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Weighted Approximation by Szász–Mirakjan Operators*

DING-XUAN ZHOU

Department of Mathematics, Zhejiang University; Advanced Institute of Mathematics, Hangzhou, Zhejiang, 310027; and Institute of Mathematics, Academia Sinica, Beijing, 100080, People's Republic of China

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In this paper we consider the weighted approximation by the Szász–Mirakjan operators. We characterize the functions with nonoptimal approximation order by smoothness.

1. INTRODUCTION

The Szász–Mirakjan operators in $[0, \infty)$ are given by

$$S_n(f, x) = \sum_{k=0}^{\infty} f(k/n)\, p_{n,k}(x), \qquad p_{n,k}(x) = e^{-nx}(nx)^k/k!. \tag{1.1}$$

In 1978, M. Becker [1] proved for $m \in N \cup \{0\}$, $(1+x)^{-m} f(x) \in L_\infty[0, \infty)$ and $0 < \alpha < 2$ that

$$\begin{aligned} &(1+x)^{-m}\,|S_n(f, x) - f(x)| \leqslant M_f (x/n)^{\alpha/2} \qquad (x \geqslant 0,\, n \in N) \\ \Leftrightarrow\ &(1+x)^{-m}\,|f(x+2h) - 2f(x+h) + f(x)| \leqslant M_f' h^\alpha \qquad (h > 0,\, x \geqslant 0). \end{aligned} \tag{1.2}$$

V. Totik [7] gave a characterization theorem for these operators in 1983. He proved for $f \in C[0, \infty) \cap L_\infty[0, \infty)$ and $0 < \alpha < 2$ that

$$\begin{aligned} &\|S_n(f) - f\|_\infty = O(n^{-\alpha/2}) \\ \Leftrightarrow\ &x^{\alpha/2}\,|f(x+2h) - 2f(x+h) + f(x)| \leqslant M_f h^\alpha \qquad (x \geqslant 0,\, h > 0). \end{aligned} \tag{1.3}$$

This result was also proved by V. Totik [6], Z. Ditzian and V. Totik [4].

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In this paper we shall consider the weighted approximation by the Szász–Mirakjan operators and give a characterization theorem. We take spaces $C_{a,b}$ via the weights $w_{a,b}$ as follows.

$$w(x) := w_{a,b}(x) = x^a(1+x)^{-b}, \qquad 1 > a > 0, b > 0, \tag{1.4}$$

$$C_{a,b} = \{f \in C[0, \infty) : wf \in L_\infty[0, \infty)\}, \tag{1.5}$$

$$\|f\|_w = \|wf\|_\infty. \tag{1.6}$$

In their monograph in 1987 [4], Z. Ditzian and V. Totik gave some weighted approximation theorems for Kantorovich type integral version of the exponential-type operators. It is strange that they did not consider the same problems for the exponential-type operators. We shall point out that the Szász–Mirakjan operators are unbounded in $C_{a,b}$ with the norm (1.6). However, since the Szász–Mirakjan operators reproduce linear functions, we only need to discuss in the space $C^0_{a,b} = \{f \in C_{a,b} : f(0) = 0\}$ first and then extend to $C_{a,b}$. Let us denote $[x]$ as the integer part of $x > 0$.

2. An Unbounded Property

We show the unbounded property for the Szász–Mirakjan operators as follows.

Lemma 2.1. *For $S_n(f, x)$ given by (1.1) and $f \in C_{a,b}$, we have*

$$\left| w(x) \sum_{k=1}^{\infty} f(k/n)\, p_{n,k}(x) \right| \leqslant M_{a,b} \|f\|_w, \tag{2.1}$$

where $M_{a,b}$ is a constant depending only on a and b. This implies for $f \in C^0_{a,b}$ that

$$\|S_n(f)\|_w \leqslant M_{a,b} \|f\|_w. \tag{2.2}$$

Proof. Note that [1] for $m \in N$

$$S_n(1 + t^m, x) \leqslant M_m(1 + x^m). \tag{2.3}$$

We have

$$S_n((1+t)^m, x) \leqslant 2^m S_n(1 + t^m, x) \leqslant 2^m M_m (1+x)^m. \tag{2.4}$$

By choosing $m=[b/(1-a)]+1$ we then have

$$\left|w(x)\sum_{k=1}^{\infty} f(k/n)\,p_{n,k}(x)\right| \leqslant w(x)\left(\sum_{k=1}^{\infty}(n/k)\,p_{n,k}(x)\right)^{a} \times\left(\sum_{k=1}^{\infty}(1+k/n)^{m}\,p_{n,k}(x)\right)^{b/m}\|f\|_{w} \leqslant 2^{a}2^{b}(M_{m})^{b/m}\|f\|_{w}.$$

Our proof is then complete.

