JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **221,** 59–72 (1998)
ARTICLE NO. AY965102

On Nilpotent Semigroups and Solutions with Finite Stopping Time*

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Submitted by Joseph A. Ball

Received April 12, 1995

We consider here the evolution equation $u'(t) = Bu(t)$, where B is some unbounded closed operator with dense domain in some separable Hilbert space. We consider the non-trivial classical solution $u(t)$ of the last equation such that $u(t) = 0$ for $t > T$. We are interested in finding conditions on operator B for this to occur. There are two cases: in the first case operator B generates a nice semigroup and the inverse to it is an abstract Volterra operator without point spectra, the Cauchy problem is well-posed in this case, and every solution will be zero in finite time; in the second case every point of the complex plane is in the spectral of operator B and so it cannot generate any semigroup and the Cauchy problem in this case is not well-posed. More precisely, there is no uniqueness for solution of the Cauchy problem in the last case. It is interesting to note that such a solution can occur only in two extreme situations: when the spectra of operator B are trivial, or when every point of the complex plane is in it. © 1998 Academic Press

INTRODUCTION

We consider here the equation

$$u'(t) = Bu(t)$$

with some closed unbounded operator B defined on some dense domain D_1 of separable Hilbert space H. We would like to know under what condition on operator B the last equation will possess a non-trivial solution $u(t)$ that becomes zero in finite time. If operator B generates a C_0 semigroup the answer is known and was given by Esterle [3, p. 102]. It

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0022-247X/98 $25.00

generalized a classical example due to Hille and Phillips [1, p. 537]. In both cases the study was limited to the case of semigroups. Here we would like to study the question from the evolution equation point of view.

In Section 1 we give the Hille–Phillips example. In Section 2 we consider the case of operator B with trivial kernel. The main result here is Theorem 1 which gives a generalization of the Hille–Phillips example and of the Esterle theorem (with necessary and sufficient conditions). In Theorem 5 one can see another type of result.

In Section 3 we consider the representation of our Hilbert space as a direct sum of some specifically constructed invariant with respect to B subspaces. In Section 4 we consider the case of operator B with non-trivial kernel. Theorem 2 is the main result here and it gives necessary and sufficient conditions in this case. Theorems 3 and 4 give some sufficient (Theorem 3) and some necessary (Theorem 4) results. There are several remarks in different sections.

What is very interesting here is the fact that there are two extreme cases for such solutions to occur. In the first case (Theorem 1) operator B generates a nice semigroup, the Cauchy problem is well-posed, every solution will became zero in finite time, and the spectra of operator B are trivial. More precisely, only points at infinity will be in the spectra of B. In the second case (Theorem 2) every point of the complex plane will be in the spectra of operator B and so B could not generate any semigroup, the Cauchy problem cannot be well-posed in this case, and not every solution will be zero in finite time; moreover, in this case one could construct solutions with compact (in t) support, i.e., we cannot have here uniqueness.

1. HILLE–PHILLIPS EXAMPLE

Let's consider the Hilbert space H closer in the norm of H_1 of C^∞ functions on $[0, 1]$ that have zeros at $t = 1$. Now let's consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \tag{1}$$

where for every $t \in \mathbb{R}_+$, $u(t) \in H$. We consider now the solution of (1). It is easy to see that $u(t) = 0$ for $t \geq 1$ for any possible initial data from H and $\partial/\partial x$ is a generator of a nilpotent semigroup of left translations $U_0(s)$. This is a well known example of Hille and Phillips [1, p. 537].

Now one can see that our operator $\partial/\partial x = B$ has the following properties

(1) $\operatorname{Ker} B|_H = \{0\}$;

(2) B^{-1} is defined on all H and it is an integration—the special type of Volterra operator;

(3) there exists a chain of strictly increasing invariant subspaces of the form $\operatorname{Ker} U_0(\tau)|_H$ for $\tau \in [0, 1]$ (actually this is the chain of invariant subspaces of function with support on the interval $[0, 1 - \tau]$),

(4) $\|B^{-n}x\| \leq A(c^n/n!)$ with constants $A > 0$, $c > 0$ depending on $x \in H$;

(5) $U_0(s)$ is defined on every $u_0 \in H$ and $U_0(s)u_0 = 0$ for $s \geq s_0$ depending on u_0 but $s_1 \leq 1$;

(6) the Cauchy problem for (1) in H has an unique solution given by $u(t) = U_0(t)u_0$.

