Weighted Approximation by Szász–Mirakjan Operators*

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In this paper we consider the weighted approximation by the Szász–Mirakjan operators. We characterize the functions with nonoptimal approximation order by smoothness. © 1994 Academic Press, Inc.

1. INTRODUCTION

The Szász–Mirakjan operators in \([0, \infty)\) are given by

\[
S_n(f, x) = \sum_{k=0}^{\infty} f(k/n) p_{n,k}(x), \quad p_{n,k}(x) = e^{-nx} (nx)^{k}/k!.
\]

In 1978, M. Becker [1] proved for \(m \in N \cup \{0\}\), \((1 + x)^{-m} f(x) \in L_\infty[0, \infty)\) and \(0 < \alpha < 2\) that

\[
(1 + x)^{-m} |S_n(f, x) - f(x)| \leq M_f (x/n)^{\alpha/2} \quad (x \geq 0, n \in N)
\]

\[
\iff (1 + x)^{-m} |f(x + 2h) - 2f(x + h) + f(x)| \leq M_f h^2 \quad (h > 0, x \geq 0).
\]

V. Totik [7] gave a characterization theorem for these operators in 1983. He proved for \(f \in C[0, \infty) \cap L_\infty[0, \infty)\) and \(0 < \alpha < 2\) that

\[
\|S_n(f) - f\|_{\infty} = O(n^{-\alpha/2})
\]

\[
\iff x^{\alpha/2} |f(x + 2h) - 2f(x + h) + f(x)| \leq M_f h^2 \quad (x \geq 0, h > 0).
\]

This result was also proved by V. Totik [6], Z. Ditzian and V. Totik [4].

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In this paper we shall consider the weighted approximation by the Szász–Mirakjan operators and give a characterization theorem. We take spaces $C_{a,b}$ via the weights $w_{a,b}$ as follows.

$$w(x) := w_{a,b}(x) = x^a(1+x)^{-b}, \quad 1 > a > 0, \ b > 0 , \quad (1.4)$$

$$C_{a,b} = \{ f \in C[0, \infty) : wf \in L_\infty[0, \infty) \}, \quad (1.5)$$

$$\|f\|_w = \|w f\|_\infty . \quad (1.6)$$

In their monograph in 1987 [4], Z. Ditzian and V. Totik gave some weighted approximation theorems for Kantorovich type integral version of the exponential-type operators. It is strange that they did not consider the same problems for the exponential-type operators. We shall point out that the Szász–Mirakjan operators are unbounded in $C_{a,b}$ with the norm (1.6). However, since the Szász–Mirakjan operators reproduce linear functions, we only need to discuss in the space $C_{a,b}^0 = \{ f \in C_{a,b} : f(0) = 0 \}$ first and then extend to $C_{a,b}$. Let us denote $\lfloor x \rfloor$ as the integer part of $x > 0$.

2. AN UNBOUNDED PROPERTY

We show the unbounded property for the Szász–Mirakjan operators as follows.

**Lemma 2.1.** For $S_n(f, x)$ given by (1.1) and $f \in C_{a,b}$, we have

$$\left| w(x) \sum_{k=1}^{\infty} f(k/n) p_{n,k}(x) \right| \leq M_{a,b} \|f\|_w, \quad (2.1)$$

where $M_{a,b}$ is a constant depending only on $a$ and $b$. This implies for $f \in C_{a,b}^0$ that

$$\|S_n(f)\|_w \leq M_{a,b} \|f\|_w . \quad (2.2)$$

**Proof.** Note that [1] for $m \in N$

$$S_n(1 + t^m, x) \leq M_m(1 + x^m). \quad (2.3)$$

We have

$$S_n((1 + t)^m, x) \leq 2^m S_n(1 + t^m, x) \leq 2^m M_m(1 + x)^m. \quad (2.4)$$
By choosing \( m = [b/(1 - a)] + 1 \) we then have
\[
\left| w(x) \sum_{k=1}^{\infty} f(k/n) p_{n,k}(x) \right| \leq w(x) \left( \sum_{k=1}^{\infty} (n/k) p_{n,k}(x) \right)^a \\
\times \left( \sum_{k=1}^{\infty} (1 + k/n)^m p_{n,k}(x) \right)^{b/m} \| f \|_w \\
\leq 2^a 2^b (M_m)^{b/m} \| f \|_w.
\]

Our proof is then complete.

