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Stochastic Modelling and Optimisation of Internet Auction Processes

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Abstract

Internet auctions are an attractive mechanism for the exchange of goods at a non-fixed price point. The operation of these auctions can be run under a variety of parameters. In this paper, we provide a theoretical analysis of fixed time forward auctions in cases where a single bid or multiple bids are accepted in a single auction. A comparison of the economic benefits and the corresponding buyer and seller surpluses between the auctions where a single bid is accepted and the auctions where multiple bids are accepted is made. These models are verified through systematic simulation experiments, based on a series of operational assumptions, which characterise the arrival rate of bids, as well as the distribution from which the private values of buyers are sampled. Decision rules for optimising surplus under different auction fee structures are also given.

Keywords: Online Auctions, Internet Auctions, Auction Income, Auction Duration, Multiple Bids.

1 Introduction and Related Work

The prevalence of the Internet has ushered in the friction-less dissemination of data, though, in more recent times, this free and centralised information is increasingly being monetised and distributed. The use of Internet auctions has become the prominent method of exchanging goods between consumers due to the primary advantage that it affords: the selling of goods of unknown value for which a variable price-point model is advantageous. In this paper, we i) develop mathematical models and characterise the properties of different algorithms that may be found in or may be built into Internet auction mechanisms, basing these models on a series of operational assumptions including the arrival rate of bids as well as the distribution from which the private values of buyers are sampled, and ii) construct and run simulation experiments in the context of the mathematical models, checking the validity of the mathematical models and using the same assumptions.

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Although this paper addresses the fixed time forward auction, which is the type of auction used by eBay, as well as auctions where two or more bids being accepted in a single auction, this is only a subset of the auctions available to buyers and sellers on the Internet. The sheer size and volume of the buyer and seller markets means that there is ample supply and demand for the markets to be efficient. Other auctions include timeshift auctions [1] where there is a initial bidding period where all buyers can participate in, and a second exclusive period where buyers can only participate in if they have submitted a bid previously. This removes the advantage caused by sniping, which see bidders submit their bids moments before the close of an auction preventing other bidders from submitting counter-bids [2]. Another type of auction is the penny auction, e.g. www.swoopo.co.uk, where bids are bought and placed on items. There is a small window of time (usually 5 seconds) after each bid, which will see the latest bid win the product at a nominal price if no-one places a bid during that window. If there is activity, however, the timer is reset.

While many eBay users believe sniping to be problematic, eBay has always maintained the policy that a bidder should bid his private value. Since the winner pays the second price, there is little reason for a bidder to shade his bid. In order to counteract sniping, other online auction websites, such as Amazon, have employed auctions with a soft close, automatically extending the length of the auction. The investigation of different types of auction terminations has been undertaken in [3], where it is found that late bidders of eBay type of hard close tend to be associated with highly experienced bidders, whereas those of Amazon type tend to be relatively inexperienced bidders. In [4], it is found that sniping often leads to winning, and it observes that many sellers tend to set the starting bid price unrealistically low to stimulate bidder participation.

The *Independent Private Values* model is often associated with auctions [5]. The characteristics of this model include the assumptions of privacy and independence where the value of the commodity in question is private to the individual buyers, and that different buyers do not know the values other buyers attached to the commodity. In addition, these values are drawn from a common distribution which is known to the buyers. In probabilistic terms, this essentially amounts to a series of values which are independent and identically distributed. Experimental studies of Internet auction behaviour have been undertaken in [6,7,8]. In [8], it concentrates on the Dutch auction and first-price sealed bid auction formats, using laboratory experiments and human subjects, where values are drawn from the uniform distribution between 0 and 100, focusing primarily on the effect of clock speed on sellers revenue. In our subsequently analysis, we shall follow the independent private values model using the uniform distribution. Here we assume the bids $\{Q_k\}$ to be ascending ordered values taken from the uniform distribution over the interval $(0, L)$, and that, as in [9], we assume that the bids arrive over time in a Poisson manner with rate λ.