2. CASE OF TRIVIAL KERNEL

Now we would like to see how different things could go in the abstract case. We will consider any abstract closed operator B (of course unbounded) defined on dense subset D_1 of Hilbert space H and such that the equation

$$\frac{du}{dt} = Bu \tag{2}$$

has a classical (or strong) solution $u(t)$ with $u(0) = u_0$, $u(t) \not\equiv 0$, but $u(t) = 0$ for $t \geq T$.

We will assume that our operator B satisfies the condition (see (1) above)

$$\operatorname{Ker} B|_H = \{0\}. \tag{3}$$

We would like to know what kind of operator B can produce this phenomenon and how much information about operator B we could obtain only from the condition (3) and the existence of such an "abnormal" solution. We have the following

THEOREM 1. *Let's assume that* $\operatorname{Ker} B|_H = 0$. *Then the following conditions are equivalent*

(1) *There exists one solution* $u(t)$ *of* (2) *such that* $u(t) \not\equiv 0$ *and* $u(t) = 0$ *for* $t \geq T$.

(2) *There exists an infinitely dimensional subspace* $\mathscr{H}$ *such that all solutions of* $\mathscr{H}$ *have the same property, i.e.,* $u(t) = 0$ *for* $t \geq T$.

(3) *There exists an infinitely dimensional subspace* $\mathscr{H}$ *such that*:

(a) $\mathscr{H}$ *is invariant with respect to* B *and* B^{-1};

(b) $\mathscr{H}$ *is a span of all* $\{u(t), t > 0\}$ *where* $u(t)$ *is a solution from* (1);

(c) B^{-1} *will be an abstract Volterra operator, moreover,*

$$\|B^{-n}x\| \leq c\frac{A^n}{n!},$$

where $c > 0$, $A > 0$ *depend on* $x \in \mathscr{H}$;

(d) $\operatorname{span}\{u_0, B^{-1}u_0, \dots\} = \operatorname{span}\{B^{-1}u_0, B^{-2}u_0, \dots\}$;

(e) $u(0)$ *will be a cyclic vector for subspace* $\mathscr{H}$ *with respect to the operator* B^{-1};

(h) *Operator* B *defined on subspace* $\mathscr{H}$ *generates a nilpotent semi-group* $\mathscr{U}(s)$ *on* $\mathscr{H}$;

(f) *The Cauchy problem for Eq.* (2) *has an unique solution given by the formula*

$$u(t) = \mathscr{U}(t)u_0;$$

(g) *There exists a strictly increasing chain of invariant subspaces with respect to* B *in* $\mathscr{H}$ *of the form*

$$\operatorname{Ker} \mathscr{U}(T-\tau)|_{\mathscr{H}}.$$

Proof. If we take the Laplace transform of (2) we obtain using closedness of operator B

$$\xi\tilde{u}(\xi) - u(0) = B\tilde{u}(\xi). \tag{4}$$

Now, the Paley–Wiener type theorem is true in this case and $\tilde{u}(\xi)$ will be an entire function of exponential type, i.e.,

$$|\tilde{u}(\xi)| \leq ce^{N|\xi|} \tag{5}$$

with c, N depending on $u(t)$ (actually $N \leq T$). From here we have that

$$\tilde{u}(\xi) = \sum_{n=0}^{\infty} \xi^n v_n \tag{6}$$

and from the estimates of the coefficients of function of exponential type we get

$$\|v_n\| \le c\frac{A^n}{n!}, \qquad n = 1, 2, \dots . \tag{7}$$

Substituting (6) in (4) one obtains

$$-u(0) = -u_0 = Bv_0, \qquad v_{n-1} = Bv_n \tag{8}$$

or

$$-u_0 = B^{n+1}v_n, \tag{9}$$

or $v_n \in D(B^{n+1}$ for $n = 0, 1, 2, \dots$.

If we consider now the closed linear hull of $\{v_n\}_{n=0}^{\infty}$ and denote it by $\mathscr{H}$, then we obtain:

(1) $u(t) \in \mathscr{H}$ for $t > 0$.

(2) Taking a limit for $t \to 0$ we see that $u(0) \in \mathscr{H}$.

