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Uncertainty principles and Balian–Low type theorems in principal shift-invariant spaces

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ABSTRACT

In this paper, we consider the time-frequency localization of the generator of a principal shift-invariant space on the real line which has additional shift-invariance. We prove that if a principal shift-invariant space on the real line is translation-invariant then any of its orthonormal (or Riesz) generators is non-integrable. However, for any $n \geqslant 2$, there exist principal shift-invariant spaces on the real line that are also $\frac{1}{n}\mathbb{Z}$-invariant with an integrable orthonormal (or a Riesz) generator ϕ, but ϕ satisfies $\int_{\mathbb{R}} |\phi(x)|^2 |x|^{1+\epsilon}\,dx = \infty$ for any $\epsilon > 0$ and its Fourier transform $\hat{\phi}$ cannot decay as fast as $(1+|\xi|)^{-r}$ for any $r > \frac{1}{2}$. Examples are constructed to demonstrate that the above decay properties for the orthonormal generator in the time domain and in the frequency domain are optimal.

1. Introduction and main results

In this paper, a *principal shift-invariant space* on the real line is a shift-invariant space $V_2(\phi)$ generated by a function $\phi \in L^2 := L^2(\mathbb{R})$,

$$V_2(\phi) := \left\{ \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k) \;\middle|\; c := \big(c(k)\big)_{k\in\mathbb{Z}} \in \ell^2 := \ell^2(\mathbb{Z}) \right\}, \tag{1.1}$$

such that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for $V_2(\phi)$, i.e., there exist positive constants A and B such that

$$A\|c\|_{\ell^2} \leqslant \left\| \sum_{k\in\mathbb{Z}} c(k)\phi(\cdot - k) \right\|_2 \leqslant B\|c\|_{\ell^2} \quad \text{for all } c := \big(c(k)\big)_{k\in\mathbb{Z}} \in \ell^2. \tag{1.2}$$

The function ϕ is called the *generator* of the principal shift-invariant space $V_2(\phi)$, and it is called the *orthonormal generator* if $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for $V_2(\phi)$, i.e., (1.2) holds for $A = B = 1$. Principal shift-invariant spaces have been widely used in approximation theory, numerical analysis, sampling theory and wavelet theory (see, e.g., [2,3,8,11,18] and the references therein).

The classical models of principal shift-invariant spaces on the real line are the Paley–Wiener space PW also known as the space of bandlimited functions (the set of all square-integrable functions bandlimited to $[-1/2, 1/2]$) and the spline space S_n^{n-1} (the set of all $(n-1)$-differentiable square-integrable functions whose restriction on any integer interval $[k, k+1]$ coincides with a polynomial of degree at most n). More precisely, the Paley–Wiener space PW is the shift-invariant space

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generated by the sinc function $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$, i.e. $PW = V_2(\mathrm{sinc})$ and the spline space S_n^{n-1} is generated by the B-spline β^n, i.e. $S_n^{n-1} = V_2(\beta^n)$ where β^0 is the characteristic function on $[0,1)$ and $\beta^n, n \geqslant 2$, are defined iteratively by $\beta^n(t) = \int_{\mathbb{R}} \beta^{n-1}(t-\tau)\beta^0(\tau)\,d\tau$.

Now we consider principal shift-invariant spaces that are invariant under additional set of translates other than $\mathbb{Z}$. The shift-invariant spaces with additional invariance have been used in the study of wavelet analysis and sampling theory [19,10,17], and have been completely characterized in [1] for $L^2(\mathbb{R})$ and in [4] for $L^2(\mathbb{R}^n)$. For a subspace V of $L^2(\mathbb{R})$, let

$$\tau(V) := \bigl\{t \in \mathbb{R} \bigm| f(\cdot - t) \text{ belong to } V \text{ for all } f \in V\bigr\}. \tag{1.3}$$

For any closed subspace V of L^2, one may verify that $\tau(V)$ is a closed additive subgroup of $\mathbb{R}$, and hence $\tau(V)$ is either $\{0\}$, or $\mathbb{R}$, or $\alpha\mathbb{Z}$ for some $\alpha > 0$. It can be shown that [1] for a principal shift-invariant space $V_2(\phi)$ on the real line

$$\tau\bigl(V_2(\phi)\bigr) = \mathbb{R} \quad \text{or} \quad \tau\bigl(V_2(\phi)\bigr) = \frac{1}{n}\mathbb{Z} \quad \text{for some } n \in \mathbb{N}. \tag{1.4}$$

We say that a shift-invariant space V on the real line has additional invariance if $\tau(V) \supsetneq \mathbb{Z}$. It is well known that the Paley–Wiener space PW are invariant under all translations. Thus,

$$\tau(PW) = \mathbb{R}.$$

A closed subspace V of L^2 with $\tau(V) = \mathbb{R}$ is usually known as a *translation-invariant space*. The fact that the space of bandlimited functions PW is translation-invariant ($\tau(PW) = \mathbb{R}$) makes it useful for modeling signals and images. However, it is known that any function ϕ that generates a Riesz basis for PW has slow spatial-decay in the sense that $\phi \notin L^1(\mathbb{R})$, e.g., $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$. This slow spatial-decay property for the generator of principal shift-invariant spaces $V_2(\phi)$ that are also translation-invariant is not unique to the space of bandlimited functions PW. In fact, in this paper, we first show that the generator ϕ of any translation-invariant principal shift-invariant space $V_2(\phi)$ on the real line is not integrable.

Theorem 1.1. *Let $\phi \in L^2$ and $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ be a Riesz basis for its generating space $V_2(\phi)$. If $V_2(\phi)$ is translation-invariant then $\phi \notin L^1 := L^1(\mathbb{R})$.*

The slow spatial-decay of the generators of shift-invariant spaces that are also translation-invariant is a disadvantage for the numerical implementation of some analysis and processing algorithms.