**Theorem 1.** For any \( n \in \mathbb{N} \), the Szász–Mirakjan operator \( S_n(f, x) \) is unbounded in \((C_{a,b}, \| \cdot \|_w)\).

**Proof.** Let \( f_m(x) = 1/(x^a + 1/m) \). Then we have \( \| f_m \|_w \leq 1 \). By Lemma 2.1 we have
\[
\| S_n(f_m) \|_w \geq \| w(x) f_m(0) p_{n,0}(x) \|_\infty - \left\| w(x) \sum_{k=1}^{\infty} f_m(k/n) p_{n,k}(x) \right\|_\infty \\
\geq m \| w(x) e^{-anx} \|_\infty - M_{a,b} \\
\rightarrow \infty \quad (m \rightarrow \infty).
\]

Hence \( S_n(f) \) is unbounded in \((C_{a,b}, \| \cdot \|_w)\).

### 3. Bernstein Type Inequalities

The main tool for the proof of the inverse theorem in the nonoptimal case is an appropriate Bernstein-type inequality. Denote \( \varphi(x) = x \).

**Lemma 3.1.** Let \( c, d \geq 0 \). Then we have
\[
\left| \sum_{k=1}^{\infty} (k/n)^{-c} (1 + k/n)^d p_{n,k}(x) \right| \leq M_{c,d} x^{-c}(1 + x)^d, \quad (x > 0), \tag{3.1}
\]
where \( M_{c,d} \) is a constant depending only on \( c \) and \( d \).

**Proof.** If \( c, d > 0 \), then we have for \( x > 0 \)
\[
\left| \sum_{k=1}^{\infty} (k/n)^{-c} (1 + k/n)^d p_{n,k}(x) \right| \\
\leq \left( \sum_{k=1}^{\infty} (k/n)^{-2c} p_{n,k}(x) \right)^{1/2} \left( \sum_{k=1}^{\infty} (1 + k/n)^{2d} p_{n,k}(x) \right)^{1/2}
\]
\[ \left( \sum_{k=1}^{\infty} \frac{(n/k)^{[2c]+1}}{p_{n,k}(x)} \right)^{c/[2c]+1} \left( S_n((1+t)^{[2d]+1}, x)\right)^{d/[2d]+1} \]

\[ \leq (([2c]+2)! x^{-([2c]+1)})^{c/[2c]+1} \times (2^{[2d]+1} M_{[2d]+1} (1+x)^{[2d]+1})^{d/[2d]+1} \]

\[ \leq M_{c,d} x^{-c}(1+x)^d. \]

The cases of \( c = 0 \) or \( d = 0 \) can be easily obtained, and our proof is then complete.

**Lemma 3.2 (Bernstein-type Inequality).** Let \( f \in C^{0}_{a,b}, \ n \in \mathbb{N}. \) Then we have

\[ \| \phi S_n^\,(f) \|_{w} \leq M_1 n \| f \|_{w}, \tag{3.2} \]

where \( M_1 \) is a constant independent of \( f \) and \( n. \)

**Proof.** Note that for \( g \in C[0, \infty) \)

\[ S_n^\,(g, x) = n^2 \sum_{k=0}^{\infty} \left( g((k+2)/n) - 2g((k+1)/n) + g(k/n) \right) p_{n,k}(x), \tag{3.3} \]

\[ S_n^\,(g, x) = (n/x)^2 \sum_{k=0}^{\infty} g(k/n) \left( (k/n - x)^2 - kn^{-2} \right) p_{n,k}(x). \tag{3.4} \]