(3) $\mathscr{H}$ is invariant with respect to B, since $Bv_0 = -u_0 \in \mathscr{H}$ and u_0 is a linear combination of $\{v_n\}_{n=0}^{\infty}$—see (1) and $Bv_n = v_{n-1}$.

(4) $\dim \mathscr{H} = \infty$, since the finite dimensional case we cannot get $u(t) = 0$ for $t \ge T$.

(5) We can define B^{-1} on v_n by the formula

$$B^{-1}v_n = v_{n+1}, \qquad B^{-1}u_0 = -v_0 \tag{10}$$

and later on we will extend it by linearity to all $\mathscr{H}$ (see (13)). We will get $\|B^{-n}x\|^{1/n} \to 0$ (from the fact that $\tilde{u}(\xi)$ is an entire function) and, moreover from (7) we have

$$\|B^{-n}x\| \le c\frac{A^n}{n!} \tag{11}$$

with constants c, A depending on $x \in \mathscr{H}$ (for now on we can claim (11) only from a dense subset of $\mathscr{H}$).

(6) $u(t)$ for $t > 0$ and its linear combinations are dense in $\mathscr{H}$, so we can consider $\mathscr{H}$ as the minimal subspace of H that is spanned by $\{u(t), t > 0\}$, or the closed linear null of $\{u(t),\ t > 0\}$. To see (6) we can consider a minimal subspace of H that is spanned by $u(t)$. We can perform the Laplace transform there and we will get that the v_n belong to that subspace, for $n = 0, 1, \dots$.

From (5) we have the following:

(7) $\operatorname{span}\{u_0, B^{-1}u_0, B^{-2}u_0, \dots\} = \operatorname{span}\{B^{-1}u_0, B^{-2}u_0, \dots\}$. Let's assume that there exists $t_0 \in [0, T)$ such that $u(t_0) = 0$ but $u(t) \not\equiv 0$ for $t > t_0$. Then we can make the same Laplace transform starting from $t = t_0$ and obtain

$$Bv_0 = -u(t_0) = 0$$

and from here ($\operatorname{Ker} B|_H = \operatorname{Ker} B|_{\mathscr{H}} = \{0\}$) we have

$$v_0 = 0, \dots, \qquad v_n = 0, \dots$$

and from (6) we conclude that $u(t) = 0$ for $t \geq t_0$. It gives us some kind of uniqueness. Now we introduce for $s \geq 0$ the operator $\mathscr{U}(s)$ by the formula

$$\mathscr{U}(s)u(t) = u(t+s)$$

and we extend it by linearity to the dense subset of $\mathscr{H}$. $\mathscr{U}(s)$ behaves as a semigroup since

$$\mathscr{U}(0) = E, \qquad \mathscr{U}(s_1)\mathscr{U}(s_2) = \mathscr{U}(s_1 + s_2).$$

Now

$$\lim_{\varepsilon \to 0} \frac{\mathscr{U}(\varepsilon) - \mathscr{U}(0)}{\varepsilon} u(t) = \lim_{\varepsilon \to 0} \frac{u(t+\varepsilon) - u(t)}{\epsilon} = \frac{du}{dt} = Bu$$

and B will be a generator of $\mathscr{U}(s)$.

(8) We have $\mathscr{U}(s)$ defined on a dense subset of $\mathscr{H}$.

(9) $u(0)$ will be a cyclic vector with respect to B^{-1} of the subspace $\mathscr{H}$, or equivalently $u(0)$ will cover $\mathscr{H}$. This follows from (7).

For our solution $u(t)$ we have the following: For any $n > 0$, $t > 0$ exists $c(n, t)$ such that

$$\|u(s+t)\| \leq c(n,t)e^{-ns}\|u(t)\| \tag{12}$$

(it follows from the fact that $u(t) = 0$ for $t = T$ and boundedness of $u(t)$). We can produce from (12) the same kind of estimate with uniform $c(n)$ for all $t \in [0, T]$, i.e.,

$$\|u(s+t)\| \leq c(n)e^{-ns}\|u(t)\|. \tag{13}$$

Now using (13) we can extend our semigroup $\mathscr{U}(s)$ from the dense subset of all $\mathscr{H}$ and we obtain the following:

(10) We have $\mathscr{U}(s)$ defined on all $\mathscr{H}$ and $\mathscr{U}(s)x = 0$ for $s \geq s_0$ depending on $x \in \mathscr{H}$, but $s_0 \leq T$, i.e., $\mathscr{U}(T)x = 0$ for all $x \in \mathscr{H}$.