On the other hand, Riesz bases for the spline spaces $S_n^{n-1} = V_2(\beta^n)$ can be generated by the compactly supported B-spline functions β^n. This is one of the reasons that spline spaces are often used in signal and image processing algorithms as well as in numerical analysis. Moreover, the B-spline functions β^n are also well localized in frequency domain, since $\hat{\beta}^n(\xi) = O(|\xi|^{-n-1})$. However, the spaces $S_n^{n-1} = V_2(\beta^n)$ have no invariance other than by integer shifts. In fact, it can be shown that any principle shift-invariant space $V_2(\phi)$ generated by a compactly supported function ϕ cannot have any invariance other than by integer shifts [17,1].

One way to circumvent some of the problems is to seek principle shift-invariant spaces $V_2(\phi)$ that are close to being translation invariant, with a generator ϕ which is well localized in both space and frequency domains, i.e., ϕ and $\hat{\phi}$ are well localized. Specifically, we ask whether we can find a shift-invariant space $V(\phi)$ such that $V(\phi)$ is also $\frac{1}{n}\mathbb{Z}$-invariant for some $2 \leqslant n \in \mathbb{N}$, and such that ϕ and $\hat{\phi}$ are well localized. It turns out that it is possible to construct functions ϕ that are well localized in time and frequency domains, that generate shift-invariant spaces $V_2(\phi)$ that are also $\frac{1}{n}\mathbb{Z}$-invariant. However, there are uncertainty and Balian–Low type obstructions, as will be described below. Specifically, the classical uncertainty principle tells us that there is a lower limit on the simultaneous time-frequency localization of functions as shown by

Theorem *(Uncertainty principle). For any function $f \in L^2(\mathbb{R})$, we have*

$$\|f\|_2^2 \leqslant 4\pi \bigl\|xf(x)\bigr\|_2 \bigl\|\xi \hat{f}(\xi)\bigr\|_2, \tag{1.5}$$

and the equality holds only if

$$f(x) = ce^{-sx^2}$$

for $s > 0$ and $c \in \mathbb{R}$.

If we impose more conditions, the time-frequency localization deteriorates even further (see, e.g., [5–7,9,13–16] and the references therein). For example, if the Gabor system $\{E_m T_n g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imx} g(x+n)\}_{m,n\in\mathbb{Z}}$ of a function g is a Riesz basis for $L^2(\mathbb{R})$, we will have the following Balian–Low theorem:

Theorem *(Balian–Low). Let $g \in L^2(\mathbb{R})$. If $\{E_m T_n g\}$ is a Riesz basis for $L^2(\mathbb{R})$, then*

$$\Biggl(\int_{-\infty}^{\infty} \bigl|xg(x)\bigr|^2\,dx\Biggr)\Biggl(\int_{-\infty}^{\infty} \bigl|\xi \hat{g}(\xi)\bigr|^2\Biggr)d\xi = \infty.$$

The Balian–Low theorem implies that if function g generates a Gabor Riesz basis, then it is not possible for the functions g and $\hat{g}$ to be simultaneously well localized. In particular

$$\left|g(x)\right| < \frac{c}{|x|^r}, \qquad \left|\hat{g}(\xi)\right| < \frac{c}{|\xi|^r}$$

cannot hold simultaneously with $r > 3/2$.

1.1. Balian–Low type results for shift-invariant spaces

For the case of a shift-invariant space $V_2(\phi)$ which is also $\frac{1}{n}\mathbb{Z}$-invariant for some $2 \leqslant n \in \mathbb{N}$, we obtain the following surprising result:

Theorem 1.2. *If $\phi \in L^2$ has the property that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V_2(\phi)$, and that $V_2(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant for some $n \geqslant 2$, then for any $\epsilon > 0$, we have*

$$\int_{\mathbb{R}} \left|\phi(x)\right|^2 |x|^{1+\epsilon}\, dx = +\infty. \tag{1.6}$$

Remark 1.

(i) Theorem 1.2 is a Balian–Low type result. If we choose $\epsilon = 1$ in (1.6) of Theorem 1.2, we get $\int_{\mathbb{R}} |x\phi(x)|^2\, dx = +\infty$. It should be noted that in the Balian–Low theorem $\int_{-\infty}^{\infty} |xg(x)|^2\, dx$ can be finite, while in the case of Theorem 1.2 $\int_{\mathbb{R}} |x\phi(x)|^2\, dx$ is always infinite. For the case $\Delta_p = \int_{\mathbb{R}} |\phi(x)|^2 |x|^p\, dx$, the theorem above should be comparable to the $(1,\infty)$ version of the Balian–Low theorem [6,13].

(ii) If we do not require other invariances besides integer shifts, then we can find $V_2(\phi)$ such that $\{\phi(\cdot - k)\colon k \in \mathbb{Z}\}$ is an orthonormal basis for V and such that ϕ decays exponentially in both time and frequency. In particular for such a ϕ it is obvious that $(\int_{-\infty}^{\infty} |x|^\alpha |g(x)|^2\, dx)(\int_{-\infty}^{\infty} |\xi|^\beta |\hat{g}(\xi)|^2\, d\xi) < \infty$, where $\alpha, \beta > 0$ are any positive real numbers.

There is also a decay restriction in the Fourier domain. Specifically, the Fourier transform of an integrable generator ϕ of a principal shift-invariant space which is $\frac{1}{n}\mathbb{Z}$-invariant for some integer $n \geqslant 2$ cannot decay faster than $|\xi|^{-1/2-\epsilon}$ for any $\epsilon > 0$.

Theorem 1.3. *Let $2 \leqslant n \in \mathbb{N}$. Let $\phi \in L^1 \cap L^2$ have the property that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V_2(\phi)$, and that $V_2(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant, then for any $\epsilon > 0$,*

$$\sup_{\xi \in \mathbb{R}} \left|\hat{\phi}(\xi)\right| |\xi|^{1/2+\epsilon} = +\infty. \tag{1.7}$$

We conclude from Theorem 1.3 that there is an obstruction to pointwise frequency (non)-localization property.

