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Positive projections onto spin factors ☆

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Abstract

We characterize unital positive projections onto spin factors in a concrete representation and show that these projections are atomic positive maps. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Suppose that A is a unital C^*-algebra and that A_h is the real vector space of hermitian elements of A. A linear map $P : A \to A$ is a unital positive projection if $P^2 = P$, $P(I) = I$, and $P(a^*a)$ is positive in A for every $a \in A$. Any positive linear map that can be expressed as a sum of 2-positive and 2-copositive maps is called a decomposable map. Positive linear maps that are not decomposable are called atomic.

If the real vector space $P(A_h)$ is norm-closed and is closed under the Jordan product $\circ$ defined by $a \circ b = \frac{1}{2}(ab + ba)$, for $a, b \in P(A_h)$, then $P(A_h)$ is a JC-algebra. The results of [8,9] show that when $P(A_h)$ is a JC-algebra, P is decomposable if and only if $P(A_h)$ is reversible, meaning that $a_1a_2 \cdots a_n + a_n \cdots a_2a_1 \in P(A_h)$ for all $a_1, a_2, \ldots, a_n \in P(A_h)$ [9]. A spin factor is a JC-algebra V_n having the form $\mathbb{R}I \oplus N_n$, where N_n is a real n-dimensional Hilbert space and where the Jordan

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product $\circ$ satisfies $s \circ t = \langle s, t\rangle I$ whenever $s, t \in N_n$. Størmer characterizes the unital positive projections onto reversible spin factors in [9]. It is known that if $n \geqslant 6$, then V_n is non-reversible [4,8]; a direct proof of this fact is given here in Section 3. On the other hand, there always exist unital positive projections onto spin factors (see [1, Lemma 2.3]). Thus, unital positive projections onto spin factors V_n with $n \geqslant 6$ are necessarily atomic maps.

In [3], most of the Choi maps are shown to be atomic by using the result of Eom and Kye [2] (used again here in Section 3). In this note, we obtain more examples of atomic maps by presenting a concrete representation for positive projections onto spin factors. These projections will be shown to be uniquely determined by the dimension of the spin factors.

2. Contractive projections onto spin factors

Now, consider a concrete representation of V_n which is found in [4]. We fix any positive integer k and denote I_m to be identity matrix in $M_{2^m}(\mathbb{C})$. Put $\sigma_1, \sigma_2, \sigma_3$ to be the following matrices in $M_2(\mathbb{C})$,

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

For each $i = 1, 2, \dots, 2k$, define $s_i \in (M_{2^k}(\mathbb{C}))_h = (M_2(\mathbb{C}))_h \otimes \cdots \otimes (M_2(\mathbb{C}))_h$ by

$$\begin{aligned} s_1 &= \sigma_1 \otimes I_{k-1}, & s_2 &= \sigma_2 \otimes I_{k-1}, \\ s_{2k-1} &= \sigma_3^{(k-1)} \otimes \sigma_1, & s_{2k} &= \sigma_3^{(k-1)} \otimes \sigma_2, \\ s_{2i-1} &= {\sigma_3}^{(i-1)} \otimes \sigma_1 \otimes I_{k-i}, & s_{2i} &= {\sigma_3}^{(i-1)} \otimes \sigma_2 \otimes I_{k-i} \quad (2 \leqslant i \leqslant k-1), \end{aligned}$$

where ${\sigma_3}^{(m)}$ means m-fold tensor product. Then $\{s_1, s_2, \dots, s_{2k}\}$ is a spin system in JC-algebra $(M_{2^k}(\mathbb{C}))_h$ since $s_i \circ s_j = \delta_{ij} I_k$. So for each $n = 2k-1$ or $2k$, $V_n = N_n + \mathbb{R}I_k$ is a spin factor of real dimension $n+1$ in $M_{2^k}(\mathbb{C})$, where N_n is the real linear span of $\{s_1, s_2, \dots, s_n\}$. Note that N_n is a real Hilbert space with the following inner product $\langle \cdot, \cdot \rangle$

$$\langle s, t\rangle I_k = s \circ t = \mathrm{tr}(st) I_k, \quad s, t \in N_n,$$

where tr is the normalized trace for matrix algebra.