Then for \( x \in (0, 1/n] \) we have by Lemma 3.1 and (3.3)

\[ |w(x) \phi(x) S_n^\,(f, x)| \leq w(x) x n^2 \sum_{k=1}^{\infty} 4(k/n)^{-a} (1+k/n)^b p_{n,k}(x) \| f \|_{w} \]

\[ \leq 4nw(x) M_{a,b} x^{-a}(1+x)^b \| f \|_{w} \]

\[ \leq M_1 n \| f \|_{w}. \]

For \( x > 1/n, \) by (3.4) we have

\[ |w(x) \phi(x) S_n^\,(f, x)| \]

\[ \leq w(x) n^2/x \sum_{k=1}^{\infty} (w(k/n))^{-1} \left( (k/n - x)^2 + kn^{-2} \right) p_{n,k}(x) \| f \|_{w} \]

\[ \leq w(x) n^2/x \left( \sum_{k=1}^{\infty} (k/n)^{-2a} (1+k/n)^b p_{n,k}(x) \right)^{1/2} \]

\[ \times \left( 2S_n((1-x)^4 + t^2n^{-2}, x) \right)^{1/2} \| f \|_{w} \]

\[ \leq \sqrt{M_{2a,2b}} n^2/x (2(xn^{-3} + 3xn^{-2} + x^2n^{-2} + xn^{-3}))^{1/2} \| f \|_{w} \]

\[ \leq 4 \sqrt{M_{2a,2b}} n \| f \|_{w} \]

\[ \leq M_1 n \| f \|_{w}. \]
here we have used the moments of the Szász–Mirakjan operators

\[ S_n((t-x)^2, x) = x/n, \]

\[ S_n((t-x)^4, x) = xn^{-3} + 3x^2n^{-2}. \]  

Thus we have proved our Bernstein-type inequality.

To prove our direct and inverse results we need the Peetre's \( K \)-functional defined in \( C_{a,b}^0 \) as

\[ K(f, t)_w = \inf_{g \in D} \{ \| f - g \|_w + t \| \varphi g'' \|_w \}, \]  

\[ D = \{ g \in C_{a,b}^0 : g' \in A.C., \| \varphi g'' \|_w < \infty \}. \]

**Lemma 3.3.** Let \( f \in D, n \in N \). Then we have

\[ \| \varphi S_n''(f) \|_w \leq M_2 \| \varphi f'' \|_w, \]

where \( M_2 \) is a constant independent of \( f \) and \( n \).

**Proof.** Let \( x > 0, n \in N \). By (3.3) and Lemma 3.1 we have

\[ |w(x) \varphi(x) S''_n(f, x)| = |w(x) xn^2 \sum_{k=0}^{\infty} \int_{0}^{1/n} \int_{0}^{1/n} f''(k/n + u + v) dv \, dp_n, k(x)| \]

\[ \leq w(x) xn^2 \sum_{k=0}^{\infty} \int_{0}^{1/n} (k/n + u + v)^{-a} \]

\[ \times (1 + k/n + u + v)^b \, du \, dp_n, k(x) \| \varphi f'' \|_w \]

\[ \leq w(x) \sum_{k=1}^{\infty} \{(1 + (k + 2)/n)^b (k/n)^{-a} p_n, k(x)\} \]

\[ \times \| \varphi f'' \|_w + w(x) xn^2(1 + 2/n)^b \]

\[ \times \int_{0}^{1/n} u^{-a/b} du \, dp_n, 0(x) \| \varphi f'' \|_w \]

\[ \leq w(x) \sum_{k=1}^{\infty} \left\{ 3^b (1 + k/n)^b (k/n)^{-1-a} p_n, k(x) \right\} \| \varphi f'' \|_w \]

\[ + w(x) xn^2 3^b (1/n)^{1-a} / (a(1-a)) p_n, 0(x) \| \varphi f'' \|_w \]

\[ \leq w(x) 3^b M_1 + a \cdot x^{-a} (1 + x)^b \| \varphi f'' \|_w \]

\[ + 3^b (n) x^{-a} e^{-nx/(a(1-a))} \| \varphi f'' \|_w \]

\[ \leq M_2 \| \varphi f'' \|_w. \]

Our proof is therefore complete.
4. A Characterization Theorem

With all the above preparations we can now give our characterization theorem. Denote \( \Delta^2_h f(x) = f(x + 2h) - 2f(x + h) + f(x) \) for \( x \geq 0 \).