Remark 2. The conclusion of Theorem 1.3 remains valid if we weaken the condition that $\phi \in L^1 \cap L^2$ to $\phi \in L^2$ and $\hat{\phi}$ is continuous.

1.2. Optimality of the Balian–Low type results

Now, we show the optimality of the results of Theorems 1.2 and 1.3.

The optimality of Theorem 1.2 is obvious since the $\phi = \text{sinc}$ function generates a translation invariant space and $\int_{\mathbb{R}} |\phi(x)|^2 |x|^{1-\epsilon}\, dx < \infty$ for any $0 < \epsilon < 1$.

The following result shows that (1.7) in Theorem 1.3 is sharp and that for any $2 \leqslant n \in \mathbb{N}$ there exists a generator $\phi \in L^1 \cap L^2$ (that depends on n) for $V_2(\phi)$ such that $\hat{\phi}$ decays like $|\xi|^{-1/2}$. This is done by constructing time-frequency localized generators ϕ that achieve the desired properties:

Theorem 1.4. *For each integer $n \geqslant 2$, there exists a function $\phi \in L^1 \cap L^2$ (and hence $\hat{\phi}$ is continuous) which depends on n, such that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for its generating space $V_2(\phi)$, $V_2(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant, and*

$$\int_{\mathbb{R}} \left|\phi(x)\right|^2 \left(1 + |x|\right)^{1-\epsilon}\, dx < \infty, \tag{1.8}$$

$$\sup_{\xi \in \mathbb{R}} \left|\hat{\phi}(\xi)\right| |\xi|^{1/2} < +\infty. \tag{1.9}$$

Remark 3.

(i) Note that by giving up the translation invariance and only allowing $1/n$ invariance as in Theorem 1.4, we are able to have an L^1 generator, while this is not possible for translation invariance as shown in Theorem 1.1.
(ii) Note that Theorem 1.4 shows the optimality of both Theorems 1.2 and 1.3 simultaneously.

We turn our attention to the integral measure of time-frequency localization for generators of $\frac{1}{n}\mathbb{Z}$-invariant spaces. Unlike what was proven for the translation-invariant case in Theorem 1.1, we prove that by sacrificing a little frequency localization, it is possible for generators of such spaces to be in L^1, even when satisfying the optimality condition (1.8).

Theorem 1.5. *For any $2 \leqslant n \in \mathbb{N}$, $\epsilon > 0$, $\gamma \geqslant 0$, $\delta > 0$, $1 \leqslant q < \infty$ with $1 + \delta - q/2 < 1/(2\gamma)$, there exists $\phi \in L^2$ (that depends on $\epsilon, \delta, q, \gamma, n$) such that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for its generating space $V_2(\phi)$, $V_2(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant and ϕ satisfies the following conditions*:

(1) $\int_{\mathbb{R}} |\phi(x)|^2 (1 + |x|)^{1-\epsilon}\, dx < \infty$,
(2) $\int_{\mathbb{R}} |\phi(x)| (1 + |x|)^{\gamma}\, dx < \infty$,
(3) $\int_{\mathbb{R}} |\hat{\phi}(\xi)|^q (1 + |\xi|)^{\delta}\, d\xi < \infty$.

Remark 4.

(i) Note that the orthonormal generator $\phi = \operatorname{sinc}$ for the Paley–Wiener space PW satisfies the first and third localization properties in Theorem 1.5. However, the sinc function does not satisfy the second time localization inequality. In fact no function ϕ generating a shift-invariant space $V_2(\phi)$ that is also translation invariant can satisfy the second inequality of Theorem 1.5, as is shown in Theorem 1.1. Thus by relaxing translation invariance to $\frac{1}{n}\mathbb{Z}$-invariance we are able to get better time localization in the sense of the second localization inequality above. For this however, we needed to trade off some frequency localization by allowing infinite support in frequency.
(ii) We do not know what happens for the case $\epsilon = 0$.
(iii) Using Lemmas 2.5, 2.6 and 2.7, Theorem 1.5 can be shown to be valid for other norms and other weights.

2. Proofs

2.1. Proof of Theorem 1.1

To prove Theorem 1.1, we recall a characterization for the Riesz (orthonormal) basis property (see, e.g., [12]) and for the translation-invariance property (see [1]).

Proposition 2.1. *Let $\phi \in L^2$. Then*

(i) *$\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V_2(\phi)$ if and only if*

$$m \leqslant \sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi + k)\right|^2 \leqslant M \quad \text{for almost all } \xi \in \mathbb{R}$$

where m and M are positive constants, and
(ii) *$\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for its generating space $V_2(\phi)$ if and only if*

$$\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi + k)\right|^2 = 1 \quad \text{for almost all } \xi \in \mathbb{R}.$$

For shift-invariant spaces that are also translation invariant, the following proposition is a special case of a general result in [1].

Proposition 2.2. *Let $\phi \in L^2$ with the property that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V_2(\phi)$. Then $V_2(\phi)$ is translation-invariant if and only if for almost all $\xi \in \mathbb{R}$,*

$$\hat{\phi}(\xi)\hat{\phi}(\xi + k) = 0 \quad \text{for all } 0 \neq k \in \mathbb{Z}.$$

Now we start to prove Theorem 1.1.