It is easy to see that our semigroup $\mathscr{U}(t)$ will be the continuous semigroup, since $\|\mathscr{U}(t)\| \leq 1$ and $t \geq 0$ and since

$$\lim_{t \to t_0} \mathscr{U}(t)u(s) = u(t_0 + s) = \mathscr{U}(t_0)u(s)$$

(it follows from the definition of solution and its continuity in $t \geq 0$). Moreover, our semigroup $\mathscr{U}(t)$ is strongly continuous (see [2, Chap. IX]) and therefore has a generator which will be our initial operator B.

(11) The Cauchy problem for Eq. (2) has an unique mild solution in $\mathscr{H}$ and this solution is given by

$$u(t) = \mathscr{U}(t)u_0 \tag{14}$$

for any $u_0 \in \mathscr{H}$.

Our semigroup $\mathscr{U}(s)$ will be a nilpotent semigroup on the subspace $\mathscr{H}$, defined on $\mathscr{H}$ (and not on the dense subset of it). We cannot claim the same for our initial space H.

(12) There exists a strictly increasing chain of invariant subspaces with respect to the operator B in $\mathscr{H}$ of the form

$$\operatorname{Im} \mathscr{U}(\tau) = \operatorname{Ker} \mathscr{U}(T - \tau)|_{\mathscr{H}} = M_\tau \tag{15}$$

for $\tau \in [0, T]$.

To prove it let's note that any subspace of form (15) is invariant with respect to the operator B.

Let's assume now that

$$M_\tau = M_\sigma$$

for $\tau, \sigma \in [0, T]$, $\tau < \sigma$. We have that $M_0 = \mathscr{H}$, $M_T = \{0\}$ and it is easy to see that if

$$M_t \neq M_{t+s} \tag{16}$$

then $M_t \neq M_{t+s/2}$ since $M_{t+s} = \mathscr{U}(s)M_t$ and if $M_t = M_{t+s/2}$ we should have

$$M_{t+s} = \mathscr{U}\left(\frac{s}{2}\right)M_{t+s/2} = \mathscr{U}\left(\frac{s}{2}\right)M_t = M_{t+s/2} = M_t.$$

The same reasoning gives us that if (16) is correct then $M_t \neq M_{t+\tau}$ for $0 < \tau \leq s$ and moreover that they are strictly decreasing. Using the fact that $M_T = \{0\}$ we obtain (12).

(13) Now from (10) we can easily prove (5), since we can start with any $u_0 \in \mathscr{H}$ and perform the same procedure as before starting with the

Laplace transform, etc. In this case we obtain that B^{-1} is defined for every $u_0 \in \mathscr{H}$ and (11) is correct for every $x \in \mathscr{H}$ and not only for a dense subset of $\mathscr{H}$. Moreover, since we can make the same procedure for $B_\lambda = \lambda E - B$ for any $\lambda \in \mathbf{C}$, we can claim (11) not only for B^{-1} but for any B_λ^{-1}.

Remarks. (1) In some sense this theorem says that essentially any nilpotent semigroup behaves in the same way as the Hille–Phillips example.

(2) If we omit our condition (3) then there are other possibilities, for example, the existence of one solution with $u(t) = 0$ for $t \geq T$ does not necessarily mean that all solutions in some subspace $\mathscr{H}$ will have the same property. Moreover, there is no way to define a semigroup in this case. We will discuss it in the following sections.

(3) The same results could be obtained in the same way for the general Banach space, in this case our theorem generalizes the theorem of Esterle [3, p. 102]. Essentially, Esterle's theorem says that our conditions (2) and (3)(c) are equivalent. Our result is more precise and our proof is different from that of Esterle. We do not use here any general result about Banach algebras; in our opinion our proof is quite elementary.

(4) It is easy to see that whatever condition operator B should satisfy for our type of solution to occur, the same condition will be satisfied by operator $B_\lambda = B + \lambda E$ for any complex λ. So we can reformulate all theorems here not only for B but for B_λ.