To characterize unital positive projections onto spin factors, we need some lemmas. The following lemma was observed by Broise and proved by Størmer [9].

Lemma 2.1. *Let A, B be C^*-algebras and $\phi : B \to A$ be a positive linear map with $\|\phi\| \leqslant 1$. Suppose $a \in B_h$ such that $\phi(a^2) = \phi(a)^2$. Then for all $b \in B$ we have $\phi(a \circ b) = \phi(a) \circ \phi(b)$.*

By defining x_1, x_2, x_3, x_4 in $(M_2(\mathbb{C}))_h$ by

$$x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad x_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x_4 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},$$

$(M_{2^k}(\mathbb{C}))_h$ is the real linear span of elements of the form $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}$, where $x_{i_\ell} \in \{x_1, x_2, x_3, x_4\}$ for $\ell = 1, 2, \ldots, k$.

Lemma 2.2. *Let $P : M_{2^k}(\mathbb{C}) \to M_{2^k}(\mathbb{C})$ be a unital positive projection with $P((M_{2^k}(\mathbb{C}))_h) = V_n$ for $n = 2k-1$ or $2k$. For any fixed $m = 0, 1, \ldots, k-1$ and $x = I_m \otimes x_{i_{m+1}} \otimes \cdots \otimes x_{i_k}$ with each $x_{i_\ell} \in \{x_1, x_2, x_3, x_4\}$, we get the following*:

(i) $s_{2m+2} \circ (s_{2m+2} \circ P(x)) = \frac{1}{2} P\big(I_{m+1} \otimes x_{i_{m+2}} \otimes \cdots \otimes x_{i_k}\big), \quad i_{m+1} = 1, 2,$

(ii) $s_{2m+1} \circ P(x) = 0, \quad i_{m+1} = 3, 4.$

Proof. Since P is a unital projection onto V_n, we see that $P(s_i^2) = P(I_k) = I_k = s_i^2 = P(s_i)^2$. By Lemma 2.1, we have $s_i \circ (s_i \circ P(x)) = P(s_i \circ (s_i \circ x))$ for each $i = 1, 2, \ldots, 2k$. If $i_{m+1} = 1, 2$, then

$$\begin{aligned} &P(s_{2m+2} \circ (s_{2m+2} \circ x)) \\ &\quad = P\big(s_{2m+2} \circ \big({\sigma_3}^{(m)} \otimes (x_{i_{m+1}} \circ \sigma_2) \otimes x_{i_{m+2}} \otimes \cdots \otimes x_{i_k}\big)\big) \\ &\quad = \tfrac{1}{2} P\big(s_{2m+2} \circ \big({\sigma_3}^{(m)} \otimes x_3 \otimes x_{i_{m+2}} \otimes \cdots \otimes x_{i_k}\big)\big) \\ &\quad = \tfrac{1}{2} P\big(I_{m+1} \otimes x_{i_{m+2}} \otimes \cdots \otimes x_{i_k}\big). \end{aligned}$$

So, we get the first identity. Since $\sigma_1 \circ x_i = 0$ for $i = 3, 4$, the last identity is easily checked. □

Proposition 2.3. *Let $P : M_{2^k}(\mathbb{C}) \to M_{2^k}(\mathbb{C})$ be a unital positive projection with $P((M_{2^k}(\mathbb{C}))_h) = V_n$ for $n = 2k-1$ or $2k$. Then we have the following*:

(i) $\mathrm{tr}(P(x)) = \mathrm{tr}(x)$ *for each* $x \in M_{2^k}(\mathbb{C})$.

(ii) $\mathrm{tr}(s \circ P(x)) = \mathrm{tr}(sx)$ *for each* $x \in M_{2^k}(\mathbb{C}), s \in N_n$.