Proof of Theorem 1.1. Suppose on the contrary that there exists a principal shift-invariant space $V_2(\phi)$ on the real line such that $V_2(\phi)$ is translation-invariant and the generator ϕ is integrable. Let

$$\mathcal{O} := \{\xi \in \mathbb{R} \mid \hat{\phi}(\xi) \neq 0\}.$$

Since $\phi \in L^1$ by assumption, $\hat{\phi}$ is continuous, and hence $\mathcal{O}$ is an open set. From Proposition 2.2 it follows that the Lebesgue measure of the set $(\mathcal{O}+j)\cap(\mathcal{O}+k)$ is zero for all $j \neq k \in \mathbb{Z}$. This together with the fact that $\mathcal{O}$ is an open set gives that

$$(\mathcal{O}+j)\cap(\mathcal{O}+k) = \emptyset \quad \text{for all } j \neq k \in \mathbb{Z}. \tag{2.1}$$

Recall that $\mathbb{R}$ is connected and that any connected set is not a union of nonempty disjoint open sets. Thus $\{\mathcal{O}+k \mid k \in \mathbb{Z}\}$ is not an open covering of the real line, i.e., $\mathbb{R}\backslash(\bigcup_{k\in\mathbb{Z}}(\mathcal{O}+k)) \neq \emptyset$, which in turn implies the existence of a real number $\xi_0 \in \mathbb{R}$ with the property that

$$\hat{\phi}(\xi_0+k) = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{2.2}$$

As $\hat{\phi}$ is uniformly continuous by the assumption that $\phi \in L^1$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\hat{\phi}(\xi+k) - \hat{\phi}(\xi_0+k)\right| < \epsilon \quad \text{for all } |\xi-\xi_0| < \delta \text{ and } k \in \mathbb{Z}. \tag{2.3}$$

By (2.1), for any $\xi \in \mathbb{R}$ there exists an integer $l(\xi)$ such that

$$\sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\xi+k)\right|^2 = \left|\hat{\phi}\big(\xi+l(\xi)\big)\right|^2. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) yields

$$\sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\xi+k)\right|^2 < \epsilon^2 \quad \text{whenever } |\xi-\xi_0| < \delta. \tag{2.5}$$

Since $\epsilon > 0$ can be chosen to be arbitrarily small, the last inequality contradicts the Riesz basis property that there exists $m > 0$ such that $m \leqslant \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+k)|^2$ for almost all $\xi \in \mathbb{R}$. □

2.2. Proof of Theorem 1.2

We need a characterization of $\frac{1}{n}\mathbb{Z}$-invariance, which is a special case of a more general result in [1].

Proposition 2.3. *(See [1].) Let $n \geqslant 2$ be an integer, and $\phi \in L^2$ with the property that $\{\phi(\cdot - k) \mid k \in \mathbb{Z}\}$ is a Riesz basis for its generating space $V_2(\phi)$. Then $V_2(\phi)$ is $\frac{1}{n}\mathbb{Z}$-invariant if and only if for almost all $\xi \in \mathbb{R}$, one and only one of the following vectors*

$$\Phi_m(\xi) := \big(\ldots, \hat{\phi}(\xi+m-n), \hat{\phi}(\xi+m), \hat{\phi}(\xi+m+n), \ldots\big), \quad 0 \leqslant m \leqslant n-1, \tag{2.6}$$

is nonzero.

Proof of Theorem 1.2. Suppose on the contrary that

$$\int_{\mathbb{R}} \left|\phi(x)\right|^2 \big(1+|x|\big)^{1+\epsilon}\, dx < \infty. \tag{2.7}$$

Then $\phi \in L^1$, which implies that $\hat{\phi}$ is a uniformly continuous function. Let $\mathcal{O}_m = \{\xi \in \mathbb{R} \mid \Phi_m(\xi) \neq 0\}$, $0 \leqslant m \leqslant n-1$, where Φ_m is defined as in (2.6). Since

$$\mathcal{O}_m = \bigcup_{k\in\mathbb{Z}} \{\xi \in \mathbb{R} \mid \hat{\phi}(\xi+m+kn) \neq 0\},$$

then $\mathcal{O}_m, 0 \leqslant m \leqslant n-1$ are open sets, and

$$\mathcal{O}_m + m = \mathcal{O}_0 \quad \text{and} \quad \mathcal{O}_m + nk = \mathcal{O}_m \quad \text{for all } 0 \leqslant m \leqslant n-1 \text{ and } k \in \mathbb{Z}. \tag{2.8}$$

Moreover, the intersection between the sets $\mathcal{O}_m$ with different m have zero Lebesgue measure (hence are empty sets) by Proposition 2.3. Therefore $\{\mathcal{O}_m \mid 0 \leqslant m \leqslant n-1\}$ is not an open covering of the real line $\mathbb{R}$, which implies that the existence of a real number $\xi_0 \in \mathbb{R}$ with the property that

$$\hat{\phi}(\xi_0+k) = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{2.9}$$

Let $N \geqslant 1$ be a sufficiently large integer, $\delta = N^{-1-\epsilon/2}$, and h be a smooth function supported on $[-2, 2]$ and satisfy $0 \leqslant h \leqslant 1$, and $h(x) = 1$ when $x \in [-1, 1]$. Define $\phi_N(x) = h(x/N)\phi(x)$. Then we obtain that

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|(\hat{\phi}-\hat{\phi}_N)(\xi_0+\xi+k)\right|^2 d\xi\right)^{1/2} \leqslant \left(\frac{1}{2\delta}\int_{\mathbb{R}}\left|(\hat{\phi}-\hat{\phi}_N)(\xi)\right|^2 d\xi\right)^{1/2} = \left(\frac{1}{2\delta}\int_{\mathbb{R}}\left|\phi(x)-\phi_N(x)\right|^2 dx\right)^{1/2}$$
$$\leqslant N^{-\epsilon/4}\left(\int_{\mathbb{R}}\left|\phi(x)\right|^2\left(1+|x|\right)^{1+\epsilon} dx\right)^{1/2}, \tag{2.10}$$