3. REPRESENTATION OF H

Here we consider the equation

$$\frac{du}{dt} = Bu \tag{17}$$

with unbounded operator B defined on the dense subset D_1 of separable Hilbert space H. We will assume that there exists a classical solution $u(t)$ of (17) such that $u(t) \not\equiv 0$ but $u(t) = 0$ for $t \geq T$. Additionally we assume that B^n is defined on the dense subset D_n of H and moreover that

$$D = \bigcap_{n=1}^{\infty} D_n$$

is also dense in H. It, in fact, follows from the denseness of D_1—see [4].

If we have the following representation of H as the direct sum

$$H = H_1 + H_2 \tag{18}$$

with $H_1 \cap H_2 = \{0\}$ and each one of H_1, H_2 is an invariant subspace with respect to the operator B (in the sense, that $H_i \cap D = D^i$ is dense in H_i and $B\colon D^i \to H_i$), then $u(t)$ could be represented as

$$u(t) = u_1(t) + u_2(t) \tag{19}$$

with

$$u_i(t) \in H_i, \qquad i = 1,2 \; t \in \mathbb{R}_+,$$

and each one of $u_i(t)$ $i = 1,2$ will have the same property: $u_i(t) = 0$ for $t \geq T$.

Now we are going to represent H as the direct sum (18) and we shall study our problem in H_2. The case of H_1 was already treated—see Section 2.

Let's fix any complex number λ and consider now the operator $B_\lambda = B - \lambda E$ and let's introduce the subspaces

$$L(\lambda, n) = \operatorname{Ker} B_\lambda^n, \qquad \lambda \in \mathbf{C}, n = 1,2 \ldots . \tag{20}$$

Let's note that all such subspaces are closed. Now we introduce

$$H_2 = \overline{\sum L(\lambda, n)} \qquad \text{for all } \lambda \in \mathbf{C}, n = 1, \ldots, \tag{21}$$

where $\sum$ stays from the direct sum. Trivially, H_2 will be an invariant subspace with respect to the operator B (in the sense stated above).

If $H = H_2$ we stop. If $H_2 \neq H$, then there exists an element $x_1 \in D$, $x_1 \notin H_2$ and let $\mathscr{H}(x_1)$ be a minimal invariant subspace with respect to the operator B that contains x_1. Obviously, it will be the closed linear hull of the elements $\{x_1, Bx_1, \ldots\}$.

If $H = H_2 + \mathscr{H}(x_1)$ we stop. If $H_2 + \mathscr{H}(x_1) \neq H$ we can find another $x_2 \in D$, $x_2 \notin H_2 + \mathscr{H}(x_1)$ and consider $\mathscr{H}(x_2)$. Let

$$\mathscr{H}(x_1, x_2) = \mathscr{H}(x_1) + \mathscr{H}(x_2). \tag{22}$$

If $H_2 + \mathscr{H}(x_1, x_2) \neq H$ we can produce x_3 and so on. In the end we produce an increasing family of invariant (with respect to B) subspaces. By the axiom of choice we can consider a maximal such subspace, i.e., that has a trivial intersection with H_2. By the definition it will be our subspace H_1. By construction it is invariant with respect to B and moreover

$$\operatorname{Ker} p(B)|_{H_1} = \{0\} \tag{23}$$

for any polynomial function $p(t)$.

We do know already the situation with our problem in H_1. Now we are going to study it in H_2.

4. CASE OF SUBSPACE H_2

Now we consider our Eq. (17) in subspace H_2. We assume that there exists a solution $u(t) \not\equiv 0$ but $u(t) = 0$ for $t \geq T$ and $u(t) \in H_2$ for $t \geq 0$. We have the following

THEOREM 2. *Equation* (17) *has solution* $u(t)$ *in* H_2 *such that* $u(t) \not\equiv 0$, *but* $u(t) = 0$ *for* $t \geq T$ *if and only if*:

(i) *there exists a sequence of linearly independent elements* $x_n \in D$ *such that* $(x_n, x_0) = 0$ *for* $n = 1, 2 \dots$ *and* $Bx_n = x_{n-1}$, $Bx_0 = 0$ (*by linear independence we mean linear independence of any finite subsystem*). *Such a system* x_n *is called a Jordan chain*;

(ii) *every* $\lambda \in \mathbf{C}$ *is an eigenvalue for operator* B *and the corresponding eigenfunction* (*which is never zero*) *is given by* $v_\lambda = \sum_{n=0}^{\infty} \lambda^n x_n$;

(iii) $\sum_{n=0}^{\infty} (1/\|x_n\|)^{1/n} < \infty$; *or*

(iii′) $\sum_{n=0}^{\infty} (\|x_{n-1}\|/\|x_n\|) < \infty$; *or*

(iii″) *there exists a* C^∞ *function* $\alpha_0(t)$ *such that* $\alpha_0(t) \not\equiv 0$ *but* $\alpha_0(t) = 0$ *for* $t \geq T$, *series* (32) *below converges to some element in* D_1, *and after differentiation term by term it converges in* H.