Proof. By the linearity of P, it suffices to consider $x = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}$ with each $x_{i_\ell} \in \{x_1, x_2, x_3, x_4\}$, and so we only consider such x's. Note that

$$\mathrm{tr}(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}) = \mathrm{tr}(x_{i_1})\mathrm{tr}(x_{i_2}) \cdots \mathrm{tr}(x_{i_k}).$$

Since $P(x)$ lies in V_n, we can write $P(x)$ as

$$P(x) = a_0 I_k + \sum_{i=1}^{n} a_i s_i \tag{2.1}$$

for some real numbers a_i. We show assertion (i) first. If $x_{i_1} = x_3$ or x_4, then $s_1 \circ P(x) = 0$ by Lemma 2.2. On the other hand, $s_1 \circ P(x) = a_0 s_1 + a_1 I_k$ in (2.1). Therefore, we have $a_0 = a_1 = 0$, and this implies that $\mathrm{tr}(P(x)) = 0 = \mathrm{tr}(x)$. If $x_{i_1} = x_1$ or x_2, then we have

$$
\begin{aligned}
a_0 s_1 + a_1 I_k &= s_1 \circ P(x) \\
&= P(s_1 \circ x) \\
&= \begin{cases} P(x) & \text{if } x_{i_1} = x_1, \\ -P(x) & \text{if } x_{i_1} = x_2 \end{cases} \\
&= \begin{cases} a_0 I_k + \sum_{i=1}^n a_i s_i & \text{if } x_{i_1} = x_1, \\ -a_0 I_k - \sum_{i=1}^n a_i s_i & \text{if } x_{i_1} = x_2. \end{cases}
\end{aligned}
$$

This implies that

$$
P(x) = \begin{cases} a_0 I_k + a_0 s_1 & \text{if } x = x_1 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}, \\ a_0 I_k - a_0 s_1 & \text{if } x = x_2 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}. \end{cases} \tag{2.2}
$$

By using (2.2) and Lemma 2.2, we obtain that

$$
\begin{aligned}
a_0 I_k &= s_2 \circ (s_2 \circ P(x)) \\
&= P(s_2 \circ (s_2 \circ x)) \\
&= \tfrac{1}{2} P\big(I_1 \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}\big).
\end{aligned} \tag{2.3}
$$

Now, applying repeatedly the above calculation to (2.3), we get

$$
a_0 I_k = P\big(x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}\big) = \begin{cases} \left(\frac{1}{2}\right)^k P(I_k) & \text{if all } x_{i_\ell} = x_1 \text{ or } x_2, \\ 0 & \text{if some } x_{i_\ell} = x_3 \text{ or } x_4. \end{cases}
$$

Since P is unital, a_0 should be $1/2^k$.

Consequently, we conclude that for any $x = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_k}$,

$$
\operatorname{tr}(P(x)) = \begin{cases} \frac{1}{2^k} = \operatorname{tr}(x) & \text{if all } x_{i_\ell} = x_1 \text{ or } x_2, \\ 0 = \operatorname{tr}(x) & \text{if some } x_{i_\ell} = x_3 \text{ or } x_4. \end{cases}
$$

This completes the proof of (i).

To prove equality (ii), it suffices to check it for spin system $\{s_i\}$. Since $s_i \circ P(x) = P(s_i \circ x)$ for all $i = 1, 2, \ldots, n$, statement (i) shows that

$$
\operatorname{tr}(s_i \circ P(x)) = \operatorname{tr}(P(s_i \circ x)) = \operatorname{tr}(s_i \circ x) = \operatorname{tr}(s_i x). \qquad \square
$$

Proposition 2.3 completely determines the positive projection P of $(M_{2^k}(\mathbb{C}))_h$ onto V_n for $n = 2k - 1$ and $n = 2k$. By identity (2.1) and Proposition 2.3, it is obvious that

$$
\begin{aligned}
P(x_{i_1} \otimes \cdots \otimes x_{i_k}) = {} & \operatorname{tr}(x_{i_1} \otimes \cdots \otimes x_{i_k}) \cdot I_k \\
& + \sum_{i=1}^n \operatorname{tr}(s_i (x_{i_1} \otimes \cdots \otimes x_{i_k})) \cdot s_i,
\end{aligned} \tag{2.4}
$$

where each $x_{i_\ell} \in \{x_1, x_2, x_3, x_4\}$. Since $M_{2^k}(\mathbb{C})$ is the complex linear span of such x's and P is linear, P is completely determined. Since the existence of such a projection is well known in [1], we have the following theorem.