and

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_N(\xi_0+\xi+k)-\hat{\phi}_N(\xi_0+k)\right|^2 d\xi\right)^{1/2}$$
$$= \left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|\int_0^{\xi}\hat{\phi}_N'\left(\xi_0+\xi'+k\right)d\xi'\right|^2 d\xi\right)^{1/2}$$
$$\leqslant \left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\xi\int_0^{\xi}\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_N'\left(\xi_0+\xi'+k\right)\right|^2 d\xi'\,d\xi\right)^{1/2}$$
$$\leqslant \left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\xi\int_0^{\xi}\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}}N^2\left|\hat{h}'(N\eta)\right|\left|\hat{\phi}\left(\xi_0+\xi'+k-\eta\right)\right|d\eta\right|^2 d\xi'\,d\xi\right)^{1/2}$$
$$\leqslant \left(\frac{N^3}{2\delta}\left\|\hat{h}'\right\|_1\int_{-\delta}^{\delta}\xi\int_0^{\xi}\int_{\mathbb{R}}\left|\hat{h}'(N\eta)\right|\left(\sum_{k\in\mathbb{Z}}\left|\hat{\phi}\left(\xi_0+\xi'+k-\eta\right)\right|^2\right)d\eta\,d\xi'\,d\xi\right)^{1/2}$$
$$\leqslant N^{-\epsilon/2}\left\|\hat{h}'\right\|_1\left(\operatorname*{ess\,sup}_{\xi\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\xi+k)\right|^2\right)^{1/2}, \tag{2.11}$$

where $\hat{\phi}_N(\xi) = N\int_{\mathbb{R}}\hat{h}(N\eta)\hat{\phi}(\xi-\eta)\,d\eta$ is used to obtain the second inequality, while the third inequality is obtained by letting $|\hat{h}'(N\eta)| = |\hat{h}'(N\eta)|^{1/2}|\hat{h}'(N\eta)|^{1/2}$ and using Hölder inequality. Also we have that

$$\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_N(\xi_0+k)\right|^2 = \sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}}e^{-2\pi i(\xi_0+k)x}\phi(x)\left(1-h(x/N)\right)dx\right|^2$$
$$\leqslant \int_0^1\left(\sum_{l\in\mathbb{Z}}\left|\phi(x+l)\right|\left|1-h\left((x+l)/N\right)\right|\right)^2 dx$$
$$\leqslant \int_0^1\left(\sum_{l\in\mathbb{Z}}\left|\phi(x+l)\right|^2\left(1+|x+l|\right)^{1+\epsilon}\right)\left(\sum_{l\in\mathbb{Z}}\left(1-h\left((x+l)/N\right)\right)^2\left(1+|x+l|\right)^{-1-\epsilon}\right)dx$$
$$\leqslant 2\left(\sum_{l=N}^{\infty}|l|^{-1-\epsilon}\right)\times\left(\int_{\mathbb{R}}\left|\phi(x)\right|^2\left(1+|x|\right)^{1+\epsilon}dx\right), \tag{2.12}$$

where the first equality follows from (2.9). Combining (2.10), (2.11) and (2.12) with Proposition 2.1 gives

$$m \leqslant \operatorname*{ess\,inf}_{\xi\in\mathbb{R}}\left(\sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\xi+k)\right|^2\right)^{1/2}$$
$$\leqslant \left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|\hat{\phi}(\xi_0+\xi+k)\right|^2 d\xi\right)^{1/2}$$
$$\leqslant \left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_N(\xi_0+\xi+k)-\hat{\phi}_N(\xi_0+k)\right|^2 d\xi\right)^{1/2}$$

$$+\left(\sum_{k\in\mathbb{Z}}\left|\hat\phi_N(\xi_0+k)\right|^2\right)^{1/2}+\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}\sum_{k\in\mathbb{Z}}\left|(\hat\phi-\hat\phi_N)(\xi_0+\xi+k)\right|^2 d\xi\right)^{1/2}$$

$$\leqslant CN^{-\epsilon/4}\to 0 \quad \text{as } N\to\infty, \tag{2.13}$$

which is a contradiction. □

2.3. Proof of Theorem 1.3

Proof. Note that $\phi\in L^1$ implies that $\hat\phi$ is uniformly continuous. Now, suppose on the contrary that

$$\left|\hat\phi(\xi)\right|\leqslant C\left(1+|\xi|\right)^{-1/2-\epsilon} \tag{2.14}$$

for some positive constants C and $\epsilon>0$. This together with the continuity of the function $\hat\phi$ implies that $G_\phi(\xi)=\sum_{k\in\mathbb{Z}}|\hat\phi(\xi+k)|^2$ is a continuous function. Therefore there exists a positive constant m such that

$$G_\phi(\xi)\geqslant m \quad \text{for all } \xi\in\mathbb{R} \tag{2.15}$$

by Proposition 2.1 and the continuity of the function G_ϕ. Using the argument in the proof of Theorem 1.2, we can find a real number $\xi_0\in\mathbb{R}$ such that $\hat\phi(\xi_0+k)=0$ for all $k\in\mathbb{Z}$, which implies that $G_\phi(\xi_0)=0$. This contradicts (2.15). □

2.4. Proof of Theorem 1.4

To prove Theorems 1.4 and 1.5, we construct a family of principal shift-invariant spaces on the real line which are $\frac{1}{n}\mathbb{Z}$-invariant for a given integer $n\geqslant 2$. Let g be an infinitely-differentiable function that satisfies $g(x)=0$ when $x\leqslant 0$, $g(x)=1$ when $x\geqslant 1$, and $(g(x))^2+(g(1-x))^2=1$ when $0\leqslant x\leqslant 1$. For positive numbers $\alpha,\beta>0$ and a natural number $n\geqslant 2$, define $\psi_{\alpha,\beta,n}$ with the help of the Fourier transform by

$$\widehat{\psi_{\alpha,\beta,n}}(\xi)=h_0(\xi)+\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1}(\beta_j)^{-1/2}h_j\bigl(\xi-n(\gamma_j+l)\bigr)+\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1}(\beta_j)^{-1/2}h_j\bigl(-\xi-n(\gamma_j+l)\bigr), \tag{2.16}$$

where $\beta_j=\lceil 2^{j\beta}\rceil$ (the smallest integer larger than or equal to $2^{j\beta}$), $\gamma_j=\sum_{k=0}^{j-1}\beta_k$, $g_0(x)=g(x+1)g(-x+1)$, $g_1(x)=g(x+1)g(-2^\alpha x+1)$, and

$$h_j(\xi)=\begin{cases} g_0(2\xi/(1-2^{-\alpha})) & \text{if } j=0,\\ g_1(2^{j\alpha}(2\xi-1+2^{-j\alpha})/(2^\alpha-1)) & \text{if } j\geqslant 1.\end{cases} \tag{2.17}$$