Proof. H_2 is a direct sum of subspaces of the form

$$B_\lambda x_n = x_{n-1}, \qquad B_\lambda x_0 = 0. \tag{24}$$

We have three cases:

(1) System $\{x_0, \dots, x_n, \dots\}$ is finite dimensional (it means that its hull is a finite dimensional subspace) and in this case there is no solution of our type.

(2) System $\{x_0, \dots\}$ is infinite dimensional and $\lambda \neq 0$.

(3) System $\{x_0, \dots\}$ is infinite dimensional and $\lambda = 0$.

In case (2) let's introduce the operator $B_\lambda = B - \lambda E$ as operator B_0. The function

$$v(t) = u(t) e^{-\lambda t} \tag{25}$$

will be a solution of the equation

$$\frac{dv}{dt} = B_0 v \tag{26}$$

and on our subspace now we will have case (3). It means that it is enough for us to consider only the last case.

Without loss of generality we can assume that the closed linear hull of $\{x_n\}_{n=0}^{\infty} \in D$ is the infinitely dimensional subspace $\mathscr{H}$ of H_2 and that

$$(x_n, x_0) = 0, \qquad n = 1, \dots . \tag{27}$$

Now we consider only $\mathscr{H}$, and we assume that there exists a solution $u(t) \in \mathscr{H}$ with $u(t) = 0$ for $t \geq T, u(t) \not\equiv 0$ (actually it will be a projection of our old $u(t)$).

Our $u(t)$ must have the form

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) x_i \tag{28}$$

and since it is a classical solution $u(t) \in D_1$, one can differentiate the last series term by term and obtain a series that converges in H. So we can substitute it into (17) and obtain

$$\sum_{n=0}^{\infty} \alpha_n'(t) x_n = \sum_{n=1}^{\infty} \alpha_n(t) x_{n-1} = \sum_{n=0}^{\infty} \alpha_{n+1}(t) x_n . \tag{29}$$

From (27) we have

$$\alpha_1(t) = \alpha_0'(t), \tag{30}$$

and by applying B n-times we obtain

$$\alpha_n(t) = \alpha_0^{(n)}(t) = \left(\frac{d}{dt}\right)^n \alpha_0(t), \qquad n = 1, \dots . \tag{31}$$

Now

$$u(t) = \sum_{i=0}^{\infty} \left(\frac{d}{dt}\right)^i \alpha_0(t) x_i \tag{32}$$

will be a formal solution of (17). It will be a mild solution of (17) if the series in (32) converges in H; it will be a classical solution of (32) converges in D_1 and after differentiation series (32) converges in H; it will be a C^∞ solution of (17) if series (32) converges to some element in D and one could differentiate it term by term any number of times and the series will continue to converge in D.

To have it as any solution we need a convergence and the series in (32), and this can be have only if the norm $\|x_n\|$ tend to zero fast enough, as $n \to \infty$. How fast? At least, faster than $1/n!\varepsilon^n$ for any $\varepsilon > 0$, since otherwise by choosing point $t = T - \varepsilon/2$ and considering existing, by assumption, $\alpha_0(t)$, we would have a contradiction with the fact that $u(t) = 0$ for $t \geq T$.

In this case we have also that for any $\lambda \in \mathbf{C}$ the series

$$v_\lambda = \sum_{n=0}^{\infty} \lambda^n x_n \tag{33}$$

converges and v_λ is an eigenfunction of the operator B for the eigenvalue $\lambda \in \mathbf{C}$. So all $\mathbf{C}$ is in the spectrum of the operator B, since $v_\lambda \neq 0$ $((x_n, x_0) = 0, n = 1, \dots)$.

From here we can see also that there is no semigroup defined in $\mathscr{H}$ with B as its generator, at least there is no C_0 semigroup.