Theorem 2.4. *Fix any positive integer k and let V_n be the real $n+1$-dimensional spin factor in $M_{2^k}(\mathbb{C})$ for each $n = 2k-1$ or $2k$. Then there exists a unique unital positive projection $P_n : M_{2^k}(\mathbb{C}) \to M_{2^k}(\mathbb{C})$ with the property $P_n((M_{2^k}(\mathbb{C}))_h) = V_n$.*

We now describe P_n recursively. Let $\{e_{ij}\}$ be the usual matrix units in $M_2(\mathbb{C})$. For $y = e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \in M_{2^k}(\mathbb{C})$, define the nonnegative integer $m = m(x)$ by the cardinal number of the set $\{(i_\ell, j_\ell)\colon (i_\ell, j_\ell) = (1, 2)\}$.

Since $e_{12} = \frac{1}{2}(x_3 - \mathrm{i}x_4)$ and $e_{21} = \frac{1}{2}(x_3 + \mathrm{i}x_4)$, it is easily deduced from (2.4) that for any $y = e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \in M_{2^k}(\mathbb{C})$ and $k = 1, 2, \ldots$

$$
\begin{aligned}
&\text{(i)}\quad P_{2k+2}(y \otimes e_{11}) \\
&\qquad = \begin{cases} \frac{1}{2} P_{2k}(y) \otimes I_1 & \text{if some } \{i_\ell, j_\ell\} \neq \{1, 2\}, \\ \frac{1}{2} P_{2k}(y) \otimes I_1 + \frac{i^k(-1)^m}{2^{k+1}} s_{2k+1} & \text{if all } \{i_\ell, j_\ell\} = \{1, 2\}, \end{cases} \\
&\text{(ii)}\quad P_{2k+2}(y \otimes e_{22}) \\
&\qquad = \begin{cases} \frac{1}{2} P_{2k}(y) \otimes I_1 & \text{if some } \{i_\ell, j_\ell\} \neq \{1, 2\}, \\ \frac{1}{2} P_{2k}(y) \otimes I_1 + \frac{i^k(-1)^{m+1}}{2^{k+1}} s_{2k+1} & \text{if all } \{i_\ell, j_\ell\} = \{1, 2\}, \end{cases} \\
&\text{(iii)}\quad P_{2k+2}(y \otimes e_{12}) = P_{2k+2}(y \otimes e_{21}) \\
&\qquad\qquad = \begin{cases} 0 & \text{if some } \{i_\ell, j_\ell\} \neq \{1, 2\}, \\ \frac{i^k(-1)^m}{2^{k+1}} s_{2k+2} & \text{if all } \{i_\ell, j_\ell\} = \{1, 2\}, \end{cases} \\
&\text{(iv)}\quad P_{2k+1}(y \otimes e_{ij}) = \begin{cases} P_{2k+2}(y \otimes e_{ij}) & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\
&\text{(v)}\quad P_2(e_{11}) = e_{11}, \quad P_2(e_{12}) = P_2(e_{21}) = \tfrac{1}{2}\sigma_2, \quad P_2(e_{22}) = e_{22}.
\end{aligned}
\tag{2.5}
$$

3. Atomic property of the projections onto spin factors

The purpose of this section is to show that the projection P_n in Theorem 2.4 is an atomic map for all $n \geqslant 6$, using the result of Eom and Kye [2]. For the convenience of readers, we briefly explain the results in [2]. Generalizing the Woronowicz's argument [10], they considered the duality between the space $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ $(= M_{nm}(\mathbb{C}))$ of all $nm \times nm$ matrices over the complex field and the space $\mathscr{L}(M_m(\mathbb{C}), M_n(\mathbb{C}))$ of all linear maps from $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$, which is given by

$$\langle A, \phi\rangle = \mathrm{Tr}\left[\sum_{i,j=1}^{m} \left(\phi(e_{ij}) \otimes e_{ij}\right) A^{\mathrm{t}}\right] = \sum_{i,j=1}^{m} \langle \phi(e_{ij}), a_{ij}\rangle \tag{3.1}$$

for $A = \sum_{i,j=1}^{m} a_{ij} \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ and a linear map $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$, where $\{e_{ij}\}$ is the matrix units of $M_m(\mathbb{C})$ and the bilinear form on the right-hand side is given by $\langle X, Y\rangle = \mathrm{Tr}(YX^{\mathrm{t}})$ for $X, Y \in M_n(\mathbb{C})$ with the usual trace Tr, that is, $\mathrm{Tr}(I_k) = 2^k \mathrm{tr}(I_k)$.

