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Some relations between certain classes consisting of α-convex type and Bazilević type functions

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ABSTRACT

Using the more general method of differential subordinations founded by Miller and Mocanu (2000) [11], several inclusion relations between certain classes consisting of α-convex type functions and Bazilević type functions are first obtained, and their several interesting or important consequences along with some examples are then given.

1. Introduction and preliminaries

Let us denote by $\mathcal{H}(\mathbb{U})$ the class of analytic functions in the unit open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}(n)$ be the collection of the functions $f(z)$ in the general class $\mathcal{H}(\mathbb{U})$ normalized such that

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots,$$

where $a_{n+1} \in \mathbb{C}$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{C}$ is the set of complex numbers.

Under the assumptions:

$$\alpha \in \mathbb{R}, \qquad \delta \in \mathbb{R}, \qquad \mu \in \mathbb{R}, \qquad \beta < 1 \quad \text{and} \quad \gamma < 1,$$

we next denote by $\mathcal{M}_n(\alpha; \beta)$ and $\mathcal{A}_n(\delta, \mu; \gamma)$ subclasses of $\mathcal{A}(n)$ consisting of functions $f(z)$ that satisfy the conditions:

$$\Re e\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \quad (z \in \mathbb{U})$$

and

$$\frac{f(z)f'(z)}{z} \neq 0, \qquad \Re e\left\{\left[f'(z)\right]^{\delta}\left[\frac{f(z)}{z}\right]^{\mu}\right\} > \gamma \quad (z \in \mathbb{U}),$$

respectively.

In particular we note that, here and throughout this paper, the values of the above complex powers are taken to be their principal values. By specifying values for α, β, γ, δ and/or μ, we then receive some well known important or interesting subclasses of analytic and/or univalent functions:

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- $\mathcal{S}_n(\gamma) \equiv \mathcal{M}_n(0; \gamma) = \mathcal{A}_n(1, -1; \gamma)$ $(0 \le \gamma < 1)$ is the class of starlike functions of order γ;
- $\mathcal{S}^* \equiv \mathcal{M}_1(0; 0)$ is the class of starlike functions;
- $\mathcal{K}_n(\gamma) \equiv \mathcal{M}_n(1; \gamma)$ $(0 \le \gamma < 1)$ is the class of convex functions of order γ;
- $\mathcal{K} \equiv \mathcal{M}_1(1; 0)$ is the class of convex functions;
- $\mathcal{B}(n, \mu, \gamma) \equiv \mathcal{A}_n(1, \mu; \gamma)$ $(\mu > -1,\ 0 \le \gamma < 1)$ is subclass of the class of Bazilević functions.

As we know, in the literature, several authors have studied functions $f(z) \in \mathcal{A}(n)$ (or $f(z) \in \mathcal{A}(1)$) as subclasses of the classes $\mathcal{B}(n, \mu, \gamma)$ and/or $\mathcal{A}_n(1, 1-\alpha; \gamma)$, and also received some valuable results. References [1–10] are relevant in that direction.

In this investigation, we first focus on certain inequalities consisting of the differential operator:

$$J(\delta, \mu; f)(z) := \mu \frac{zf'(z)}{f(z)} + \delta \left(1 + \frac{zf''(z)}{f'(z)}\right)$$

that generalizes the expression used in the definition of class $\mathcal{M}_n(\alpha; \beta)$ and we then receive several properties of the expression

$$\left[f'(z)\right]^{\delta} \left[\frac{f(z)}{z}\right]^{\mu}$$

including relations between classes $\mathcal{M}_n(\alpha; \beta)$ and $\mathcal{A}_n(\delta, \mu; \gamma)$.

For that purpose, the following two important lemmas (Lemmas 1.1 and 1.2), which are both (nearly) new for using as in this investigation for the literature and more general form for the theory of differential subordinations, obtained by Miller and Mocanu in [11, pp. 33–35] (and see also (as example) [12]) will be required to prove our main results.