The functions $\widehat{\psi_{\alpha,\beta,n}}(\xi)$ with $\alpha=1$, $\beta=2$ and $n=2$ and $h_i(\xi)$, $0\leqslant i\leqslant 3$, with $\alpha=1$ are plotted in Fig. 1.

Lemma 2.4. *For $\alpha,\beta>0$ and an integer $n\geqslant 2$, let $\psi_{\alpha,\beta,n}$ be defined as in* (2.16)*. Then $\psi_{\alpha,\beta,n}$ is an orthonormal generator of its generating space $V_2(\psi_{\alpha,\beta,n})$ and the principal shift-invariant space $V_2(\psi_{\alpha,\beta,n})$ is $\frac{1}{n}\mathbb{Z}$-invariant.*

Proof. As each h_j, for $j\geqslant 0$, is supported in $(-1/2,1/2)$ by construction,

$$\left|\widehat{\psi_{\alpha,\beta,n}}(\xi)\right|^2=\left|h_0(\xi)\right|^2+\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1}(\beta_j)^{-1}\left|h_j\bigl(\xi-n(\gamma_j+l)\bigr)\right|^2+\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1}(\beta_j)^{-1}\left|h_j\bigl(-\xi-n(\gamma_j+l)\bigr)\right|^2,$$

which implies that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi_{\alpha,\beta,n}}(\xi+k)\right|^2&=\sum_{k\in\mathbb{Z}}\left|h_0(\xi+k)\right|^2+\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\left(\left|h_j(\xi+k)\right|^2+\left|h_j(-\xi+k)\right|^2\right)\\ &=\left|h_0(\xi)\right|^2+\sum_{j=1}^{\infty}\left|h_j(\xi)\right|^2+\sum_{j=1}^{\infty}\left|h_j(-\xi)\right|^2\end{aligned} \tag{2.18}$$

for any $\xi\in(-1/2,1/2)$. Set

$$H(\xi):=\left|h_0(\xi)\right|^2+\sum_{j=1}^{\infty}\left|h_j(\xi)\right|^2+\sum_{j=1}^{\infty}\left|h_j(-\xi)\right|^2.$$

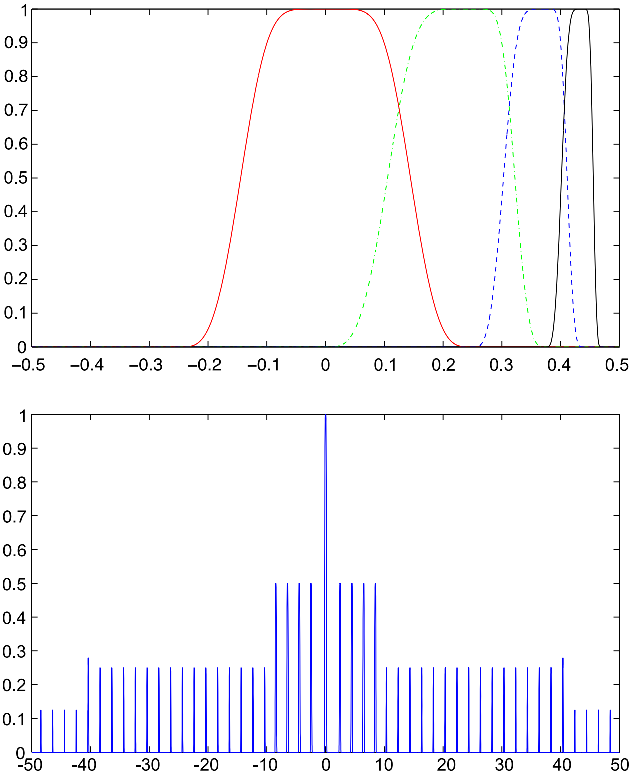

Fig. 1. The functions h_i, $0 \leqslant i \leqslant 3$ with $\alpha = 1$ on the top, and the function $\widehat{\psi_{\alpha,\beta,n}}$ with $\alpha = 1$, $\beta = 2$ and $n = 2$ on the bottom.

Then $H(\xi)$ is a symmetric function supported on $(-1/2, 1/2)$ and for any $\xi \in [1 - 2^{-j\alpha}, 1 - 2^{-(j+1)\alpha}]/2$ with $j \geqslant 0$,

$$\begin{aligned} H(\xi) &= \left|h_j(\xi)\right|^2 + \left|h_{j+1}(\xi)\right|^2 \\ &= \left|g\left(-2^{(j+1)\alpha}\left(2\xi - 1 + 2^{-j\alpha}\right)/\left(2^\alpha - 1\right) + 1\right)\right|^2 + \left|g\left(2^{(j+1)\alpha}\left(2\xi - 1 + 2^{-(j+1)\alpha}\right)/\left(2^\alpha - 1\right) + 1\right)\right|^2 \\ &= 1 \end{aligned} \tag{2.19}$$

by the construction of the functions g and h_j, $j \geqslant 0$. Therefore $H(\xi) = 1$ for all $\xi \in (-1/2, 1/2)$, which together with (2.18) implies that

$$\sum_{k\in\mathbb{Z}} \left|\widehat{\psi_{\alpha,\beta,n}}(\xi + k)\right|^2 = 1 \quad \text{for all } \xi \in \mathbb{R}\backslash(1/2 + \mathbb{Z}). \tag{2.20}$$

Then $\psi_{\alpha,\beta,n}$ is an orthonormal generator for its generating space $V_2(\psi_{\alpha,\beta,n})$ by (2.20) and Proposition 2.1.