THEOREM 1. *For any $n\in N$, the Szász–Mirakjan operator $S_n(f,x)$ is unbounded in $(C_{a,b},\|\cdot\|_w)$.*

Proof. Let $f_m(x)=1/(x^a+1/m)$. Then we have $\|f_m\|_w\leqslant 1$. By Lemma 2.1 we have

$$\begin{aligned}\|S_n(f_m)\|_w &\geqslant \|w(x)\,f_m(0)\,p_{n,0}(x)\|_\infty-\left\|w(x)\sum_{k=1}^{\infty} f_m(k/n)\,p_{n,k}(x)\right\|_\infty\\ &\geqslant m\,\|w(x)\,e^{-nx}\|_\infty - M_{a,b}\\ &\to\infty \qquad (m\to\infty).\end{aligned}$$

Hence $S_n(f)$ is unbounded in $(C_{a,b},\|\cdot\|_w)$.

3. BERNSTEIN TYPE INEQUALITIES

The main tool for the proof of the inverse theorem in the nonoptimal case is an appropriate Bernstein-type inequality. Denote $\varphi(x)=x$.

LEMMA 3.1. *Let $c,d\geqslant 0$. Then we have*

$$\left|\sum_{k=1}^{\infty}(k/n)^{-c}\,(1+k/n)^{d}\,p_{n,k}(x)\right|\leqslant M_{c,d}\,x^{-c}(1+x)^{d},\qquad (x>0),\tag{3.1}$$

where $M_{c,d}$ is a constant depending only on c and d.

Proof. If $c,d>0$, then we have for $x>0$

$$\left|\sum_{k=1}^{\infty}(k/n)^{-c}\,(1+k/n)^{d}\,p_{n,k}(x)\right| \leqslant\left(\sum_{k=1}^{\infty}(k/n)^{-2c}\,p_{n,k}(x)\right)^{1/2}\left(\sum_{k=1}^{\infty}(1+k/n)^{2d}\,p_{n,k}(x)\right)^{1/2}$$

$$\leqslant \left(\sum_{k=1}^{\infty} (n/k)^{[2c]+1} p_{n,k}(x) \right)^{c/([2c]+1)} (S_n((1+t)^{[2d]+1}, x))^{d/([2d]+1)}$$
$$\leqslant (([2c]+2)!\, x^{-([2c]+1)})^{c/([2c]+1)}$$
$$\times (2^{[2d]+1} M_{[2d]+1} (1+x)^{[2d]+1})^{d/([2d]+1)}$$
$$\leqslant M_{c,d}\, x^{-c}(1+x)^d.$$

The cases of $c=0$ or $d=0$ can be easily obtained, and our proof is then complete.

LEMMA 3.2 (Bernstein-type Inequality). *Let $f \in C^0_{a,b}$, $n \in N$. Then we have*

$$\|\varphi S''_n(f)\|_w \leqslant M_1 n\, \|f\|_w, \tag{3.2}$$

where M_1 is a constant independent of f and n.

Proof. Note that for $g \in C[0, \infty)$

$$S''_n(g, x) = n^2 \sum_{k=0}^{\infty} (g((k+2)/n) - 2g((k+1)/n) + g(k/n))\, p_{n,k}(x), \tag{3.3}$$

$$S''_n(g, x) = (n/x)^2 \sum_{k=0}^{\infty} g(k/n)((k/n - x)^2 - kn^{-2})\, p_{n,k}(x). \tag{3.4}$$

Then for $x \in (0, 1/n]$ we have by Lemma 3.1 and (3.3)

$$|w(x)\, \varphi(x)\, S''_n(f, x)| \leqslant w(x)\, xn^2 \sum_{k=1}^{\infty} 4(k/n)^{-a} (1+k/n)^b\, p_{n,k}(x)\, \|f\|_w$$
$$\leqslant 4nw(x)\, M_{a,b} x^{-a}(1+x)^b\, \|f\|_w$$
$$\leqslant M_1 n\, \|f\|_w.$$

