Order regularity of two-node Birkhoff interpolation with lacunary polynomials

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A B S T R A C T

In this short work we study the existence and uniqueness of solution for some Birkhoff interpolation problems with lacunary polynomials. First we solve the one-node problem; next we solve the two-node problem in the restricted case where one of the nodes is null.

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1. Introduction

Let \( K = (k_1, \ldots, k_n) \) be an \( n \)-tuple of integers such that \( 0 \leq k_1 < \cdots < k_n \). We call \( K \) a degree sequence and denote by \( \mathcal{P}_K \) the real linear space of lacunary polynomials spanned by the powers \( x^{k_1}, \ldots, x^{k_n} \). When \( k_j = j - 1 \) for \( 1 \leq j \leq n \), \( \mathcal{P}_K \) is the space \( \mathcal{P}_{n-1} \) of polynomials of degree less than \( n \); otherwise \( \mathcal{P}_K \) is a nontrivial linear subspace of \( \mathcal{P}_{n-1} \).

Given an interpolation matrix \( E = (e_{ij})_{i=1,j=0}^{n,m} \) with \( e_{ij} \in \{0, 1\} \) and exactly \( n \) ones, a node system \( X = (x_1, \ldots, x_m) \) made up of \( m \) different real points (nodes) and a system of \( n \) real values \( C = (c_{ij} : e_{ij} = 1) \), the quartet \((X, E, K, C)\) defines a \( K \)-algebraic Birkhoff interpolation problem whose objective is to find a lacunary polynomial \( p(x) = \sum_{i=1}^{n} a_i x^{k_i} \) that satisfies

\[
D^{(j)} p(x_i) = c_{ij}, \quad e_{ij} = 1.
\]

For a given arrangement of the elements \( e_{ij} = 1 \), the \( n \) interpolation conditions (1) determine a system of \( n \) linear equations with \( n \) unknowns. The coefficient matrix of this linear system is the generalized Vandermonde matrix

\[
V(X, E, K) = \begin{pmatrix}
\frac{x_i^{k_1-j}}{(k_1-j)!} & \cdots & \frac{x_i^{k_n-j}}{(k_n-j)!} & e_{ij} = 1
\end{pmatrix},
\]

where we agree on \( 1/k! = 0 \) if \( k < 0 \).

The interpolation problem \((X, E, K, C)\) has a unique solution if and only if the matrix \( V(X, E, K) \) is regular. An interpolation matrix \( E \) is said to be conditionally \( K \)-regular if there exists a node system \( X \) such that \( V(X, E, K) \) is regular. When \( V(X, E, K) \) is regular for every node system \( X = (x_1, \ldots, x_m) \) such that \( a \leq x_1 < \cdots < x_m \leq b \), the matrix \( E \) is said to be order \( K \)-regular on \([a, b]\); otherwise, \( E \) is said to be order \( K \)-singular on \([a, b]\).

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The algebraic Birkhoff interpolation problem [1] is obtained as a particular case of the $K$-algebraic problem when the degree sequence is $K = (0, 1, \ldots, n - 1)$. There exists a well-established theory on the regularity problem of algebraic Birkhoff interpolation; a good review can be found in [2]. In contrast, the knowledge of the $K$-algebraic case is very incomplete. The more generally known results [3,2] are the necessary and sufficient conditions for order $K$-regularity of the Hermite–Sylvester interpolation when the nodes can vary all over the real line, that is, when $-\infty < x_{1} < \cdots < x_{m} < +\infty$. In [4] we stated a new approach to the $K$-algebraic interpolation problem, extended the Pólya condition to the $K$-algebraic problem and characterized the conditionally $K$-regular interpolation matrices. In this work we show how the ideas introduced in [4] can be successfully applied in solving the one-node $K$-regularity problem, and the two-node order $K$-regularity problem in the restricted case where one of the nodes is null.

2. The Pólya $K$-condition

The derivative sequence of an interpolation matrix is the nondecreasing sequence $Q(E) = (q_{1}, \ldots, q_{n})$ whose elements are the derivative orders specified in $E$. We say that an interpolation matrix $E$ with $n$ ones satisfies the Pólya $K$-condition with respect to a degree sequence $K = (k_{1}, \ldots, k_{n})$ if $q_{i} \leq k_{i}$ for $1 \leq i \leq n$; in this case, we write $Q(E) \leq K$. In [4], we proved the following result.

