{R^2})$ is zero, the method in [11] fails to produce second order differential equations and the resulting dynamical realization involves higher derivative terms.

4. $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry in Pais–Uhlenbeck oscillator

In two preceding sections we were mainly concerned with the construction of dynamical realizations for the l -conformal Newton–Hooke group in terms of second order differential equations. Yet, under certain circumstances, higher derivative formulations may turn out to be useful as well. Below we demonstrate that the multi-dimensional Pais–Uhlenbeck oscillator [24] enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry for a special choice of its frequencies.

Let us consider the case of a negative cosmological constant and stick to $l = \frac{3}{2}$. The dynamical equations (14) for the vector coordinates read

$$\begin{aligned} \rho^2 \ddot{x}_i^{(0)} &= x_i^{(1)}, & \rho^2 \dot{x}_i^{(1)} &= 2x_i^{(2)} - 3\gamma^2 x_i^{(0)}, \\ \rho^2 \dot{x}_i^{(2)} &= 3x_i^{(3)} - 2\gamma^2 x_i^{(1)}, & \rho^2 \ddot{x}_i^{(3)} &= -\gamma^2 x_i^{(2)}. \end{aligned} \quad (28)$$

Instead of rewriting them as two second order differential equations invariant under the $l = \frac{3}{2}$ -conformal Newton–Hooke group, let us eliminate $x_i^{(1)}$, $x_i^{(2)}$, $x_i^{(3)}$ via the algebraic equations entering (28), and derive the fourth order differential equation for the remaining $x_i^{(0)}$ (in what follows we omit the superscript attached to $x_i^{(0)}$)

$$\begin{aligned} \rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} x_i \right) \right) \right) + 10\gamma^2 \rho^2 \frac{d}{dt} \left(\rho^2 \frac{d}{dt} x_i \right) \\ + 9\gamma^4 x_i = 0. \end{aligned} \quad (29)$$

It is assumed that ρ obeys the leftmost equation in (18). In particular, choosing the static solution for the conformal mode $\rho = \sqrt{\gamma R}$, which means that $\rho(t)$ is at rest at the minimum of the corresponding potential $\frac{\tilde{\gamma}^2}{\rho^2} + \frac{\rho^2}{R^2}$, one arrives at

$$x_i^{(4)} + \frac{10}{R^2} \ddot{x}_i + \frac{9}{R^4} x_i = 0. \quad (30)$$

This is a particular case of the multi-dimensional Pais–Uhlenbeck oscillator $x_i^{(4)} + \alpha^2 \ddot{x}_i + \beta^2 x_i = 0$, which in general involves two arbitrary frequencies α^2 and β^2 .

Let us demonstrate that (30) enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry. The action functional reproducing (30) reads

$$S = \int \left(\dot{x}_i^2 - \frac{10}{R^2} \dot{x}_i^2 + \frac{9}{R^4} x_i^2 \right) dt. \quad (31)$$

Taking into account the explicit form of the vector generators in (3), the relations (8), (9) which link u and z to ρ , and the explicit form of the static ρ above, one obtains the following transformation laws

$$x'_i(t') = x_i(t) + \lambda_i^{(0)} \cos^3 \frac{t}{R} + \lambda_i^{(1)} R \cos^2 \frac{t}{R} \sin \frac{t}{R} + \lambda_i^{(2)} R^2 \cos \frac{t}{R} \sin^2 \frac{t}{R} + \lambda_i^{(3)} R^3 \sin^3 \frac{t}{R}, \quad (32)$$

which are associated with the generators $C_i^{(0)}$, $C_i^{(1)}$, $C_i^{(2)}$ and $C_i^{(3)}$ in the algebra. The Noether theorem then yields constants of the motion

$$\begin{aligned} C_i^{(0)} &= x_i^{(3)} \cos^3 \frac{t}{R} + 3\ddot{x}_i \cos^2 \frac{t}{R} \sin \frac{t}{R} + 7\dot{x}_i \frac{1}{R^2} \cos^3 \frac{t}{R} \\ &\quad + 6\dot{x}_i \frac{1}{R^2} \cos \frac{t}{R} \sin^2 \frac{t}{R} + 9x_i \frac{1}{R^3} \cos^2 \frac{t}{R} \sin \frac{t}{R} \\ &\quad + 6x_i \frac{1}{R^3} \sin^3 \frac{t}{R}, \\ C_i^{(1)} &= x_i^{(3)} R \cos^2 \frac{t}{R} \sin \frac{t}{R} + 2\ddot{x}_i \cos \frac{t}{R} \sin^2 \frac{t}{R} - \ddot{x}_i \cos^3 \frac{t}{R} \\ &\quad + 3\dot{x}_i \frac{1}{R} \cos^2 \frac{t}{R} \sin \frac{t}{R} + 2\dot{x}_i \frac{1}{R} \sin^3 \frac{t}{R} - 3x_i \frac{1}{R^2} \cos^3 \frac{t}{R}, \\ C_i^{(2)} &= R^2 x_i^{(3)} \cos \frac{t}{R} \sin^2 \frac{t}{R} + \ddot{x}_i R \sin^3 \frac{t}{R} - 2\ddot{x}_i R \cos^2 \frac{t}{R} \sin \frac{t}{R} \\ &\quad + 3\dot{x}_i \cos \frac{t}{R} \sin^2 \frac{t}{R} + 2\dot{x}_i \cos^3 \frac{t}{R} + 3x_i \frac{1}{R} \sin^3 \frac{t}{R}, \\ C_i^{(3)} &= x_i^{(3)} R^3 \sin^3 \frac{t}{R} - 3\ddot{x}_i R^2 \sin^2 \frac{t}{R} \cos \frac{t}{R} + 7\dot{x}_i R \sin^3 \frac{t}{R} \\ &\quad + 6\dot{x}_i R \sin \frac{t}{R} \cos^2 \frac{t}{R} - 6x_i \cos^3 \frac{t}{R} - 9x_i \sin^2 \frac{t}{R} \cos \frac{t}{R}. \end{aligned} \quad (33)$$

This is a set of linear algebraic equations, which allows one to fix x_i (and its derivatives up to the third order) by purely algebraic means. In particular, the general solution of (30) proves to be a linear combination of the four functions $\cos^3 \frac{t}{R}$, $\sin^3 \frac{t}{R}$, $\cos \frac{t}{R} \sin^2 \frac{t}{R}$ and $\sin \frac{t}{R} \cos^2 \frac{t}{R}$ with arbitrary coefficients.

