Positive projections onto spin factors

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Received 17 August 1999; accepted 4 November 2001
Submitted by C.-K. Li

Abstract

We characterize unital positive projections onto spin factors in a concrete representation and show that these projections are atomic positive maps. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 46L05; 15A30

Keywords: Positive projection; Spin factor; Atomic map

1. Introduction

Suppose that $A$ is a unital $C^*$-algebra and that $A_h$ is the real vector space of hermitian elements of $A$. A linear map $P : A \to A$ is a unital positive projection if $P^2 = P$, $P(I) = I$, and $P(a^*a)$ is positive in $A$ for every $a \in A$. Any positive linear map that can be expressed as a sum of 2-positive and 2-copositive maps is called a decomposable map. Positive linear maps that are not decomposable are called atomic.

If the real vector space $P(A_h)$ is norm-closed and is closed under the Jordan product $\circ$ defined by $a \circ b = \frac{1}{2}(ab + ba)$, for $a, b \in P(A_h)$, then $P(A_h)$ is a $JC$-algebra. The results of [8,9] show that when $P(A_h)$ is a $JC$-algebra, $P$ is decomposable if and only if $P(A_h)$ is reversible, meaning that $a_1a_2 \cdots a_n + a_n \cdots a_2a_1 \in P(A_h)$ for all $a_1, a_2, \ldots, a_n \in P(A_h)$ [9]. A spin factor is a $JC$-algebra $V_n$ having the form $\mathbb{R}I \oplus N_n$, where $N_n$ is a real $n$-dimensional Hilbert space and where the Jordan...
product $\circ$ satisfies $s \circ t = \langle s, t \rangle I$ whenever $s, t \in N_n$. Størmer characterizes the unital positive projections onto reversible spin factors in [9]. It is known that if $n \geq 6$, then $V_n$ is non-reversible [4,8]; a direct proof of this fact is given here in Section 3. On the other hand, there always exist unital positive projections onto spin factors (see [1, Lemma 2.3]). Thus, unital positive projections onto spin factors $V_n$ with $n \geq 6$ are necessarily atomic maps.

In [3], most of the Choi maps are shown to be atomic by using the result of Eom and Kye [2] (used again here in Section 3). In this note, we obtain more examples of atomic maps by presenting a concrete representation for positive projections onto spin factors. These projections will be shown to be uniquely determined by the dimension of the spin factors.

### 2. Contractive projections onto spin factors

Now, consider a concrete representation of $V_n$ which is found in [4]. We fix any positive integer $k$ and denote $I_m$ to be identity matrix in $M_{2^m}(\mathbb{C})$. Put $\sigma_1, \sigma_2, \sigma_3$ to be the following matrices in $M_{2^k}(\mathbb{C})$, $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

For each $i = 1, 2, \ldots, 2k$, define $s_i \in (M_{2^k}(\mathbb{C}))_h = (M_2(\mathbb{C}))_h \otimes \cdots \otimes (M_2(\mathbb{C}))_h$ by

$s_1 = \sigma_1 \otimes I_{k-1}, \quad s_2 = \sigma_2 \otimes I_{k-1},$

$s_{2^{k-1}} = \sigma_3^{(k-1)} \otimes \sigma_1, \quad s_{2^{k}} = \sigma_3^{(k-1)} \otimes \sigma_2,$

$s_{2^{i-1}} = \sigma_3^{(i-1)} \otimes \sigma_1 \otimes I_{k-i}, \quad s_{2^{i}} = \sigma_3^{(i-1)} \otimes \sigma_2 \otimes I_{k-i} \quad (2 \leq i \leq k - 1),$

where $\sigma_3^{(m)}$ means $m$-fold tensor product. Then $\{s_1, s_2, \ldots, s_{2^k}\}$ is a spin system in $JC$-algebra $(M_{2^k}(\mathbb{C}))_h$ since $s_i \circ s_j = \delta_{ij} I_k.$ So for each $n = 2k - 1$ or $2k, V_n = N_n + R I_k$ is a spin factor of real dimension $n + 1$ in $M_{2^k}(\mathbb{C})$, where $N_n$ is the real linear span of $\{s_1, s_2, \ldots, s_n\}$. Note that $N_n$ is a real Hilbert space with the following inner product $\langle \cdot, \cdot \rangle$

$\langle s, t \rangle I_k = s \circ t = \text{tr}(st) I_k, \quad s, t \in N_n,$

where $\text{tr}$ is the normalized trace for matrix algebra.