Lemma 1.1. *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ satisfies $\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega$ for all $K \ge Mn$, $\theta \in \mathbb{R}$, and $z \in \mathbb{U}$. If $p(z) \in \mathcal{H}[a, n] \equiv \{p \in \mathcal{H}(\mathbb{U}) : p(z) = a + a_n z^n + \cdots, z \in \mathbb{U}\}$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $|p(z)| < M$ $(z \in \mathbb{U})$.*

Lemma 1.2. *Let $\Omega \subset \mathbb{C}$ and suppose that the function $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$ satisfies $\psi(ix, y; z) \notin \Omega$ for all $x \in \mathbb{R}$, $y \le -n(1+x^2)/2$, and $z \in \mathbb{U}$. If $p(z) \in \mathcal{H}[a, n]$ and $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{U}$, then $\Re e\{p(z)\} > 0$ $(z \in \mathbb{U})$.*

2. Main results and their certain consequences

We now state and then prove each of our main results given by Theorems 2.1 and 2.2.

Theorem 2.1. *Let $f(z) \in \mathcal{A}(n)$ with $f'(z)f(z)/z \ne 0$ for all $z \in \mathbb{U}$, and also let $\delta \in \mathbb{R}$ and $\mu \in \mathbb{R}$. If*

$$\Re e\{J(\delta, \mu; f)(z)\} < \delta + \mu + \frac{nM}{M+1} \quad (z \in \mathbb{U}),$$

where $M \ge 1$, then

$$\left|\left[f'(z)\right]^{\delta} \left[\frac{f(z)}{z}\right]^{\mu} - 1\right| < M \quad (z \in \mathbb{U}),$$

where the powers are the principal ones.

Proof. Let us define $p(z)$ in the form

$$p(z) = \left[f'(z)\right]^{\delta} \left[\frac{f(z)}{z}\right]^{\mu} - 1.$$

Then, from the assumptions $f(z) \in \mathcal{A}(n)$ and $f'(z)f(z)/z \ne 0$ for all $z \in \mathbb{U}$, we easily observe that $p(z)$ is in the class $\mathcal{H}[0, n]$, and simple computation shows that

$$J(\delta, \mu; f)(z) = \delta + \mu + \frac{zp'(z)}{p(z)+1}.$$

Letting

$$\psi(r, s; z) := \delta + \mu + \frac{s}{r+1}$$

and

$$\Omega := \left\{w \in \mathbb{C} : \Re\{w\} < \delta + \mu + \frac{nM}{M+1}\right\},$$

we then receive that $\psi(p(z), zp'(z); z) = J(\delta, \mu; f)(z) \in \Omega$ for all $z \in \mathbb{U}$. Further, for any $\theta \in \mathbb{R}$, $K \geq nM$ and $z \in \mathbb{U}$, since $M \geq 1$, we also have

$$\Re e\left\{\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right)\right\} = \delta + \mu + K\Re e\left(\frac{1}{M + e^{-i\theta}}\right) \geq \delta + \mu + \frac{nM}{M+1},$$

i.e., $\psi\left(Me^{i\theta}, Ke^{i\theta}; z\right) \notin \Omega$. Therefore, according to Lemma 1.1, we obtain $|p(z)| < M$ $(z \in \mathbb{U})$. This completes the proof of Theorem 2.1. $\square$

For the special case $M := \gamma + 1$, the above theorem reduces to the next result, which represents a sufficient condition for a function $f(z) \in \mathcal{A}(n)$ to be in the class $A_n(\delta, \mu; -\gamma)$:

Corollary 2.1. *Let $f(z) \in \mathcal{A}_n$ with $f'(z)f(z)/z \neq 0$ for all $z \in \mathbb{U}$, and also let $\delta \in \mathbb{R}$ and $\mu \in \mathbb{R}$. If*

$$\Re e\{J(\delta, \mu; f)(z)\} < \delta + \mu + \frac{n(\gamma+1)}{\gamma+2} \quad (z \in \mathbb{U}),$$

where $\gamma \geq 0$, then

$$\left|\left[f'(z)\right]^{\delta}\left[\frac{f(z)}{z}\right]^{\mu} - 1\right| < \gamma + 1 \quad (z \in \mathbb{U}),$$

where the powers are the principal ones, and furthermore $f(z) \in \mathcal{A}_n(\delta, \mu; -\gamma)$.