If we consider the following condition on the norms $\|x_n\|$

$$\|x_n\| \leq A \frac{C^n}{(n!)^{1+\varepsilon}} \tag{34}$$

then we can choose $\alpha_0(t)$ from the corresponding Gevrey class $\gamma^{1+\varepsilon/2}$ such that $\alpha_0(t) = 0$ for $t \geq T$ and

$$\left| \left(\frac{d}{dt} \right)^n \alpha_0(t) \right| \leq A' c'^n (n!)^{1+\varepsilon/2} \tag{35}$$

and the series (32) converges to the solution of (17) such that $u(t) = 0$ for $t \geq T$. The existence of $\alpha_0(t) \in \gamma^{1+\varepsilon/2}$ with our properties is a well known fact.

From our estimates from below for the norms $\|x_n\|$ one can see that there is no first order partial differential operator B that could satisfy our conditions. This could be obtained also as a simple corollary of Holmgren's theorem. This is the first big difference with our results about the corresponding situation in H_1. The second difference is the fact that in our $\mathscr{H}$ there are other solutions; for example, take $\alpha_0(t)$ as a polynomial and get a solution that belongs to a finite dimensional subspace of $\mathscr{H}$ and of course, not "abnormal." The third is the fact that in H_2 all $\lambda \in \mathbf{C}$ are eigenvalues; in H_1 we do not have any of them.

By the way, if our norms $\|x_n\|$ satisfy condition (34) one could easily produce $\alpha_0(t)$ with compact support such that series (32) converges to the solution with compact support, i.e., $u(t) \not\equiv 0$ but $u(t) = 0$ for $t \leq T_1$ or $t \geq T_2$. It means, in this case we have no uniqueness for the Cauchy problem in $\mathscr{H}$. From (32) one can obtain that in this case a correct problem should be an Eq. (17) with a given C^∞ function $\alpha_0(t)$, since $\alpha_0(t)$ defines a solution in a unique way.

By using the Denjoy–Carleman theorem about quasi-analytic classes (see [5]) we can substitute our condition for convergence of (32) for one of the following necessary and sufficient conditions:

$$\text{(i)} \quad \sum_{n=0}^{\infty} \left(\frac{1}{\|x_n\|} \right)^{1/n} < \infty, \text{ or} \tag{36}$$

$$\text{(i}'\text{)} \quad \sum_{n=0}^{\infty} \frac{\|x_{n-1}\|}{\|x_n\|} < \infty. \tag{37}$$

Each one of them is necessary and sufficient for the existence of $\alpha_0(t) \in C^\infty$ such that the series (32) converges to the solution of (17) with assumed properties.

From the proof of the last theorem we can produce related results with necessary or sufficient conditions.

THEOREM 3. *Let us assume that there exists a Jordan chain x_n with estimates* (34). *Then Eq.* (17) *has at least one solution in form* (32) *such that $u(t)$ is not trivial but $u(t) = 0$ for $t > T$ for some $T > 0$.*

THEOREM 4. *Let us assume that there exists in subspace H_2 non-trivial solution $u(t)$ such that $u(t) = 0$ for $t > T$ for some $T > 0$. Then there exists a Jordan chain x_n such that $\|x_n\|$ tends to zero faster than $1/n!\varepsilon^n$ for any $\varepsilon > 0$.*

Remark 5. In this case one can produce a lot of examples with the differential operator B, one of them is the famous Tikhonoff's example for the heat equation. Others could be obtained more or less in the same way as examples of flat solutions given in [6]. One can prove corresponding theorems (as in [6]) for the second (or higher) order differential (or partial differential) operator B. This is also a big difference with the case of H_1, where a Hille–Phillips example is the only one known and essentially the only one that exists. I hope to discuss this subject more precisely elsewhere. Another interesting result related to Theorem 1 is the following

THEOREM 4. *Under condition* (3) *the two following conditions are equivalent*:

(i) *Equation* (2) *has at least one non-trivial solution $u(t)$ such that $u(t) = 0$ for $t > T$ for some $T > 0$;*

(ii) *There exists at least one x such that condition* (11) *will be satisfied.*

Proof. For (i) → (ii) see Theorem 1. Now let us assume that (11) is satisfied for some $x = u_0$. We can produce v_n as in (9). With them we construct $u(\xi)$ as in (6). This function will satisfy estimate (5) and Eq. (4). From here we can see that its inverse Laplace transform exists, satisfies (2), and by the Paley−Wiener type theorem will be zero after finite time.

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