By (2.16), $\widehat{\psi_{\alpha,\beta,n}}$ is supported on $(-1/2, 1/2) + n\mathbb{Z}$. Then $V_2(\psi_{\alpha,\beta,n})$ is $\frac{1}{n}\mathbb{Z}$-invariant by (2.20) and Proposition 2.3. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\psi_{\alpha,\beta,n}$ be as in (2.16) for $\alpha, \beta > 0$, and set $\phi = \psi_{\alpha,\beta,n}$. Then by Lemma 2.4 it suffices to prove (1.9) for the function ϕ just defined. From (2.16) it follows that

$$\begin{aligned} \left|\hat{\phi}(\xi)\right||\xi|^{1/2} &= \left|\widehat{\psi_{\alpha,\beta,n}}(\xi)\right||\xi|^{1/2} \\ &\leqslant \sup\Big\{\left|h_0(\xi)\right||\xi|^{1/2}, \sup_{j\geqslant 1,\, 0\leqslant l\leqslant \beta_j - 1} \beta_j^{-1/2}\left|h_j\left(\xi - n(\gamma_j + l)\right)\right||\xi|^{1/2}, \\ &\qquad \sup_{j\geqslant 1,\, 0\leqslant l\leqslant \beta_j - 1} \beta_j^{-1/2}\left|h_j\left(-\xi - n(\gamma_j + l)\right)\right||\xi|^{1/2}\Big\}. \end{aligned}$$

Note that, from its definition, $h_j(\xi - n(\gamma_j + l))$ has support in $[n(\gamma_j + l), n(\gamma_j + l) + 1]$ and has maximal value 1. Thus the term $|h_j(\xi - n(\gamma_j + l))||\xi|^{1/2}$ can be bounded above by $(n(\gamma_j + l) + 1)^{1/2}$ for all ξ and $0 \leqslant l \leqslant \beta_j - 1$. Thus, it follows from the last inequality and the relation $\gamma_j + \beta_j = \gamma_{j+1}$ that

$$\left|\hat{\phi}(\xi)\right||\xi|^{1/2} \leqslant 1 + C\sup_{j\geqslant 1}(\beta_j)^{-1/2}(\gamma_{j+1})^{1/2} < \infty, \tag{2.21}$$

where C is a positive constant. Hence (1.9) holds. In particular, we can show that

$$0 < \limsup_{|\xi|\to\infty}\left|\hat{\phi}(\xi)\right||\xi|^{1/2} < \infty. \tag{2.22}$$

This proves the pointwise frequency localization of the theorem. The time localization inequality is a direct consequence of Lemma 2.5 below. The fact that ϕ is also in L^1 follows from Lemma 2.6 choosing $p = 1$, $\gamma = 0$. □

2.5. Proof of Theorem 1.5

Theorem 1.5 is an immediate consequence of the following three lemmas:

Lemma 2.5. *Let $\epsilon \in (0, 1)$, $\alpha, \beta > 0$, n be an integer with $n \geqslant 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (2.16)*. Then*

$$\int_{\mathbb{R}}\left|\psi_{\alpha,\beta,n}(x)\right|^2|x|^{1-\epsilon}\,dx < \infty. \tag{2.23}$$

Lemma 2.6. *Let $\gamma \geqslant 0$, $1 \leqslant p < 2$, n be an integer with $n \geqslant 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (2.16) *for positive numbers $\alpha, \beta > 0$ with $\beta(1/p - 1/2) + \alpha(p - 1 - \gamma)/p > 0$. Then*

$$\int_{\mathbb{R}}\left|\psi_{\alpha,\beta,n}(x)\right|^p\left(1 + |x|\right)^\gamma\,dx < \infty. \tag{2.24}$$

Lemma 2.7. *Let $\delta > 0$, $1 \leqslant q < \infty$, n be an integer with $n \geqslant 2$, and $\psi_{\alpha,\beta,n}$ be defined as in* (2.16) *for positive numbers $\alpha, \beta > 0$ with $\alpha > \beta(1 + \delta - q/2)$. Then*

$$\int_{\mathbb{R}}\left|\widehat{\psi_{\alpha,\beta,n}}(\xi)\right|^q\left(1 + |\xi|\right)^\delta\,d\xi < \infty. \tag{2.25}$$

Proof of Lemma 2.5. Taking the inverse Fourier transform of both sides of (2.16) yields

$$\begin{aligned}\psi_{\alpha,\beta,n}(x) = {} & \frac{1-2^{-\alpha}}{2}g_0^\vee\left(\frac{1-2^{-\alpha}}{2}x\right) + \frac{2^\alpha - 1}{2}\sum_{j=1}^\infty(\beta_j)^{-1/2}2^{-j\alpha}g_1^\vee\left(\frac{2^\alpha-1}{2^{j\alpha+1}}x\right) \\ & \times e^{\pi ix(1-2^{-j\alpha})}\left(\sum_{l=0}^{\beta_j-1}e^{2\pi ixn(\gamma_j+l)}\right) + \frac{2^\alpha-1}{2}\sum_{j=1}^\infty(\beta_j)^{-1/2}2^{-j\alpha} \\ & \times g_1^\vee\left(-\frac{2^\alpha-1}{2^{j\alpha+1}}x\right) \times e^{-\pi ix(1-2^{-j\alpha})}\left(\sum_{l=0}^{\beta_j-1}e^{-2\pi ixn(\gamma_j+l)}\right),\end{aligned} \tag{2.26}$$