**Theorem 2.** Let \( 0 < a < 1, b > 0, w(x) = x^a(1 + x)^{-b}, f \in C_{a,b} \). Then for \( 0 < a < 1 \), the following statements are equivalent:

1. \( w(x) |S_n(f, x) - f(x)| \leq M_f n^{-a} \) \( (n \in N, x \geq 0) \). (4.1)
2. \( K(f, t)_w \leq M'_f t^a \) \( (t > 0) \). (4.2)
3. \[ \sup_{x \geq 0} |x^a + x(1 + x + 2h)^{-b} \Delta^2_h f(x)| \leq M''_f h^{2a} \text{ (} h > 0 \text{);} \] (4.3)
   \[ |w(x) x^a \Delta^2_x f(0)| \leq M''_f x^{2a} \text{ (} x > 0 \text{).} \]

**Proof.** It is sufficient to prove this theorem for \( f \in C_{a,b}^0 \).

By the standard method for the inverse results \([4, 5]\) we have the implication (1) \( \Rightarrow \) (2) from Lemmas 2.1, 3.2, and 3.3.

Now suppose (2) holds. We want to prove (3).
Let \( x > 0 \). Then we have

\[
| x^a(1 + x + 2h)^{-b} \Delta^2_h f(x) | \\
\leq x^a(1 + x + 2h)^{-b} (1/w(x) + 2/w(x + h) + 1/w(x + 2h)) \\
\times \| f - g \|_w + x^a(1 + x + 2h)^{-b} \int_0^h | g''(x + u + v) | \, du \, dv \\
\leq 4 \| f - g \|_w + x^a(1 + x + 2h)^{-b} \\
\times \left\{ \int_0^h (x + u + v)^{-a-1} (1 + x + u + v)^b \, du \, dv \, \| \phi g'' \|_w \right\}.
\]

By taking infimum for \( g \in D \) we obtain

\[
x^a(1 + x + 2h)^{-b} | \Delta^2_h f(x) | \leq 4K(f, h^2/x)_w \leq 4M'_f (h^2/x)^a.
\]

Hence the first statement of (4.3) is valid. The second statement can be proved in the same way.

We now want to prove the final implication (3) \( \Rightarrow \) (1).

Introducing the Steklov type means for \( h > 0 \) by

\[
f_a(x) = (2/h)^2 \int_0^{h/2} (2f(x + u + v) - f(x + 2u + 2v)) \, du \, dv, \quad (4.4)
\]
one has [1]

\[ f(x) - f_h(x) = (2/h)^2 \int_0^{h/2} A_{h/4}^2 f(x) \, du \, dv, \]

\[ f_h''(x) = h^{-2}(8A_{h/2}^2 f(x) - A_h^2 f(x)). \]  (4.5)

Suppose that (4.3) holds. For \( x > 0, \ n \in \mathbb{N}, \) let \( h = (x/n)^{1/2}. \) Note that \((t - u)u^{-a} \) is monotone for \( u \in [t, x] \) or \([x, t].\) Then we have

\[ w(x) |S_n(f_h, x) - f_h(x)| \]

\[ \leq w(x) S_n \left( \int_x^t (t - u) |f_h''(u)| \, du, x \right) \]

\[ \leq 9M_f''w(x) h^{-2} S_n \left( \int_x^t (t - u)h^{2a}u^{-a-x}(1 + u + 2h)^b \, du, x \right) \]

\[ \leq 9M_f''w(x) h^{2a-2} \]

\[ \times S_n((t - x)x^{-a} \int_x^t u^{-x}(1 + x + 2h)^b + (1 + t + 2h)^b), x). \]

If \( x \leq 1/n, \) then we have \( h \leq 1 \) and

\[ (t - x) \int_x^t u^{-x} \, du = (t - x)(t^{1 - a} - x^{1 - a})/(1 - a) \leq (t - x)^{2 - a/(1 - a)}. \]