To characterize unital positive projections onto spin factors, we need some lemmas. The following lemma was observed by Broise and proved by Størmer [9].

**Lemma 2.1.** Let $A, B$ be $C^*$-algebras and $\phi : B \to A$ be a positive linear map with $\|\phi\| \leq 1$. Suppose $a \in B_h$ such that $\phi(a^2) = \phi(a)^2$. Then for all $b \in B$ we have $\phi(a \circ b) = \phi(a) \circ \phi(b)$. 

Lemma 2.2. Let \( x_1, x_2, x_3, x_4 \) in \( (M_2(\mathbb{C}))_h \) by
\[
x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]
\((M_2(\mathbb{C}))_h\) is the real linear span of elements of the form \( x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \), where \( x_{i_\ell} \in \{x_1, x_2, x_3, x_4\} \) for \( \ell = 1, 2, \ldots, k \).

Lemma 2.2. Let \( P : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) be a unital positive projection with \( P((M_{2k}(\mathbb{C}))_h) = V_n \) for \( n = 2k - 1 \) or \( 2k \). For any fixed \( m = 0, 1, \ldots, k - 1 \) and \( x = I_m \otimes x_{i_{m+1}} \otimes \cdots \otimes x_k \) with each \( x_{i_\ell} \in \{x_1, x_2, x_3, x_4\} \), we get the following:

(i) \( s_{2m+2} \circ (s_{2m+2} \circ P(x)) = \frac{1}{2} P(I_{m+1} \otimes x_{i_{m+2}} \otimes \cdots \otimes x_k), \quad i_{m+1} = 1, 2, \ldots, k - 1 \)

(ii) \( s_{2m+1} \circ P(x) = 0, \quad i_{m+1} = 3, 4. \)

Proof. Since \( P \) is a unital projection onto \( V_n \), we see that \( P(s_i^2) = P(I_k) = I_k = s_i^2 = P(s_i)^2 \). By Lemma 2.1, we have \( s_i \circ (s_j \circ P(x)) = P(s_i \circ (s_j \circ x)) \) for each \( i = 1, 2, \ldots, 2k \). If \( i_{m+1} = 1, 2 \),

\[
P(s_{2m+2} \circ (s_{2m+2} \circ x)) = P(s_{2m+2} \circ (\sigma_3 \circ (x_{i_{m+1}} \circ x_{i_{m+2}} \circ \cdots \circ x_k) ))
\]

\[
= \frac{1}{2} P(s_{2m+2} \circ (\sigma_3 \circ (x_{i_{m+1}} \circ x_{i_{m+2}} \circ \cdots \circ x_k) ))
\]

\[
= \frac{1}{2} P((I_{m+1} \otimes x_{i_{m+2}} \otimes \cdots \otimes x_k).
\]

So, we get the first identity. Since \( \sigma_1 \circ x_i = 0 \) for \( i = 3, 4 \), the last identity is easily checked.

\[

Proposition 2.3. Let \( P : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) be a unital positive projection with \( P((M_{2k}(\mathbb{C}))_h) = V_n \) for \( n = 2k - 1 \) or \( 2k \). Then we have the following:

(i) \( \text{tr}(P(x)) = \text{tr}(x) \) for each \( x \in M_{2k}(\mathbb{C}). \)

(ii) \( \text{tr}(s \circ P(x)) = \text{tr}(sx) \) for each \( x \in M_{2k}(\mathbb{C}), s \in N_n. \)

Proof. By the linearity of \( P \), it suffices to consider \( x = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k} \) with each \( x_{i_\ell} \in \{x_1, x_2, x_3, x_4\} \), and so we only consider such \( x \)'s. Note that

\[
\text{tr}(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}) = \text{tr}(x_{i_1}) \text{tr}(x_{i_2}) \cdots \text{tr}(x_{i_k}).
\]