**Theorem 1.** The interpolation matrix $E$ is conditionally $K$-regular if and only if $E$ satisfies the Pólya $K$-condition.

In the proof of the main result of this work, we will use the following lemma.

**Lemma 1.** Let $K = (k_{1}, \ldots, k_{n})$ be a degree sequence and $Q = (q_{1}, \ldots, q_{n})$ a derivative sequence such that there exist indices $i_{1}, i_{2}$ with $q_{i_{1}} \geq k_{i_{1}}$ and let $K' = (k_{1}', \ldots, k_{n}')$. $Q' = (q_{1}', \ldots, q_{n}')$ represent the sequences obtained after removing $q_{i_{1}}$ from $Q$ and $k_{i_{1}}$ from $K$. If $Q \leq K$, then $Q' \leq K'$.

**Proof.** We have

$q'_{i} = \begin{cases} q_{i} & \text{if } i = 1, \ldots, i_{1} - 1 \\ q_{i+1} & \text{if } i = i_{1}, \ldots, n - 1 \end{cases}$ and $k'_{i} = \begin{cases} k_{i} & \text{if } i = 1, \ldots, i_{2} - 1 \\ k_{i+1} & \text{if } i = i_{2}, \ldots, n - 1. \end{cases}$

If $i_{1} < i_{2}$, then we get the contradiction $q_{i_{1}} \leq k_{i_{1}} < k_{i_{2}}$, so it must be that $i_{1} \geq i_{2}$. For $i = 1, \ldots, i_{2} - 1$, we have $q'_{i} = q_{i} \leq k_{i} = k'_{i}$. In the case where $i = i_{2}, \ldots, i_{1} - 1$, we have $q'_{i} = q_{i} \leq k_{i} < k_{i+1} = k'_{i}$. Finally, when $i \geq i_{2}$, we obtain $q'_{i} = q_{i+1} \leq k_{i+1} = k'_{i}$ and the lemma is proven. □

3. The one-node problem

Given an interpolation matrix $E$ with derivative sequence $Q(E) = (q_{1}, \ldots, q_{n})$ and a degree sequence $K = (k_{1}, \ldots, k_{n})$, we define the $K$-degree excess of $E$ as the number

$\rho_{K}(E) = \sum_{i=1}^{n} (k_{i} - q_{i}).$

**Lemma 2.** Let $E$ be an interpolation matrix that satisfies the Pólya $K$-condition; then the determinant $D(X, E, K) = \det \{V(X, E, K)\}$, considered as a function of the nodes $x_{1}, \ldots, x_{m}$, is a homogeneous polynomial of total degree $\rho_{K}(E)$.

**Proof.** We arrange the elements $e_{ij} = 1$ according to the lexicographic order of the pairs $(i, j)$ with prevalence of the second coordinate; then the $r$-th row of the Vandermonde matrix has the structure

$$
\begin{pmatrix}
t^{k_{1}-q_{r}} \\
t^{k_{2}-q_{r}} \\
\vdots \\
t^{k_{n}-q_{r}}
\end{pmatrix},
$$

where $t_{r}$ is one of the variables $x_{1}, \ldots, x_{m}$. The determinant $D(X, E, K)$ is a sum of terms

$$
\alpha_{\sigma} t_{1}^{k_{\sigma(1)}-q_{1}} t_{2}^{k_{\sigma(2)}-q_{2}} \cdots t_{n}^{k_{\sigma(n)}-q_{n}},
$$

where $\sigma$ is a permutation of $\{1, \ldots, n\}$ and $\alpha_{\sigma}$ is a constant that vanishes in the case where any of the $k_{\sigma(t)} - q_{t}$ are negative. For every nonzero term in (2), the total degree is $\sum_{i=1}^{n} (k_{\sigma(t)} - q_{t}) = \rho_{K}(E)$ and, from Theorem 1, we get that the determinant $D(X, E, K)$ does not vanish identically. □

As a result of Lemma 2, we can solve the problem of $K$-regularity of one-row interpolation matrices. We say that $E$ satisfies the $K$-inclusive property if all the elements of $Q(E)$ are in $K$. 