For a matrix $A = \sum_{i,j=1}^{m} x_{ij} \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$, we denote by A^{T} the block-transpose $\sum_{i,j=1}^{m} x_{ji} \otimes e_{ij}$ of A. We say that a vector $z = \sum_{i=1}^{m} z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m$ is an s-simple if the linear span of $\{z_1, \ldots, z_m\}$ has the dimension $\leqslant s$, where $\{e_1, \ldots, e_m\}$ is the usual orthonormal basis of $\mathbb{C}^m$.

Let $\mathbb{P}_s[M_n(\mathbb{C})]$ (respectively, $\mathbb{P}^s[M_n(\mathbb{C})]$) be the convex cone of all s-positive (respectively, s-copositive) linear maps between $M_n(\mathbb{C})$, and $\mathbb{V}_s[M_n(\mathbb{C})]$ (respectively, $\mathbb{V}^s[M_n(\mathbb{C})]$) denote the convex cone in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ generated by $zz^* \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ (respectively, $(zz^*)^{\mathrm{T}} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$) with all s-simple vectors $z \in \mathbb{C}^n \otimes \mathbb{C}^n$. It turns out that $\mathbb{V}_s[M_n(\mathbb{C})]$ (respectively, $\mathbb{V}^s[M_n(\mathbb{C})]$) is the dual cone of $\mathbb{P}_s[M_n(\mathbb{C})]$ (respectively, $\mathbb{P}^s[M_n(\mathbb{C})]$) with respect to the pairing (3.1). With this machinery, the maximal faces of $\mathbb{P}_s[M_n(\mathbb{C})]$ and $\mathbb{P}^s[M_n(\mathbb{C})]$ are characterized in terms of s-simple vectors (see also [5–7]). Another consequence is a characterization of the cone $\mathbb{P}_s[M_n(\mathbb{C})] + \mathbb{P}^{\mathrm{t}}[M_n(\mathbb{C})]$: For a linear map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, the map ϕ is the sum of an s-positive linear map and a t-copositive linear map if and only if $\langle A, \phi\rangle \geqslant 0$ for each $A \in \mathbb{V}_s[M_n(\mathbb{C})] \cap \mathbb{V}^{\mathrm{t}}[M_n(\mathbb{C})]$. From the result, we show the atomic property of a positive linear map depends on a tedious matrix manipulation (see [2,3]).

Throughout this section, every vector in the space $\mathbb{C}^n$ will be considered as an $n \times 1$ matrix. The usual orthonormal basis of $\mathbb{C}^n$ and matrix units of $M_n(\mathbb{C})$ will be denoted by $\{e_i\colon i = 1, \ldots, n\}$ and $\{e_{ij}\colon i, j = 1, \ldots, n\}$, respectively, regardless of the dimension n.

From relations (2.5), we calculate $(P_4(e_{ij})) \in M_4(M_4(\mathbb{C}))$ as follows:

$$(P_4(e_{ij})) = \begin{pmatrix} \left(\frac{1}{2}\right)^2(I_2 + s_1) & \left(\frac{1}{2}\right)^2(s_2 - \mathrm{i}s_3) & 0 & -\left(\frac{1}{2}\right)^2\mathrm{i}s_4 \\ \left(\frac{1}{2}\right)^2(s_2 + \mathrm{i}s_3) & \left(\frac{1}{2}\right)^2(I_2 - s_1) & \left(\frac{1}{2}\right)^2\mathrm{i}s_4 & 0 \\ 0 & -\left(\frac{1}{2}\right)^2\mathrm{i}s_4 & \left(\frac{1}{2}\right)^2(I_2 + s_1) & \left(\frac{1}{2}\right)^2(s_2 + \mathrm{i}s_3) \\ \left(\frac{1}{2}\right)^2\mathrm{i}s_4 & 0 & \left(\frac{1}{2}\right)^2(s_2 - \mathrm{i}s_3) & \left(\frac{1}{2}\right)^2(I_2 - s_1) \end{pmatrix}.$$