Since \( P(x) \) lies in \( V_n \), we can write \( P(x) \) as

\[
P(x) = a_0 I_k + \sum_{i=1}^n a_i s_i
\]

for some real numbers \( a_i \). We show assertion (i) first. If \( x_{i_1} = x_3 \) or \( x_4 \), then \( s_1 \circ P(x) = 0 \) by Lemma 2.2. On the other hand, \( s_1 \circ P(x) = a_0 s_1 + a_1 I_k \) in (2.1). Therefore, we have \( a_0 = a_1 = 0 \), and this implies that \( \text{tr}(P(x)) = 0 = \text{tr}(x) \). If \( x_{i_1} = x_1 \) or \( x_2 \), then we have
\[ a_0 x_1 + a_1 I_k = s_1 \circ P(x) \]
\[ = P(s_1 \circ x) \]
\[ = \begin{cases} 
  P(x) & \text{if } x_{i_1} = x_1, \\
  -P(x) & \text{if } x_{i_1} = x_2 
\end{cases} \]
\[ = \begin{cases} 
  a_0 I_k + \sum_{i=1}^n a_is_i & \text{if } x_{i_1} = x_1, \\
  -a_0 I_k - \sum_{i=1}^n a_is_i & \text{if } x_{i_1} = x_2. 
\end{cases} \]

This implies that
\[ P(x) = \begin{cases} 
  a_0 I_k + a_0 s_1 & \text{if } x = x_1 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}, \\
  a_0 I_k - a_0 s_1 & \text{if } x = x_2 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}. 
\end{cases} \tag{2.2} \]

By using (2.2) and Lemma 2.2, we obtain that
\[ a_0 I_k = s_2 \circ (s_2 \circ P(x)) \]
\[ = P(s_2 \circ (s_2 \circ x)) \]
\[ = \frac{1}{2} P(I_1 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}). \tag{2.3} \]

Now, applying repeatedly the above calculation to (2.3), we get
\[ a_0 I_k = P\left(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}\right) = \begin{cases} 
  \left(\frac{1}{2}\right)^k P(I_k) & \text{if all } x_{i_\ell} = x_1 \text{ or } x_2, \\
  0 & \text{if some } x_{i_\ell} = x_3 \text{ or } x_4. 
\end{cases} \]

Since \( P \) is unital, \( a_0 \) should be \( 1/2^k \).

Consequently, we conclude that for any \( x = x_1 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}, \)
\[ \text{tr}(P(x)) = \begin{cases} 
  \frac{1}{2^k} = \text{tr}(x) & \text{if all } x_{i_\ell} = x_1 \text{ or } x_2, \\
  0 = \text{tr}(x) & \text{if some } x_{i_\ell} = x_3 \text{ or } x_4. 
\end{cases} \]

This completes the proof of (i).

To prove equality (ii), it suffices to check it for spin system \( \{s_i\}. \) Since \( s_i \circ P(x) = P(s_i \circ x) \) for all \( i = 1, 2, \ldots, n, \) statement (i) shows that
\[ \text{tr}(s_i \circ P(x)) = \text{tr}(P(s_i \circ x)) = \text{tr}(s_i \circ x) = \text{tr}(s_i x). \]

Proposition 2.3 completely determines the positive projection \( P \) of \( (M_{2^k} (\mathbb{C}))^h \) onto \( V_n \) for \( n = 2k - 1 \) and \( n = 2k. \) By identity (2.1) and Proposition 2.3, it is obvious that
\[ P(x_{i_1} \otimes \cdots \otimes x_{i_k}) = \text{tr}(x_{i_1} \otimes \cdots \otimes x_{i_k}) \cdot I_k \]
\[ + \sum_{i=1}^n \text{tr}(s_i(x_{i_1} \otimes \cdots \otimes x_{i_k})) \cdot s_i. \tag{2.4} \]
where each \( x_\ell \in \{x_1, x_2, x_3, x_4\} \). Since \( M_{2k}(\mathbb{C}) \) is the complex linear span of such \( x \)'s and \( P \) is linear, \( P \) is completely determined. Since the existence of such a projection is well known in [1], we have the following theorem.