For $x > 1/n$, by (3.4) we have

$$|w(x)\, \varphi(x)\, S''_n(f, x)|$$
$$\leqslant w(x)\, n^2/x \sum_{k=1}^{\infty} (w(k/n))^{-1} ((k/n - x)^2 + kn^{-2})\, p_{n,k}(x)\, \|f\|_w$$
$$\leqslant w(x)\, n^2/x \left(\sum_{k=1}^{\infty} (k/n)^{-2a} (1+k/n)^{2b}\, p_{n,k}(x) \right)^{1/2}$$
$$\times (2S_n((t-x)^4 + t^2 n^{-2}, x))^{1/2}\, \|f\|_w$$
$$\leqslant \sqrt{M_{2a,2b}}\, n^2/x(2(xn^{-3} + 3x^2n^{-2} + x^2n^{-2} + xn^{-3}))^{1/2}\, \|f\|_w$$
$$\leqslant 4\sqrt{M_{2a,2b}}\, n\, \|f\|_w$$
$$\leqslant M_1 n\, \|f\|_w,$$

here we have used the moments of the Szász–Mirakjan operators

$$\begin{aligned} S_n((t-x)^2, x) &= x/n, \\ S_n((t-x)^4, x) &= xn^{-3} + 3x^2n^{-2}. \end{aligned} \tag{3.5}$$

Thus we have proved our Bernstein-type inequality.

To prove our direct and inverse results we need the Peetre's K-functional defined in $C^0_{a,b}$ as

$$K(f, t)_w = \inf_{g \in D} \{ \|f-g\|_w + t\, \|\varphi g''\|_w \}, \tag{3.6}$$

$$D = \{ g \in C^0_{a,b} : g' \in \text{A.C.}_{\text{loc}}, \|\varphi g''\|_w < \infty \}. \tag{3.7}$$

LEMMA 3.3. *Let $f \in D$, $n \in N$. Then we have*

$$\|\varphi S_n''(f)\|_w \leqslant M_2\, \|\varphi f''\|_w, \tag{3.8}$$

where M_2 is a constant independent of f and n.

Proof. Let $x > 0$, $n \in N$. By (3.3) and Lemma 3.1 we have

$$\begin{aligned} |w(x)\, \varphi(x)\, S_n''(f, x)| &= \left| w(x)\, xn^2 \sum_{k=0}^{\infty} \iint_0^{1/n} f''(k/n + u + v)\, du\, dv\, p_{n,k}(x) \right| \\ &\leqslant w(x)\, xn^2 \sum_{k=0}^{\infty} \iint_0^{1/n} (k/n + u + v)^{-1-a} \\ &\quad \times (1 + k/n + u + v)^b\, du\, dv\, p_{n,k}(x)\, \|\varphi f''\|_w \\ &\leqslant w(x)\, x \sum_{k=1}^{\infty} \{(1 + (k+2)/n)^b\, (k/n)^{-1-a}\, p_{n,k}(x)\} \\ &\quad \times \|\varphi f''\|_w + w(x)\, xn^2(1 + 2/n)^b \\ &\quad \times \int_0^{1/n} u^{-a}/a\, du\, p_{n,0}(x)\, \|\varphi f''\|_w \\ &\leqslant w(x)\, x \sum_{k=1}^{\infty} \{3^b(1 + k/n)^b\, (k/n)^{-1-a}\, p_{n,k}(x)\}\, \|\varphi f''\|_w \\ &\quad + w(x)\, xn^2 3^b (1/n)^{1-a}/(a(1-a))\, p_{n,0}(x)\, \|\varphi f''\|_w \\ &\leqslant w(x)\, x 3^b M_{1+a,b} x^{-1-a}(1+x)^b\, \|\varphi f''\|_w \\ &\quad + 3^b(nx)^{1+a} e^{-nx}/(a(1-a))\, \|\varphi f''\|_w \\ &\leqslant M_2\, \|\varphi f''\|_w. \end{aligned}$$

Our proof is therefore complete.

4. A Characterization Theorem

With all the above preparations we can now give our characterization theorem. Denote $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$ for $x \geqslant 0$.