Taking $\delta := \alpha$ and $\mu := 1 - \alpha$ in the above corollary, we then obtain the following result.

Corollary 2.2. *Let $f(z) \in \mathcal{A}_n$ with $f'(z)f(z)/z \neq 0$ for all $z \in \mathbb{U}$, and let also $\alpha \in \mathbb{R}$. If*

$$\Re e\{J(\alpha, 1-\alpha; f)(z)\} < 1 + \frac{n(\gamma+1)}{\gamma+2} \quad (z \in \mathbb{U}),$$

where $\gamma \geq 0$, then

$$\left|\left[f'(z)\right]^{\alpha}\left[\frac{f(z)}{z}\right]^{1-\alpha} - 1\right| < \gamma + 1 \quad (z \in \mathbb{U}),$$

and furthermore $f(z) \in \mathcal{A} - n(\alpha, 1-\alpha; -\gamma) = \mathcal{M}_n(\alpha; -\gamma)$.

For $\alpha := 1$ and $\alpha := 0$, the above corollary reduces respectively to the following examples.

Example 2.1. Let $\gamma \geq 0$ and $f(z) \in \mathcal{A}(n)$. Then,

(i) If

$$\Re e\left(\frac{zf''(z)}{f'(z)}\right) < \frac{n(\gamma+1)}{\gamma+2} \quad (z \in \mathbb{U}),$$

then $\left|f'(z) - 1\right| < \gamma + 1$ $(z \in \mathbb{U})$.

(ii) If

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{n(\gamma+1)}{\gamma+2} \quad (z \in \mathbb{U}),$$

then $\left|\frac{f(z)}{z} - 1\right| < \gamma + 1$ $(z \in \mathbb{U})$.

Remark 2.1. For $\gamma = 0$, Example 2.1(i) gives us the next criterion for univalence: if

$$f(z) \in \mathcal{A}(n) \quad \text{and} \quad \Re e\left(\frac{zf''(z)}{f'(z)}\right) < \frac{n}{2} \quad (z \in \mathbb{U}),$$

then

$$\left|f'(z) - 1\right| < 1 \quad (z \in \mathbb{U}),$$

i.e., $\Re e\{f'(z)\} > 0$ $(z \in \mathbb{U})$, hence the function $f(z)$ is univalent in the disc $\mathbb{U}$.

Theorem 2.2. *Let $f(z) \in \mathcal{A}(n)$ with $f'(z)f(z)/z \neq 0$, $z \in \mathbb{U}$, and also let $\delta, \mu \in \mathbb{R}$ and $\gamma \in [0, 1)$. If $\Re e\{J(\delta, \mu; f)(z)\} > \beta(\delta, \mu; \gamma)$ $(z \in \mathbb{U})$, where*

$$\beta(\delta, \mu; \gamma) = \begin{cases} \delta + \mu - \dfrac{n\gamma}{2(1-\gamma)} & \text{if } \gamma \in \left[0, \dfrac{1}{2}\right] \\ \delta + \mu - \dfrac{n(1-\gamma)}{2\gamma} & \text{if } \gamma \in \left[\dfrac{1}{2}, 1\right), \end{cases} \tag{1}$$

then $f(z) \in \mathcal{A}_n(\delta, \mu; \gamma)$.

Proof. If we take $p(z)$ as

$$p(z) = \frac{1}{1-\gamma}\left(\left[f'(z)\right]^{\delta}\left[\frac{f(z)}{z}\right]^{\mu} - \gamma\right),$$

where the powers are the principal ones, we then easily observe $p(z) \in \mathcal{H}[1, n]$ and also

$$\delta + \mu + \frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z) + \gamma} := J(\delta, \mu; f)(z).$$