Hence

\[ w(x) |S_n(f_h, x) - f_h(x)| \leq 9M_f''(1 + x)^{-b} h^{2a - 2}/(1 - a)(S_n((t - x)^2, x))^{1 - a/2} \]

\[ \times 2(S_n((1 + x + 2h)^{2b/a} + (1 + t + 2h)^{2b/a}), x))^{a/2} \]

\[ \leq 18M_f''(1 + x)^{-b} (x/n)^{a - 1/(1 - a)}(x/n)^{1 - a/2} \]

\[ \times ((1 + x + 2h)^{2b/a} + 3^{2b/a}S_n((1 + t)^{2b/a}, x))^{a/2} \]

\[ \leq 18M_f''(1 + x)^{-b}/(1 - a)(x/n)^{a/2} \]

\[ \times 3^b((1 + x)^{2b/a} + M_{a, b}(1 + x)^{2b/a})^{a/2} \]

\[ \leq M_{f, n}^{-a}. \]

Let

\[ S_n^*(g, y) = \sum_{k=1}^{\infty} g(k/n) p_{n, k}(y). \]  (4.6)
If \( x > 1/n \), then we have

\[
  w(x) |S_n(f_h, x) - f_h(x)| \leq 9M_n^* h^{2x-2}(1+x)^{-b} \\
  \times (S_n(((1 + x + 2h)^b + (1 + t + 2h)^b)^2, x))^{1/2} \\
  \times \left(S_n\left(\left(\int_x^t u^{-z} \, du\right)^2, x\right)\right)^{1/2} \\
  (4.7)
\]

Note that \( 1 + x + 2h \leq 3(1 + x) \) and

\[
  (1 + t + 2h)^{2b} \leq 2^{2b}((1 + t)^{2b} + (2h)^{2b}).
\]

We have

\[
  (S_n(((1 + x + 2h)^b + (1 + t + 2h)^b)^2, x))^{1/2} \\
  \leq (2((1 + x + 2h)^{2b} + 2^{2b}S_n((2h)^{2b} + (1 + t)^{2b}, x)))^{1/2} \\
  \leq M_b(1 + x)^b.
\]

By the moments of the Szász–Mirakjan operators \([4]\) we have

\[
  S_n\left(\left(\int_x^t u^{-z} \, du\right)^2, x\right) \leq S_n^*\left(((t-x)^2(x^{-z} + t^{-z}))^2, x\right) \\
  + \left(\int_0^x u^{-z} \, du\right)^2 p_{n,0}(x) \\
  \leq (S_n((t-x)^8, x))^{1/2} \\
  \times (16S_n^*((x^{-4z} + t^{-4z}), x))^{1/2} \\
  + (x^{2-3/(1-\alpha)})^2 e^{-nx} \\
  \leq 4M_d(x/n)^2(x^{-4z} + M_{4z} x^{-4z})^{1/2} \\
  + 2(1-\alpha)^{-2} x^{2-2z}(nx)^2 e^{-nx}/2! n^{-2} \\
  \leq M_5 x^{2-2z} n^{-2},
\]

where \( M_d \) and \( M_5 \) are constants independent of \( x \) and \( n \).

Thus, for \( x > 1/n \), we also have

\[
  w(x) |S_n(f_h, x) - f_h(x)| \leq 9M_n^* h^{2x-2}(1+x)^{-b} M_b(1+x)^b \sqrt{M_5} x^{1-z} n^{-1} \\
  \leq 9M_n^* M_b \sqrt{M_5} n^{-2}.
\]