where $g_0^\vee$ and $g_1^\vee$ are the inverse Fourier transforms of the functions g_0 and g_1 respectively. Since both g_0 and g_1 are compactly supported and infinitely differentiable, their inverse Fourier transforms $g_0^\vee$ and $g_1^\vee$ have polynomial decay at infinity. In particular

$$\left|g_0^\vee(x)\right| + \left|g_1^\vee(x)\right| \leqslant C\left(1 + |x|\right)^{-2}, \quad x \in \mathbb{R}$$

for some positive constant C. Hence

$$\begin{aligned}&\left(\int_{\mathbb{R}}\left|\psi_{\alpha,\beta,n}(x)\right|^2\left(1+|x|\right)^{1-\epsilon}dx\right)^{1/2} \\ &\quad\leqslant \left(\frac{1-2^{-\alpha}}{2}\right)\left(\int_{\mathbb{R}}\left|g_0^\vee(x)\right|^2\left(1+|x|\right)^{1-\epsilon}dx\right)^{1/2} \\ &\qquad + \left(2^\alpha - 1\right)\sum_{j=1}^\infty(\beta_j)^{-1/2}2^{-j\alpha}\left(\int_{\mathbb{R}}\left|g_1^\vee\left(\frac{2^\alpha-1}{2^{j\alpha+1}}x\right)\right|^2\left(\frac{\sin\beta_jn\pi x}{\sin n\pi x}\right)^2\left(1+|x|\right)^{1-\epsilon}dx\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\leqslant C + C\sum_{j=1}^{\infty} 2^{-j(\beta+\alpha+\alpha\epsilon)/2}\biggl(\int_{\mathbb{R}} \bigl(1+2^{-j\alpha}|x|\bigr)^{-2}\biggl(\frac{\sin\beta_j\pi x}{\sin\pi x}\biggr)^2 dx\biggr)^{1/2} \\
&= C + C\sum_{j=1}^{\infty} 2^{-j(\beta+\alpha+\alpha\epsilon)/2}\biggl(\int_{-1/2}^{1/2} \biggl(\sum_{l\in\mathbb{Z}}\bigl(1+2^{-j\alpha}|x+l|\bigr)^{-2}\biggr)\biggl(\frac{\sin\beta_j\pi x}{\sin\pi x}\biggr)^2 dx\biggr)^{1/2} \\
&\leqslant C + C\sum_{j=1}^{\infty} 2^{-j(\beta/2+\alpha\epsilon/2)}\biggl(\int_{-1/2}^{1/2} \biggl(\frac{\sin\beta_j\pi x}{\sin\pi x}\biggr)^2 dx\biggr) \\
&\leqslant C + C\sum_{j=1}^{\infty} 2^{-j(\beta+\alpha\epsilon)/2}\biggl(\int_{-1/2}^{1/2} \biggl(\min\biggl(\beta_j, \frac{1}{2|x|}\biggr)\biggr)^2 dx\biggr)^{1/2} \\
&\leqslant C + C\sum_{j=1}^{\infty} 2^{-j\alpha\epsilon/2} < \infty,
\end{aligned}
\tag{2.27}
$$

where C is a positive constant which could be different at different occurrences. □

Proof of Lemma 2.6. Similar to the argument in Lemma 2.5 we have

$$
\begin{aligned}
\biggl(\int_{\mathbb{R}} \bigl|\psi_{\alpha,\beta,n}(x)\bigr|^p \bigl(1+|x|\bigr)^\gamma dx\biggr)^{1/p} &\leqslant C + C\sum_{j=1}^{\infty} 2^{-j(\beta/2+\alpha(1-(1+\gamma)/p))}\biggl(\int_{-1/2}^{1/2} \biggl(\frac{\sin\beta_j\pi x}{\sin\pi x}\biggr)^p dx\biggr)^{1/p} \\
&\leqslant \begin{cases} C + C\sum_{j=1}^{\infty} 2^{-j(\beta(1/p-1/2)+\alpha(p-1-\gamma)/p)} & \text{if } 1<p<2, \\ C + C\sum_{j=1}^{\infty} j2^{-j(\beta/2-\gamma\alpha)} & \text{if } p=1 \end{cases} \\
&< \infty,
\end{aligned}
$$

from which the lemma follows. □

Proof of Lemma 2.7. By (2.16), we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \bigl|\widehat{\psi_{\alpha,\beta,n}}(\xi)\bigr|^q \bigl(1+|\xi|\bigr)^\delta d\xi \\
&\quad = \int_{\mathbb{R}} \bigl|h_0(\xi)\bigr|^q \bigl(1+|\xi|\bigr)^\delta d\xi + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} \beta_j^{-q/2}\int_{\mathbb{R}} \bigl|h_j\bigl(\xi-n(\gamma_j+l)\bigr)\bigr|^q \bigl(1+|\xi|\bigr)^\delta d\xi \\
&\qquad + \sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} \beta_j^{-q/2}\int_{\mathbb{R}} \bigl|h_j\bigl(-\xi-n(\gamma_j+l)\bigr)\bigr|^q \bigl(1+|\xi|\bigr)^\delta d\xi \\
&\quad \leqslant C + C\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} 2^{-\beta j(q/2-\delta)}\int_{\mathbb{R}} \bigl|h_j\bigl(\xi-n(\gamma_j+l)\bigr)\bigr|^q d\xi \\
&\qquad + C\sum_{j=1}^{\infty}\sum_{l=0}^{\beta_j-1} 2^{-\beta j(q/2-\delta)}\int_{\mathbb{R}} \bigl|h_j\bigl(-\xi-n(\gamma_j+l)\bigr)\bigr|^q d\xi \\
&\quad \leqslant C + C\sum_{j\geqslant 1} 2^{j\beta(\delta-q/2+1)-\alpha j} < \infty,
\end{aligned}
\tag{2.28}
$$

where C is a positive constant which could be different at different occurrences. Hence the lemma is established. □

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