**Theorem 2.4.** Fix any positive integer \( k \) and let \( V_n \) be the real \( n + 1 \)-dimensional spin factor in \( M_{2k}(\mathbb{C}) \) for each \( n = 2k - 1 \) or \( 2k \). Then there exists a unique unital positive projection \( P_n : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) with the property \( P_n((M_{2k}(\mathbb{C}))_h) = V_n \).

We now describe \( P_n \) recursively. Let \( \{e_{ij}\} \) be the usual matrix units in \( M_2(\mathbb{C}) \). For \( y = e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \in M_{2k}(\mathbb{C}) \), define the nonnegative integer \( m = m(x) \) by the cardinal number of the set \( \{(i_\ell, j_\ell): (i_\ell, j_\ell) = (1, 2)\} \).

Since \( e_{12} = \frac{1}{2}(x_3 - ix_4) \) and \( e_{21} = \frac{1}{2}(x_3 + ix_4) \), it is easily deduced from (2.4) that for any \( y = e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \in M_{2k}(\mathbb{C}) \) and \( k = 1, 2, \ldots \)

\[
\begin{align*}
(i) \quad & P_{2k+2}(y \otimes e_{11}) = \begin{cases} 
\frac{1}{2} P_{2k}(y) \otimes I_1 & \text{if } i_\ell, j_\ell \neq 1, 2, \\
\frac{1}{2} P_{2k}(y) \otimes I_1 + \frac{i^k(-1)^m}{2^{2k+1}} s_{2k+1} & \text{if } i_\ell, j_\ell = 1, 2,
\end{cases} \\
(ii) \quad & P_{2k+2}(y \otimes e_{22}) = \begin{cases} 
\frac{1}{2} P_{2k}(y) \otimes I_1 & \text{if } i_\ell, j_\ell \neq 1, 2, \\
\frac{1}{2} P_{2k}(y) \otimes I_1 + \frac{i^k(-1)^{m+1}}{2^{2k+1}} s_{2k+1} & \text{if } i_\ell, j_\ell = 1, 2,
\end{cases} \\
(iii) \quad & P_{2k+2}(y \otimes e_{12}) = P_{2k+2}(y \otimes e_{21}) = \begin{cases} 
0 & \text{if } i_\ell, j_\ell \neq 1, 2, \\
\frac{i^k(-1)^m}{2^{2k+1}} s_{2k+2} & \text{if } i_\ell, j_\ell = 1, 2,
\end{cases} \\
(iv) \quad & P_{2k+1}(y \otimes e_{ij}) = \begin{cases} 
P_{2k+2}(y \otimes e_{ij}) & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases}
\end{align*}
\]

\[ (2.5) \]

\[
\begin{align*}
(v) \quad & P_2(e_{11}) = e_{11}, \quad P_2(e_{12}) = P_2(e_{21}) = \frac{1}{2} \sigma_2, \quad P_2(e_{22}) = e_{22}.
\end{align*}
\]

3. **Atomic property of the projections onto spin factors**

The purpose of this section is to show that the projection \( P_n \) in Theorem 2.4 is an atomic map for all \( n \geq 6 \), using the result of Eom and Kye [2]. For the convenience of readers, we briefly explain the results in [2]. Generalizing the Woronowicz's argument [10], they considered the duality between the space \( M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) (= M_{nm}(\mathbb{C})) \) of all \( nm \times nm \) matrices over the complex field and the space \( \mathcal{L}(M_m(\mathbb{C}), M_n(\mathbb{C})) \) of all linear maps from \( M_m(\mathbb{C}) \) into \( M_n(\mathbb{C}) \), which is given by
\[
\langle A, \phi \rangle = \text{Tr} \left[ \sum_{i,j=1}^{m} (\phi(e_{ij}) \otimes e_{ij}) A^i \right] = \sum_{i,j=1}^{m} \langle \phi(e_{ij}), a_{ij} \rangle
\] (3.1)

for \( A = \sum_{i,j=1}^{m} a_{ij} \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \) and a linear map \( \phi : M_m(\mathbb{C}) \to M_n(\mathbb{C}) \), where \( \{ e_{ij} \} \) is the matrix units of \( M_m(\mathbb{C}) \) and the bilinear form on the right-hand side is given by \( \langle X, Y \rangle = \text{Tr}(YX^t) \) for \( X, Y \in M_n(\mathbb{C}) \) with the usual trace \( \text{Tr} \), that is, \( \text{Tr}(I_k) = 2^k \text{tr}(I_k) \).