Theorem 2. Let $E$ be an interpolation matrix with one row and $n$ ones, $K = (k_1, \ldots, k_n)$ a degree sequence and $X = (x)$. (i) If $x \neq 0$, the generalized Vandermonde matrix $V(X, E, K)$ is regular if and only if $E$ satisfies the Pólya $K$-condition. (ii) If $x = 0$, $V(X, E, K)$ is regular if and only if $E$ satisfies the $K$-inclusive property.

**Proof.** (i) We know that if $E$ does not satisfy the Pólya $K$-condition, then $V(X, E, K)$ is singular. On the other hand, if $E$ satisfies the Pólya $K$-condition, from Lemma 2 there results $D(X, E, K) = \alpha x^\alpha(E)$ with $\alpha \neq 0$, and hence $D(X, E, K) \neq 0$.

(ii) In this case, $x = 0$. If $E$ does not satisfy the inclusive $K$-condition we can find an element $e_{ij} = 1$ such that $j \neq k_s$ for every degree $k_s$ in $K$. Then, the row in the generalized Vandermonde matrix that corresponds to $e_{ij}$ contains only 0 entries and it turns out that $V(X, E, K)$ is singular. Let us now suppose that all the elements of $Q(E)$ are in $K$. In this case we have $Q(E) = K$ and $\rho_K(E) = 0$; hence $D(X, E, K) = \alpha \neq 0$. □

4. The two-node problem

Let $Q(E)$ represent the derivative sequence corresponding to the first row of $E$. We say that the interpolation matrix $E$ satisfies the upper $K$-inclusive property if all the elements of $Q_1(E)$ are in $K$.

Theorem 3. Let $K = (k_1, \ldots, k_n)$ be a degree sequence, $E$ an interpolation matrix with two rows and $n$ ones that satisfies the Pólya $K$-condition and $X = (x_1, x_2)$ a system of nodes such that $0 = x_1 < x_2$. $V(X, E, K)$ is regular if and only if $E$ satisfies the upper $K$-inclusive property.

**Proof.** If $E$ does not satisfy the upper inclusive $K$-condition we can proceed as in case (ii) of Theorem 2, and it turns out that there exists a row in the generalized Vandermonde matrix that has only 0 entries. Let us assume that $E$ satisfies the upper inclusive $K$-condition and let $E_1$ and $E_2$ be the first and second row of $E$. $n_1$, the number of ones in $E_1$, $n_2$ the number of ones in $E_2$, $Q_1(E) = (q_1^1, \ldots, q_{n_1}^1)$, and $K^* = (k_1^*, \ldots, k_{n_2}^*)$ the degree sequence made up of the elements of $K$ which are not in $Q_1(E)$. We rearrange the basis of $\mathcal{F}_K$ into the form

$$\frac{x^{q_1^1}}{q_1^1!}, \ldots, \frac{x^{q_1^{n_1-1}}}{q_1^{n_1-1}!}, \frac{x^{q_2^1}}{q_2^1!}, \ldots, \frac{x^{q_2^{n_2-1}}}{q_2^{n_2-1}!}$$

and organize the elements $e_{ij} = 1$ in the lexicographic order of the pairs $(i, j)$ with prevalence of the first coordinate. The generalized Vandermonde matrix

$$V(X, E, K) = \begin{pmatrix} x_{1}^{q_1^1-j} \cdots x_{1}^{q_1^{n_1-1}-j} (q_1^1 - j)! & \cdots & x_{2}^{q_2^1-j} \cdots x_{2}^{q_2^{n_2-1}-j} (q_2^1 - j)! \end{pmatrix}$$

will have the following lower triangular block structure:

$$V(X, E, K) = \begin{pmatrix} I_{n_1} & 0_{n_1 \times n_2} \alpha_{n_1 \times n_2} \end{pmatrix} \beta_{n_1 \times n_2}$$