Now, we fix an integer $k \geqslant 3$, and apply relations (2.5) to the above calculation to get

$$\begin{aligned} P_{2k-1}(e_{22}) &= P_{2k}(e_{22}) \\ &= P_{2k-1}(e_{44}) \\ &= P_{2k}(e_{44}) \\ &= \left(\tfrac{1}{2}\right)^{k-1} e_{22} \otimes I_{k-1}, \end{aligned}$$

$$\begin{aligned}
P_{2k-1}(e_{23}) &= P_{2k}(e_{23})\\
&= (P_{2k-1}(e_{32}))^{\mathrm{t}}\\
&= (P_{2k}(e_{32}))^{\mathrm{t}}\\
&= \left(\tfrac{1}{2}\right)^k(\sigma_3\otimes\sigma_3\otimes\sigma_1+\mathrm{i}\sigma_3\otimes\sigma_2\otimes I_1)\otimes I_{k-3},\\
P_{2k-1}(e_{34}) &= P_{2k}(e_{34})\\
&= (P_{2k-1}(e_{43}))^{\mathrm{t}}\\
&= (P_{2k}(e_{43}))^{\mathrm{t}}\\
&= \left(\tfrac{1}{2}\right)^k(\sigma_2\otimes I_1\otimes I_1+\mathrm{i}\sigma_3\otimes\sigma_1\otimes I_1)\otimes I_{k-3},\\
P_{2k-1}(e_{24}) &= P_{2k}(e_{24})\\
&= (P_{2k-1}(e_{42}))^{\mathrm{t}}\\
&= (P_{2k}(e_{42}))^{\mathrm{t}}\\
&= 0,\\
P_{2k-1}(e_{33}) &= P_{2k}(e_{33})\\
&= \left(\tfrac{1}{2}\right)^{k-1}e_{11}\otimes I_{k-1}.
\end{aligned}$$

For each $1 \leqslant i \leqslant 18$, define $z_i^k \in (\mathbb{C}^{2^3}\otimes\mathbb{C}^{2^{k-3}})\otimes\mathbb{C}^{2^k}=\mathbb{C}^{2^k}\otimes\mathbb{C}^{2^k}$ by

$$\begin{aligned}
z_1^k &= (e_1\otimes e_1)\otimes e_2+(e_4\otimes e_1)\otimes e_3, & z_2^k &= (e_3\otimes e_1)\otimes e_2,\\
z_3^k &= (e_7\otimes e_1)\otimes e_2+(e_6\otimes e_1)\otimes e_3, & z_4^k &= (e_5\otimes e_1)\otimes e_2,\\
z_5^k &= (e_2\otimes e_1)\otimes e_3-(e_1\otimes e_1)\otimes e_4, & z_6^k &= (e_1\otimes e_1)\otimes e_3,\\
z_7^k &= (e_6\otimes e_1)\otimes e_3-(e_5\otimes e_1)\otimes e_4, & z_8^k &= (e_2\otimes e_1)\otimes e_4,\\
z_9^k &= (e_2\otimes e_1)\otimes e_2-(e_3\otimes e_1)\otimes e_3, & z_{10}^k &= (e_4\otimes e_1)\otimes e_2,\\
z_{11}^k &= (e_8\otimes e_1)\otimes e_2-(e_5\otimes e_1)\otimes e_3, & z_{12}^k &= (e_6\otimes e_1)\otimes e_2,\\
z_{13}^k &= (e_3\otimes e_1)\otimes e_3-(e_4\otimes e_1)\otimes e_4, & z_{14}^k &= (e_8\otimes e_1)\otimes e_3,\\
z_{15}^k &= (e_7\otimes e_1)\otimes e_3-(e_8\otimes e_1)\otimes e_4, & z_{16}^k &= (e_3\otimes e_1)\otimes e_4,\\
z_{17}^k &= (e_6\otimes e_1)\otimes e_4, & z_{18}^k &= (e_7\otimes e_1)\otimes e_4.
\end{aligned}$$