For a matrix \( A = \sum_{i,j=1}^{m} x_{ij} \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \), we denote by \( A^T \) the block-transpose \( \sum_{i,j=1}^{m} x_{ji} \otimes e_{ij} \) of \( A \). We say that a vector \( z = \sum_{i=1}^{m} z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m \) is an \( s \)-simple if the linear span of \( \{ z_1, \ldots, z_m \} \) has the dimension \( \leq s \), where \( \{ e_1, \ldots, e_m \} \) is the usual orthonormal basis of \( \mathbb{C}^m \).

Let \( P_s[M_n(\mathbb{C})] \) (respectively, \( P^t_s[M_n(\mathbb{C})] \)) be the convex cone of all \( s \)-positive (respectively, \( s \)-copositive) linear maps between \( M_n(\mathbb{C}) \), and \( \mathbb{V}_s[M_n(\mathbb{C})] \) (respectively, \( \mathbb{V}^t_s[M_n(\mathbb{C})] \)) denote the convex cone in \( M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) generated by \( zz^* \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) (respectively, \( (zz^*)^T \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \)) with all \( s \)-simple vectors \( z \in \mathbb{C}^n \otimes \mathbb{C}^m \). It turns out that \( \mathbb{V}_s[M_n(\mathbb{C})] \) (respectively, \( \mathbb{V}^t_s[M_n(\mathbb{C})] \)) is the dual cone of \( P_s[M_n(\mathbb{C})] \) (respectively, \( P^t_s[M_n(\mathbb{C})] \)) with respect to the pairing (3.1). With this machinery, the maximal faces of \( P_s[M_n(\mathbb{C})] \) and \( P^t_s[M_n(\mathbb{C})] \) are characterized in terms of \( s \)-simple vectors (see also [5–7]). Another consequence is a characterization of the cone \( P_s[M_n(\mathbb{C})] + P^t_s[M_n(\mathbb{C})] \): For a linear map \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \), the map \( \phi \) is the sum of an \( s \)-positive linear map and a \( t \)-copositive linear map if and only if \( \langle A, \phi \rangle \geq 0 \) for each \( A \in \mathbb{V}_s[M_n(\mathbb{C})] \cap \mathbb{V}^t[M_n(\mathbb{C})] \). From the result, we show the atomic property of a positive linear map depends on a tedious matrix manipulation (see [2,3]).

Throughout this section, every vector in the space \( \mathbb{C}^n \) will be considered as an \( n \times 1 \) matrix. The usual orthonormal basis of \( \mathbb{C}^n \) and matrix units of \( M_n(\mathbb{C}) \) will be denoted by \( \{ e_i : i = 1, \ldots, n \} \) and \( \{ e_{ij} : i, j = 1, \ldots, n \} \), respectively, regardless of the dimension \( n \).

From relations (2.5), we calculate \( (P_4(e_{ij})) \in M_4(M_4(\mathbb{C})) \) as follows:

\[
(P_4(e_{ij})) = \\
\begin{pmatrix}
(\frac{1}{2})^2 (I_2 + s_1) & (\frac{1}{2})^2 (s_2 - is_3) & 0 & -(\frac{1}{2})^2 is_4 \\
(\frac{1}{2})^2 (s_2 + is_3) & (\frac{1}{2})^2 (I_2 - s_1) & (\frac{1}{2})^2 is_4 & 0 \\
0 & -(\frac{1}{2})^2 is_4 & (\frac{1}{2})^2 (I_2 + s_1) & (\frac{1}{2})^2 (s_2 + is_3) \\
(\frac{1}{2})^2 is_4 & 0 & (\frac{1}{2})^2 (s_2 - is_3) & (\frac{1}{2})^2 (I_2 - s_1)
\end{pmatrix}
\]