THEOREM 2. *Let* $0 < a < 1$, $b > 0$, $w(x) = x^a(1+x)^{-b}$, $f \in C_{a,b}$. *Then for* $0 < \alpha < 1$, *the following statements are equivalent*:

$$(1) \quad w(x)\,|S_n(f, x) - f(x)| \leqslant M_f n^{-\alpha} \qquad (n \in N, x \geqslant 0). \tag{4.1}$$

$$(2) \quad K(f, t)_w \leqslant M_f' t^{\alpha} \qquad (t > 0). \tag{4.2}$$

$$(3) \quad \sup_{x \geqslant 0} |x^{a+\alpha}(1+x+2h)^{-b}\, \Delta_h^2 f(x)| \leqslant M_f'' h^{2\alpha} \qquad (h > 0);$$
$$|w(x)\, x^{\alpha} \Delta_x^2 f(0)| \leqslant M_f'' x^{2\alpha} \qquad (x > 0). \tag{4.3}$$

Proof. It is sufficient to prove this theorem for $f \in C_{a,b}^0$.

By the standard method for the inverse results [4, 5] we have the implication (1) $\Rightarrow$ (2) from Lemmas 2.1, 3.2, and 3.3.

Now suppose (2) holds. We want to prove (3).

Let $x > 0$. Then we have

$$\begin{aligned} &|x^a(1+x+2h)^{-b}\, \Delta_h^2 f(x)| \\ &\quad \leqslant x^a(1+x+2h)^{-b}\,(1/w(x) + 2/w(x+h) + 1/w(x+2h)) \\ &\qquad \times \|f-g\|_w + x^a(1+x+2h)^{-b} \iint_0^h |g''(x+u+v)|\, du\, dv \\ &\quad \leqslant 4\,\|f-g\|_w + x^a(1+x+2h)^{-b} \\ &\qquad \times \iint_0^h (x+u+v)^{-a-1}\,(1+x+u+v)^b\, du\, dv\, \|\varphi g''\|_w \\ &\quad \leqslant 4\{\|f-g\|_w + h^2/x\, \|\varphi g''\|_w\}. \end{aligned}$$

By taking infimum for $g \in D$ we obtain

$$x^a(1+x+2h)^{-b}\,|\Delta_h^2 f(x)| \leqslant 4K(f, h^2/x)_w \leqslant 4M_f'(h^2/x)^{\alpha}.$$

Hence the first statement of (4.3) is valid. The second statement can be proved in the same way.

We now want to prove the final implication (3) $\Rightarrow$ (1).

Introducing the Steklov type means for $h > 0$ by

$$f_h(x) = (2/h)^2 \iint_0^{h/2} (2f(x+u+v) - f(x+2u+2v))\, du\, dv, \tag{4.4}$$

one has [1]

$$f(x)-f_h(x)=(2/h)^2\iint_0^{h/2}\Delta_{u+v}^2 f(x)\,du\,dv,$$
$$f_h''(x)=h^{-2}(8\Delta_{h/2}^2 f(x)-\Delta_h^2 f(x)). \tag{4.5}$$

Suppose that (4.3) holds. For $x>0$, $n\in N$, let $h=(x/n)^{1/2}$. Note that $(t-u)\,u^{-a}$ is monotone for $u\in[t,x]$ or $[x,t]$. Then we have

$$\begin{aligned}
&w(x)\,|S_n(f_h,x)-f_h(x)|\\
&\leqslant w(x)\,S_n\left(\int_x^t (t-u)\,|f_h''(u)|\,du,x\right)\\
&\leqslant 9M_f''w(x)\,h^{-2}S_n\left(\int_x^t (t-u)\,h^{2\alpha}u^{-a-\alpha}(1+u+2h)^b\,du,x\right)\\
&\leqslant 9M_f''w(x)\,h^{2\alpha-2}\\
&\quad\times S_n((t-x)\,x^{-a}\int_x^t u^{-\alpha}\,du((1+x+2h)^b+(1+t+2h)^b),x).
\end{aligned}$$

If $x\leqslant 1/n$, then we have $h\leqslant 1$ and

$$(t-x)\int_x^t u^{-\alpha}\,du=(t-x)(t^{1-\alpha}-x^{1-\alpha})/(1-\alpha)\leqslant|t-x|^{2-\alpha}/(1-\alpha).$$