Although constants of the motion associated with other generators in the $l = \frac{3}{2}$ -conformal Newton–Hooke algebra are redundant for integrating (30), let us display them in explicit form. The invariance of the action functional (31) under time translations and spatial rotations yields

$$\mathcal{H} = 2x_i^{(3)} \dot{x}_i - \ddot{x}_i^2 + \frac{10}{R^2} \dot{x}_i^2 + \frac{9}{R^4} x_i^2,$$

$$\mathcal{M}_{ij} = x_i^{(3)} x_j - x_j^{(3)} x_i + \frac{10}{R^2} (\dot{x}_i x_j - \dot{x}_j x_i) - \ddot{x}_i \dot{x}_j + \ddot{x}_j \dot{x}_i.$$

In order to find a realization of the dilatations and the special conformal transformations, it is important to observe that the vector generators corresponding to the transformations (32) fit in the general construction of the l -conformal Newton–Hooke algebra in [13]. In particular, taking the missing generators as in [13]

$$\begin{aligned} D &= \frac{1}{2} R \sin(2t/R) \frac{\partial}{\partial t} + \frac{3}{2} \cos(2t/R) x_i \frac{\partial}{\partial x_i}, \\ K &= R^2 \sin^2 \frac{t}{R} \frac{\partial}{\partial t} + \frac{3}{2} R \sin(2t/R) x_i \frac{\partial}{\partial x_i} \end{aligned} \quad (34)$$

one can verify that the action functional (31) is invariant under the coordinate transformations generated by (34) and the corresponding constants of the motion read

$$\begin{aligned} \mathcal{D} &= -\frac{R}{4} \sin \frac{2t}{R} \ddot{x}_i^2 + \frac{9}{2R} \sin \frac{2t}{R} \dot{x}_i^2 + \frac{R}{2} \sin \frac{2t}{R} \dot{x}_i x_i^{(3)} + \frac{1}{2} \cos \frac{2t}{R} \dot{x}_i \ddot{x}_i \\ &\quad - \frac{3}{2} \cos \frac{2t}{R} x_i x_i^{(3)} - \frac{9}{R^2} \cos \frac{2t}{R} x_i \dot{x}_i - \frac{3}{R} \sin \frac{2t}{R} x_i \ddot{x}_i \\ &\quad - \frac{27}{4R^3} \sin \frac{2t}{R} x_i^2, \\ \mathcal{K} &= -\frac{R^2}{2} \sin^2 \frac{t}{R} \ddot{x}_i^2 + \left(5 \sin^2 \frac{t}{R} - 2 \cos \frac{2t}{R}\right) \dot{x}_i^2 \\ &\quad + \frac{9}{R^2} \left(\frac{1}{2} \sin^2 \frac{t}{R} + \cos \frac{2t}{R}\right) x_i^2 + R^2 \sin^2 \frac{t}{R} \dot{x}_i x_i^{(3)} \\ &\quad - \frac{9}{R} \sin \frac{2t}{R} x_i \dot{x}_i - \frac{3R}{2} \sin \frac{2t}{R} x_i x_i^{(3)} + 3 \cos \frac{2t}{R} x_i \ddot{x}_i \\ &\quad + \frac{R}{2} \sin \frac{2t}{R} \dot{x}_i \ddot{x}_i. \end{aligned} \quad (35)$$

Thus, we have demonstrated that for a particular choice of frequencies such that $\frac{\alpha^2}{\beta^2} = \frac{10}{9} R^2$, the multi-dimensional Pais–Uhlenbeck oscillator enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry. It should be mentioned that a possibility to reveal such a symmetry in the Pais–Uhlenbeck oscillator (without mentioning the particular choice of the frequencies, however) was anticipated in [7].

5. Conclusion

To summarize, in this Letter the method of nonlinear realizations and the technique previously developed in [11] were used to construct a dynamical system without higher derivative terms, which holds invariant under the l -conformal Newton–Hooke group. A configuration space of the model involves coordinates, which parametrize a particle moving in d spatial dimensions and a conformal mode. The latter gives rise to an effective external field. In accord with evolution of the conformal mode, such a dynamical system describes a generalized multi-dimensional oscillator, which undergoes phases of accelerated and decelerated motion in an ellipse. We also considered higher derivative formulations and demonstrated that the multi-dimensional Pais–Uhlenbeck oscillator enjoys the $l = \frac{3}{2}$ -conformal Newton–Hooke symmetry for a particular choice of its frequencies.

Turning to possible further developments, it would be interesting to investigate whether the formulations considered in this work could be deformed by interaction terms in a way compatible with the l -conformal Newton–Hooke symmetry. It is interesting to construct a Lagrangian formulation for the models in Section 2 and Section 3. A possibility to realize the l -conformal Newton–Hooke symmetry in the multi-dimensional Pais–Uhlenbeck oscillator for other choices of frequencies is worth studying as well.

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