Finally, we define the matrix $A(k) \in M_{2^k}(\mathbb{C})\otimes M_{2^k}(\mathbb{C})$ by

$$A(k)=\sum_{i=1}^{18} z_i^k z_i^{k*}+z_6^k z_6^{k*}+z_{14}^k z_{14}^{k*}.$$

Then $A(k)\in\mathbb{V}_2[M_{2^k}(\mathbb{C})]$. Since only $a_{ij}(k)\neq 0$ for $2\leqslant i,\ j\leqslant 4, k\geqslant 3,$ where $A(k)=\sum_{i,j=1}^{2^k}a_{ij}(k)\otimes e_{ij}$, we can easily calculate the pairing (3.1) as follows:

$$\langle A(k), P_{2k}\rangle=\langle A(k), P_{2k-1}\rangle=\sum_{i,j=2}^{4}\langle P_{2k}(e_{ij}), a_{ij}\rangle=-\left(\frac{1}{2}\right)^{k-2}.$$

For $1 \leqslant i \leqslant 18$, define $w_i^k \in (\mathbb{C}^{2^3} \otimes \mathbb{C}^{2^{k-3}}) \otimes \mathbb{C}^{2^k} = \mathbb{C}^{2^k} \otimes \mathbb{C}^{2^k}$ by

$$\begin{aligned}
w_1^k &= (e_3 \otimes e_1) \otimes e_2 - (e_2 \otimes e_1) \otimes e_3, & w_2^k &= (e_1 \otimes e_1) \otimes e_2,\\
w_3^k &= (e_4 \otimes e_1) \otimes e_2 + (e_1 \otimes e_1) \otimes e_3, & w_4^k &= (e_2 \otimes e_1) \otimes e_2,\\
w_5^k &= (e_5 \otimes e_1) \otimes e_2 - (e_8 \otimes e_1) \otimes e_3, & w_6^k &= (e_7 \otimes e_1) \otimes e_2,\\
w_7^k &= (e_6 \otimes e_1) \otimes e_2 + (e_7 \otimes e_1) \otimes e_3, & w_8^k &= (e_8 \otimes e_1) \otimes e_2,\\
w_9^k &= (e_1 \otimes e_1) \otimes e_3 - (e_2 \otimes e_1) \otimes e_4, & w_{10}^k &= (e_3 \otimes e_1) \otimes e_3,\\
w_{11}^k &= (e_4 \otimes e_1) \otimes e_3 - (e_3 \otimes e_1) \otimes e_4, & w_{12}^k &= (e_6 \otimes e_1) \otimes e_3,\\
w_{13}^k &= (e_5 \otimes e_1) \otimes e_3 - (e_6 \otimes e_1) \otimes e_4, & w_{14}^k &= (e_1 \otimes e_1) \otimes e_4,\\
w_{15}^k &= (e_8 \otimes e_1) \otimes e_3 - (e_7 \otimes e_1) \otimes e_4, & w_{16}^k &= (e_4 \otimes e_1) \otimes e_4,\\
w_{17}^k &= (e_5 \otimes e_1) \otimes e_4, & w_{18}^k &= (e_8 \otimes e_1) \otimes e_4.
\end{aligned}$$

Then we see that

$$A(k)^{\mathrm{T}} = \sum_{i=1}^{18} w_i^k w_i^{k*} + w_{10}^k w_{10}^{k*} + w_{12}^k w_{12}^{k*},$$

and so $A(k) \in \mathbb{V}^2[M_{2^k}(\mathbb{C})]$. Consequently we have shown that for any $k \geqslant 3$

$$\langle A(k), P_{2k} \rangle = \langle A(k), P_{2k-1} \rangle = -\left(\tfrac{1}{2}\right)^{k-2} < 0,$$

with some $A(k) \in \mathbb{V}_2[M_{2^k}(\mathbb{C})] \cap \mathbb{V}^2[M_{2^k}(\mathbb{C})]$. By the result in [2] mentioned in this section, we conclude the following:

Theorem 3.1. *Fix any positive integer $k \geqslant 3$ and let V_n be the real $(n+1)$-dimensional spin factor in $M_{2^k}(\mathbb{C})$ for each $n = 2k-1$ or $2k$. Then the unital positive projections $P_n : M_{2^k}(\mathbb{C}) \to M_{2^k}(\mathbb{C})$ with the property $P_n((M_{2^k}(\mathbb{C}))_h) = V_n$ are all atomic positive maps.*

The spin factor of dimension 6 (up to isomorphism) can be both reversible and non-reversible in concrete representations. Theorem 3.1 provides a representation in which V_5 is non-reversible.

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