Hence

$$\begin{aligned}
w(x)\,|S_n(f_h,x)-f_h(x)|&\leqslant 9M_f''(1+x)^{-b}\,h^{2\alpha-2}/(1-\alpha)(S_n((t-x)^2,x))^{1-\alpha/2}\\
&\quad\times 2(S_n((1+x+2h)^{2b/\alpha}+(1+t+2h)^{2b/\alpha},x))^{\alpha/2}\\
&\leqslant 18M_f''(1+x)^{-b}\,(x/n)^{\alpha-1}/(1-\alpha)(x/n)^{1-\alpha/2}\\
&\quad\times((1+x+2h)^{2b/\alpha}+3^{2b/\alpha}S_n((1+t)^{2b/\alpha},x))^{\alpha/2}\\
&\leqslant 18M_f''(1+x)^{-b}/(1-\alpha)(x/n)^{\alpha/2}\\
&\quad\times 3^b((1+x)^{2b/\alpha}+M_{a,b}(1+x)^{2b/\alpha})^{\alpha/2}\\
&\leqslant M_f n^{-\alpha}.
\end{aligned}$$

Let

$$S_n^*(g,y)=\sum_{k=1}^{\infty} g(k/n)\,p_{n,k}(y). \tag{4.6}$$

If $x > 1/n$, then we have

$$w(x)\,|S_n(f_h, x) - f_h(x)| \leqslant 9M_f''h^{2\alpha-2}(1+x)^{-b} \times (S_n(((1+x+2h)^b + (1+t+2h)^b)^2, x))^{1/2} \times \left(S_n\left(\left((t-x)\int_x^t u^{-\alpha}\,du\right)^2, x\right)\right)^{1/2} \tag{4.7}$$

Note that $1 + x + 2h \leqslant 3(1+x)$ and

$$(1+t+2h)^{2b} \leqslant 2^{2b}((1+t)^{2b} + (2h)^{2b}).$$

We have

$$\begin{aligned}(S_n(((1+x+2h)^b + (1+t+2h)^b)^2, x))^{1/2} &\leqslant (2((1+x+2h)^{2b} + 2^{2b}S_n((2h)^{2b} + (1+t)^{2b}, x)))^{1/2}\\ &\leqslant M_b(1+x)^b.\end{aligned}$$

By the moments of the Szász–Mirakjan operators [4] we have

$$\begin{aligned}S_n\left(\left((t-x)\int_x^t u^{-\alpha}\,du\right)^2, x\right) &\leqslant S_n^*(((t-x)^2\,(x^{-\alpha} + t^{-\alpha}))^2, x)\\ &\quad + \left(x\int_0^x u^{-\alpha}\,du\right)^2 p_{n,0}(x)\\ &\leqslant (S_n((t-x)^8, x))^{1/2}\\ &\quad \times (16S_n^*((x^{-4\alpha} + t^{-4\alpha}), x))^{1/2}\\ &\quad + (x^{2-\alpha}/(1-\alpha))^2\,e^{-nx}\\ &\leqslant 4M_4(x/n)^2\,(x^{-4\alpha} + M_{4\alpha}x^{-4\alpha})^{1/2}\\ &\quad + 2(1-\alpha)^{-2}\,x^{2-2\alpha}(nx)^2\,e^{-nx}/2!\,n^{-2}\\ &\leqslant M_5x^{2-2\alpha}n^{-2},\end{aligned}$$

where M_4 and M_5 are constants independent of x and n.

Thus, for $x > 1/n$, we also have

$$\begin{aligned}w(x)\,|S_n(f_h, x) - f_h(x)| &\leqslant 9M_f''h^{2\alpha-2}(1+x)^{-b}\,M_b(1+x)^b\,\sqrt{M_5}\,x^{1-\alpha}n^{-1}\\ &\leqslant 9M_f''M_6\,\sqrt{M_5}\,n^{-\alpha}.\end{aligned}$$