Therefore, we have

\[
  \sup_{x \geq 0} \{ w(x) |S_n(f_h, x) - f_h(x)| \} = O(n^{-2}).
\]
From (4.5) we have
\[
w(x) |S_n^*(f_h - f, x)| \leq w(x) S_n^*(2/h)^2 \int_0^{h/2} M_j'' t^{-a-\alpha} \times (1 + t + 2u + 2v)^b (u + v)^{2\alpha} \, du \, dv, x
\]
\[
\leq w(x) M_j'' \, h^{2\alpha} S_n^*(t^{-a-\alpha}(1 + t + 2h)^b, x)
\]
\[
\leq M_j'' w(x) \, h^{2\alpha}(S_n^*(t^{-2a-2\alpha}, x))^{1/2} \times (2^{b} S_n^*((2h)^{2b} + (1 + t)^{2b}, x))^{1/2}
\]
\[
\leq M_j'' w(x) \, h^{2\alpha} M_{a,\alpha} x^{-a-\alpha-2bences} ((2h)^{2b} + M_b (1 + x)^{2b})^{1/2}
\]
\[
\leq M_j'' M_{a,\alpha} 2^{b}(4^{b} + M_b)^{1/2} n^{-\alpha}.
\]

Note that \( f \in C_{a, b}^0 \). We also have
\[
w(x) |S_n(f_h - f, x)| \leq w(x) |S_n^*(f_h - f, x)| + w(x)(2/h)^2 \times \int_0^{h/2} |A_{u+v} f(0)| \, du \, dv \, p_{n,0}(x)
\]
\[
\leq w(x) |S_n^*(f_h - f, x)| + w(x)(2/h)^2 M_j'' \times \int_0^{h/2} (u + v)^{2\alpha} (1 + u + v)^b \, du \, dv \, p_{n,0}(x).
\]

For the second term we have
\[
w(x)(2/h)^2 M_j'' \int_0^{h/2} (u + v)^{2\alpha} (1 + u + v)^b \, du \, dv \, p_{n,0}(x)
\]
\[
\leq M_j'' w(x)(2/h)^2 h^2 (1 + h)^b \left( \int_0^{h/2} (u + v)^{-1} \, du \, dv \right)^{a} p_{n,0}(x)
\]
\[
\leq M_j'' 2^{2a} x^a (1 + x)^{-b} h^{a-2a-2b} (1 + x)^b M_\delta^a(h/2)^{2a} h^{-a} p_{n,0}(x)
\]
\[
\leq 2^b M_j'' M_\delta^a (2 h) x^{a+b} (a+b) / (a+b+\alpha)^{(a+\alpha)/2} (a+\alpha)^{(a+\alpha)/2} n^{-\alpha}
\]
\[
\leq 2^b M_j'' M_\delta^a (a+\alpha)^{(a+\alpha)/2} n^{-\alpha},
\]

here we have used the following inequality in [1]
\[
\int_0^t (x + u + v)^{-1} \, du \, dv \leq M_\delta t^2 (x + 2t)^{-1} \quad (0 < t \leq 1), \quad (4.8)
\]
and for \( t \in [2^m, 2^{m+1}) \), \( m \in \mathbb{N} \),
\[
\int_0^1 \frac{(u+v)^{-1}}{u+v} \, du \, dv \leq \sum_{i,j = -\infty}^m \int_{2^i}^{2^{i+1}} \int_{2^j}^{2^{j+1}} (u+v)^{-1} \, du \, dv \\
\leq \sum_{i = -\infty}^m 2^i = 2^{m+1} \leq 4t^2/(2t),
\]
hence (4.8) holds for \( x = 0 \) and any \( t > 0 \).

Finally, by (4.5) we have for \( x > 0 \)
\[
w(x) |f(x) - f_h(x)| \leq w(x)(2/h)^2 \int_0^{h/2} |A_{u+e}^2 f(x)| \, du \, dv \\
\leq w(x)(2/h)^2 \int_0^{h/2} M_f x^{-a-2} \\
\times (1 + x + 2u + 2v)^b (u+v)^{2z} \, du \, dv \\
\leq M_f (2/h)^2 (1 + x)^{-b} (1 + x + 2h)^b x^{-2} h^{2z}(h/2)^2 \\
\leq 3^b M_f n^{-z}.
\]
Combining all the above discussions we obtain
\[
w(x) |S_n(f, x) - f(x)| \leq M_f n^{-z},
\]
where \( M_f \) is a constant independent of \( n \) and \( x \).
The proof of our main result is complete.

References