Now, we fix an integer \( k \geq 3 \), and apply relations (2.5) to the above calculation to get

\[
P_{2k-1}(e_{22}) = P_{2k}(e_{22}) = P_{2k-1}(e_{44}) = P_{2k}(e_{44}) = (\frac{1}{2})^{k-1} e_{22} \otimes I_{k-1}.
\]
Then $A(k) = (P_{2k}(e_{23}))^t$

$P_{2k-1}(e_{33}) = (P_{2k-1}(e_{32}))^t$

$= (P_{2k}(e_{32}))^t$

$= \left( \frac{1}{2} \right)^k (\sigma_3 \otimes \sigma_3 \otimes \sigma_1 + i\sigma_3 \otimes \sigma_2 \otimes I_1) \otimes I_{k-3},$

$P_{2k-1}(e_{34}) = (P_{2k}(e_{43}))^t$

$= (P_{2k}(e_{43}))^t$

$= (P_{2k}(e_{42}))^t$

$= (P_{2k}(e_{42}))^t$

$= 0,$

$P_{2k-1}(e_{24}) = (P_{2k}(e_{24}))^t$

$= (P_{2k}(e_{24}))^t$

$= (P_{2k}(e_{24}))^t$

$= (P_{2k}(e_{23}))^t$

$= \left( \frac{1}{2} \right)^{k-1} e_{11} \otimes I_{k-1}.$

For each $1 \leq i \leq 18,$ define $z_i^k \in (C^{2^3} \otimes C^{2^k-3}) \otimes C^{2^k} = C^{2^k} \otimes C^{2^k}$ by

$z_1^k = (e_1 \otimes e_1) \otimes e_2 + (e_4 \otimes e_1) \otimes e_3,$

$z_2^k = (e_3 \otimes e_1) \otimes e_2,$

$z_3^k = (e_7 \otimes e_1) \otimes e_2 + (e_6 \otimes e_1) \otimes e_3,$

$z_4^k = (e_5 \otimes e_1) \otimes e_2,$

$z_5^k = (e_2 \otimes e_1) \otimes e_3 - (e_1 \otimes e_1) \otimes e_4,$

$z_6^k = (e_1 \otimes e_1) \otimes e_3,$

$z_7^k = (e_6 \otimes e_1) \otimes e_3 - (e_5 \otimes e_1) \otimes e_4,$

$z_8^k = (e_2 \otimes e_1) \otimes e_4,$

$z_9^k = (e_2 \otimes e_1) \otimes e_2 - (e_3 \otimes e_1) \otimes e_3,$

$z_{10}^k = (e_4 \otimes e_1) \otimes e_2,$

$z_{11}^k = (e_8 \otimes e_1) \otimes e_2 - (e_5 \otimes e_1) \otimes e_3,$

$z_{12}^k = (e_6 \otimes e_1) \otimes e_2,$

$z_{13}^k = (e_3 \otimes e_1) \otimes e_3 - (e_4 \otimes e_1) \otimes e_4,$

$z_{14}^k = (e_8 \otimes e_1) \otimes e_3,$

$z_{15}^k = (e_7 \otimes e_1) \otimes e_3 - (e_8 \otimes e_1) \otimes e_4,$

$z_{16}^k = (e_3 \otimes e_1) \otimes e_4,$

$z_{17}^k = (e_6 \otimes e_1) \otimes e_4,$

$z_{18}^k = (e_7 \otimes e_1) \otimes e_4.$

Finally, we define the matrix $A(k) \in M_{2^k}(C) \otimes M_{2^k}(C)$ by

$A(k) = \sum_{i=1}^{18} z_i^k z_i^{k*} + z_6^k z_6^{k*} + z_{14}^k z_{14}^{k*}.$

Then $A(k) \in \mathbb{V}_2[M_{2^k}(C)].$ Since only $a_{ij}(k) \neq 0$ for $2 \leq i, j \leq 4, k \geq 3,$ where $A(k) = \sum_{i,j=1}^{2^k} a_{ij}(k) \otimes e_{ij},$ we can easily calculate the pairing (3.1) as follows:

$\langle A(k), P_{2k} \rangle = \langle A(k), P_{2k-1} \rangle = \sum_{i,j=2}^{4} \langle P_{2k}(e_{ij}), a_{ij} \rangle = -\left( \frac{1}{2} \right)^{k-2}.$
For $1 \leq i \leq 18$, define $w_k^i \in (\mathbb{C}^3 \otimes \mathbb{C}^{2^k-3}) \otimes \mathbb{C}^{2^k} = \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$ by

$$
\begin{align*}
  w_1^k &= (e_3 \otimes e_1) \otimes e_2 - (e_2 \otimes e_1) \otimes e_3, \\
  w_2^k &= (e_1 \otimes e_1) \otimes e_2, \\
  w_3^k &= (e_4 \otimes e_1) \otimes e_2 + (e_1 \otimes e_1) \otimes e_3, \\
  w_4^k &= (e_2 \otimes e_1) \otimes e_2, \\
  w_5^k &= (e_5 \otimes e_1) \otimes e_2 - (e_8 \otimes e_1) \otimes e_3, \\
  w_6^k &= (e_7 \otimes e_1) \otimes e_2, \\
  w_7^k &= (e_6 \otimes e_1) \otimes e_2 + (e_7 \otimes e_1) \otimes e_3, \\
  w_8^k &= (e_8 \otimes e_1) \otimes e_2, \\
  w_9^k &= (e_1 \otimes e_1) \otimes e_3 - (e_2 \otimes e_1) \otimes e_4, \\
  w_{10}^k &= (e_3 \otimes e_1) \otimes e_3, \\
  w_{11}^k &= (e_4 \otimes e_1) \otimes e_3 - (e_3 \otimes e_1) \otimes e_4, \\
  w_{12}^k &= (e_6 \otimes e_1) \otimes e_3, \\
  w_{13}^k &= (e_5 \otimes e_1) \otimes e_3 - (e_6 \otimes e_1) \otimes e_4, \\
  w_{14}^k &= (e_1 \otimes e_1) \otimes e_4, \\
  w_{15}^k &= (e_8 \otimes e_1) \otimes e_3 - (e_7 \otimes e_1) \otimes e_4, \\
  w_{16}^k &= (e_4 \otimes e_1) \otimes e_4, \\
  w_{17}^k &= (e_5 \otimes e_1) \otimes e_4, \\
  w_{18}^k &= (e_8 \otimes e_1) \otimes e_4.
\end{align*}
$$

Then we see that

$$A(k)^T = \sum_{i=1}^{18} w_i^k w_i^{k*} + w_{10}^k w_{10}^{k*} + w_{12}^k w_{12}^{k*}.$$ 

and so $A(k) \in \mathbb{V}^2[M_{2^k}(\mathbb{C})]$. Consequently we have shown that for any $k \geq 3$

$$\langle A(k), P_{2k} \rangle = \langle A(k), P_{2k-1} \rangle = -(\frac{1}{2})^{k-2} < 0,$$ 

with some $A(k) \in \mathbb{V}^2[M_{2^k}(\mathbb{C})] \cap \mathbb{V}^2[M_{2k}(\mathbb{C})]$. By the result in [2] mentioned in this section, we conclude the following:

**Theorem 3.1.** Fix any positive integer $k \geq 3$ and let $V_n$ be the real $(n + 1)$-dimensional spin factor in $M_{2^k}(\mathbb{C})$ for each $n = 2k - 1$ or $2k$. Then the unital positive projections $P_n : M_{2^k}(\mathbb{C}) \to M_{2^k}(\mathbb{C})$ with the property $P_n((M_{2^k}(\mathbb{C}))_h) = V_n$ are all atomic positive maps.

The spin factor of dimension 6 (up to isomorphism) can be both reversible and non-reversible in concrete representations. Theorem 3.1 provides a representation in which $V_5$ is non-reversible.

**References**