Therefore, we have

$$\sup_{x \geqslant 0}\,\{w(x)\,|S_n(f_h, x) - f_h(x)|\} = O(n^{-\alpha}).$$

From (4.5) we have

$$w(x)\,|S_n^*(f_h-f,x)| \leqslant w(x)\,S_n^*\left((2/h)^2\iint_0^{h/2} M_f'' t^{-a-\alpha}\right.$$
$$\left.\times(1+t+2u+2v)^b\,(u+v)^{2\alpha}\,du\,dv,x\right)$$
$$\leqslant w(x)\,M_f'' h^{2\alpha} S_n^*(t^{-a-\alpha}(1+t+2h)^b,x)$$
$$\leqslant M_f'' w(x)\,h^{2\alpha}(S_n^*(t^{-2a-2\alpha},x))^{1/2}$$
$$\times(2^{2b}S_n^*((2h)^{2b}+(1+t)^{2b},x))^{1/2}$$
$$\leqslant M_f'' w(x)\,h^{2\alpha}M_{a,\alpha}x^{-a-\alpha}2^b((2h)^{2b}+M_b(1+x)^{2b})^{1/2}$$
$$\leqslant M_f'' M_{a,\alpha}2^b(4^b+M_b)^{1/2}\,n^{-\alpha}.$$

Note that $f\in C^0_{a,b}$. We also have

$$w(x)\,|S_n(f_h-f,x)| \leqslant w(x)\,|S_n^*(f_h-f,x)| + w(x)(2/h)^2$$
$$\times\iint_0^{h/2} |\Delta^2_{u+v}f(0)|\,du\,dv\,p_{n,0}(x)$$
$$\leqslant w(x)\,|S_n^*(f_h-f,x)| + w(x)(2/h)^2\,M_f''$$
$$\times\iint_0^{h/2}(u+v)^{\alpha-a}\,(1+u+v)^b\,du\,dv\,p_{n,0}(x).$$

For the second term we have

$$w(x)(2/h)^2\,M_f''\iint_0^{h/2}(u+v)^{\alpha-a}\,(1+u+v)^b\,du\,dv\,p_{n,0}(x)$$
$$\leqslant M_f'' w(x)(2/h)^{2a}\,h^\alpha(1+h)^b\left(\iint_0^{h/2}(u+v)^{-1}\,du\,dv\right)^a p_{n,0}(x)$$
$$\leqslant M_f'' 2^{2a}x^a(1+x)^{-b}\,h^{\alpha-2a}2^b(1+x)^b\,M_6^a(h/2)^{2a}\,h^{-a}\,p_{n,0}(x)$$
$$\leqslant 2^bM_f''M_6^a(nx)^{(a+\alpha)/2}\,e^{-nx}n^{-\alpha}$$
$$\leqslant 2^bM_f''M_6^a(2nxe^{-2nx/(a+\alpha)}/(a+\alpha))^{(a+\alpha)/2}\,(a+\alpha)^{(a+\alpha)/2}\,n^{-\alpha}$$
$$\leqslant 2^bM_f''M_6^a(a+\alpha)^{(a+\alpha)/2}\,n^{-\alpha},$$

here we have used the following inequality in [1]

$$\iint_0^t (x+u+v)^{-1}\,du\,dv \leqslant M_6t^2(x+2t)^{-1}\qquad(0<t\leqslant 1),\qquad(4.8)$$

and for $t\in[2^m, 2^{m+1})$, $m\in N$,

$$\iint_0^t (u+v)^{-1}\,du\,dv \leqslant \sum_{i,j=-\infty}^{m} \int_{2^i}^{2^{i+1}} \int_{2^j}^{2^{j+1}} (u+v)^{-1}\,du\,dv$$

$$\leqslant \sum_{i=-\infty}^{m} 2^i = 2^{m+1} \leqslant 4t^2/(2t),$$

hence (4.8) holds for $x=0$ and any $t>0$.

Finally, by (4.5) we have for $x>0$

$$\begin{aligned} w(x)\,|f(x)-f_h(x)| &\leqslant w(x)(2/h)^2 \iint_0^{h/2} |\Delta_{u+v}^2 f(x)|\,du\,dv \\ &\leqslant w(x)(2/h)^2 \iint_0^{h/2} M_f'' x^{-a-\alpha} \\ &\quad\times(1+x+2u+2v)^b\,(u+v)^{2\alpha}\,du\,dv \\ &\leqslant M_f''(2/h)^2\,(1+x)^{-b}\,(1+x+2h)^b\,x^{-\alpha}h^{2\alpha}(h/2)^2 \\ &\leqslant 3^b M_f'' n^{-\alpha}. \end{aligned}$$

Combining all the above discussions we obtain

$$w(x)\,|S_n(f,x)-f(x)| \leqslant M_f n^{-\alpha},$$

where M_f is a constant independent of n and x.

The proof of our main result is complete.

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