Further, since

$$\psi(r, s; z) := \delta + \mu + \frac{s(1-\gamma)}{r(1-\gamma) + \gamma}$$

and

$$\Omega := \{w \in \mathbb{C} : \Re\{w\} > \beta(\delta, \mu; \gamma)\},$$

it leads to $\psi(p(z), zp'(z); z) = J(\delta, \mu; f)(z) \in \Omega$ for all $z \in \mathbb{U}$. Also, for any $x \in \mathbb{R}$, $y \leq -n(1 + x^2)/2$ and $z \in \mathbb{U}$, we have

$$\begin{aligned} \Re e\{\psi(ix, y; z)\} &= \delta + \mu + \frac{\gamma(1-\gamma)y}{(1-\gamma)^2x^2 + \gamma^2} \\ &\leq \delta + \mu - \frac{n\gamma(1-\gamma)}{2}\,\frac{x^2+1}{(1-\gamma)^2x^2 + \gamma^2} \equiv h(x) \\ &\leq \beta(\delta, \mu; \gamma) = \begin{cases} \lim\limits_{x\to+\infty} h(x) & \text{if } \gamma \in \left[0, \dfrac{1}{2}\right] \\ h(0) & \text{if } \gamma \in \left[\dfrac{1}{2}, 1\right), \end{cases} \end{aligned}$$

i.e., $\psi(ix, y; z) \notin \Omega$. Finally, by Lemma 1.2, we obtain that $\Re\{p(z)\} > 0$ $(z \in \mathbb{U})$ which completes the proof of Theorem 2.2. $\square$

Letting $\delta := \alpha$ and $\mu := 1 - \alpha$ in Theorem 2.2, we then obtain the following corollaries.

Corollary 2.3. *Let $f(z) \in \mathcal{A}(n)$ with $f'(z)f(z)/z \neq 0$ for all $z \in \mathbb{U}$, and let $\alpha \in \mathbb{R}$ and $\gamma \in [0, 1)$. If $f(z) \in \mathcal{M}_n(\alpha; \widetilde{\beta}(\gamma))$, where*

$$\widetilde{\beta}(\gamma) \equiv \beta(\alpha, 1-\alpha; \gamma) = \begin{cases} 1 - \dfrac{n\gamma}{2(1-\gamma)} & \text{if } \gamma \in \left[0, \dfrac{1}{2}\right] \\ 1 - \dfrac{n(1-\gamma)}{2\gamma} & \text{if } \gamma \in \left[\dfrac{1}{2}, 1\right), \end{cases}$$

then $f \in \mathcal{A}_n(\alpha, 1-\alpha; \gamma)$.

Putting $\delta := -1$ and $\mu := 1$ in Theorems 2.1 and 2.2, respectively, we next get the following corollaries.

Corollary 2.4. *Let $f(z) \in \mathcal{A}(n)$ with $f'(z)f(z)/z \neq 0$ for all $z \in \mathbb{U}$, and also let $M \geq 1$. Then,*

$$\Re e\{J(-1, 1; f)(z)\} < \frac{nM}{M+1} \Rightarrow \left|\frac{f(z)}{zf'(z)} - 1\right| < M \quad (z \in \mathbb{U}),$$

i.e.,

$$\begin{cases} \left|\dfrac{zf'(z)}{f(z)} + \dfrac{1}{M^2-1}\right| > \dfrac{M}{M^2-1} \quad (z \in \mathbb{U}) & \text{when } M > 1 \\ \Re e\left(\dfrac{zf'(z)}{f(z)}\right) > \dfrac{1}{2} \quad (z \in \mathbb{U}) & \text{when } M = 1. \end{cases}$$

Corollary 2.5. *Let $f(z) \in \mathcal{A}(n)$ with $f'(z)f(z)/z \neq 0$ for all $z \in \mathbb{U}$, and also let $\gamma \in [0, 1)$. If $\Re e\{J(-1, 1; f)(z)\} > \beta(-1, 1; \gamma)$ $(z \in \mathbb{U})$, where $\beta(-1, 1; \gamma)$ is given by (1), then*

$$\Re e\left(\frac{f(z)}{zf'(z)}\right) > \gamma \quad (z \in \mathbb{U}),$$

i.e.,

$$\begin{cases} \left|\dfrac{zf'(z)}{f(z)} - \dfrac{1}{2\gamma}\right| < \dfrac{1}{2\gamma} \quad (z \in \mathrm{U}) & \text{when } \gamma \in (0, 1) \\ \Re e\left(\dfrac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathrm{U}) & \text{when } \gamma = 0. \end{cases}$$

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