where $I_{n_1}$ is the $n_1$ unity matrix, $0_{n_1 \times n_2}$ is a zero matrix, $M$ is a $n_2 \times n_1$ matrix and $V ((x_2), E_2, K^*)$ is the generalized Vandermonde matrix of the triplet $((x_2), E_2, K^*)$. We know that $Q(E) \leq K$. If we use $n_1$ times Lemma 1 to remove the elements $q_1^1, \ldots, q_1^{n_1}$ from $Q(E)$ and $K$, then we get $Q(E) \leq K^*$. In conclusion, we have the one-node problem $((x_2), E_2, K^*)$ with $x_2 \neq 0$ and with $E_2$ satisfying the Pólya $K^*$-condition. From Theorem 2, the matrix $V ((x_2), E_2, K^*)$ is regular and, in consequence, $V(X, E, K)$ is also regular. □

When the second node is null, an analogous result can be formulated. Let $Q_m(E)$ be the derivative sequence corresponding to the last row of $E$. We say that the interpolation matrix $E$ satisfies the lower $K$-inclusive property if all the elements of $Q_m(E)$ are in $K$.

Theorem 4. Let $K = (k_1, \ldots, k_n)$ be a degree sequence, $E$ an interpolation matrix with two rows and $n$ ones that satisfies the Pólya $K$-condition and $X = (x_1, x_2)$ a system of nodes such that $x_1 < x_2 = 0$. $V(X, E, K)$ is regular if and only if $E$ satisfies the lower $K$-inclusive property.

**Proof.** If $E$ does not satisfy the lower $K$-inclusive property we can choose an element $e_{2j} = 1$ such that $j \neq k_s$ for every degree of $K$; then the corresponding row in $V(X, E, K)$ is a zero row. Let us suppose that $E$ satisfies the lower $K$-inclusive property. In this case we take $Q_2(E) = (q_1^1, \ldots, q_m^1)$, $K^* = (k_1^*, \ldots, k_n^*)$ the degree sequence of the elements of $K$ which are not in $Q_2(E)$, and we proceed as in the proof of Theorem 3. We rearrange now the basis of $\mathcal{F}_K$ in the form

$$\frac{x^{k_1^*}}{k_1^*!} \cdots \frac{x^{k_1^{n_1-1}}}{k_1^{n_1-1}!}, \frac{x^{q_2^1}}{q_2^1!}, \ldots, \frac{x^{q_2^{n_2-1}}}{q_2^{n_2-1}!}$$
and obtain that the generalized Vandermonde matrix has the following lower triangular block structure:

\[ V(X, E, K) = \begin{pmatrix} V((x_1), E_1, K^*) & 0_{n_1 \times n_2} \\ \hline M & I_{n_2} \end{pmatrix}. \]

Finally, we use Lemma 1 and Theorem 2 to conclude that \( V(X, E, K) \) is regular. \( \square \)

In the case \( 0 < x_1 < x_2 \), the upper \( K \)-inclusive property is irrelevant for order regularity. As for the Pólya \( K \)-condition, it obviously is a necessary condition for order regularity, but it is not sufficient. To illustrate this fact, let us consider

\[ E = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad K = (0, 2, 3), \quad X = (x_1, x_2) \quad \text{with} \quad 0 < x_1 < x_2. \]

We observe that \( Q(E) = (0, 0, 2) \leq K \), that is, \( E \) satisfies the Pólya \( K \)-condition. From Lemma 2, we know that the regularity of the generalized Vandermonde matrix is not affected by homothetic transformation of the nodes. Hence, in the study of the order regularity of \( V(X, E, K) \) we can suppose without loss of generality that \( 0 < x_1 < x_2 = 1 \). Now, we get

\[ V(X, E, K) = \begin{pmatrix} 1 & x_1^3/2 & x_1^3/6 \\ 0 & 1 & x_1 \\ 1 & 1/2 & 1/6 \end{pmatrix}. \]

If we compute the determinant, we obtain the polynomial

\[ D(X, E, K) = \frac{1}{6} - \frac{1}{2}x_1 + \frac{1}{3}x_1^3 \]

which has a root \( x_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{3} \) in \((0, 1)\).

References