Price variation characteristics and consumer surplus are studied in [10,11]. The work is extended in this paper, particularly using the seller surplus to calculate the appropriate trade-off between time and money for auctions that accept two or more bids in a single auction. The field of Internet auctions is broad and can range from statistical analysis involving the use of various types of curves for fitting price data [12] to empirical investigations [13] where eBay auctions of coins are conducted. It makes use of regression models to estimate the price of items and examines the influence of seller ratings (which measures the reliability and services provided by the seller) on the final price. It has also found that the effect of positive and negative ratings is not symmetrical, with the latter having a much greater (adverse) influence on the price. This will be addressed in future work. It also suggests that longer auctions tend to have a beneficial effect in achieving a higher price.

In this paper, we describe the economic benefits to buyers and sellers when there are multiple identical lots available for sale and when multiple bids are accepted in an auction. In Section 2, a stochastic framework for analysing Internet auctions is described, which is then extended to include multiple accepted bids from an auction. Section 3 introduces the concept of allocative efficiency, using the buyer and seller surpluses to compare the performance of a multiple accepted bid auction with that of a singularly accepted bid auction. Finally, these theoretical findings are verified by way of simulation in Section 4, and the conclusions presented in Section 5.

2 Basic Stochastic Model of Internet Auctions

If there are N bids, we denote these ordered values by $Q_{(1)} < Q_{(2)} < \ldots < Q_{(N)}$. A forward auction is an electronic auction where buyers compete for items or services, with the price going up over time, and the items or services for sale are displayed and specified in a particular website (e.g. uBid.com). In the present model, we assume that the auction time is fixed with duration T. Let N be the number of bids received, and the largest bid $Q_{(N)}$ received over the time interval $(0, T)$ is accepted. A high value for T will produce a larger average accepted bid but the auction duration will be longer. For practical meaningful auction operation, T should be significantly greater than the mean bid inter-arrival time $1/\lambda$, i.e. $T \gg 1/\lambda$. At the close of the auction, the auctioning mechanism will select the maximum bid $Q_{(N)}$ to be accepted. The exact algorithm is shown in Figure 1.

From the results of order statistics, it can be shown [14] that the conditional income per auction is

$$E[Q_{(N)}|\text{Number of bids} = N] = \frac{NL}{N+1}. \tag{1}$$

Since

$$\frac{dE[Q_{(N)}|\text{Number of bids} = N]}{dN} = \frac{L}{(N+1)^2} > 0, \tag{2}$$

we see that, as the number of bids N increases, the corresponding average income per auction will also increase. To determine the average income $E[Q_{(N)}]$, we remove the condition on N in Equation 1 using the Poisson probabilities; i.e.

```
begin
L = 0;
accept_id = null;
while clock < T do
    begin
    for an arriving bid of magnitude R,
    if L < R, then do
        begin
        L = R;
        accept_id = bidder_id;
        end;
    end;
return bid L offered by accept_id;
end;
```

Fig. 1. Fixed time forward auction.

$$
\begin{aligned}
&E[Q_{(N)}] \\
&= \sum_{N=1}^{\infty} \frac{NL}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= \sum_{N=1}^{\infty} \left[L - \frac{L}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= \sum_{N=1}^{\infty} L \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \sum_{N=1}^{\infty} \frac{L}{N+1} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= L\left(1 - e^{-\lambda T}\right) - \frac{Le^{-\lambda T}}{\lambda T} \sum_{N=1}^{\infty} \frac{(\lambda T)^{N+1}}{(N+1)!} \\
&= L\left(1 - e^{-\lambda T}\right) - \frac{Le^{-\lambda T}}{\lambda T}\left(e^{\lambda T} - 1 - \lambda T\right).
\end{aligned} \tag{3}
$$

We omit the term $N = 0$ in the above, since when $N = 0$, the income will be zero. This gives an average income per auction of

$$E[Q_{(N)}] = \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right), \tag{4}$$

and an income rate, i.e. income per unit time, of

$$\frac{L}{\lambda T^2}\left(\lambda T + e^{-\lambda T} - 1\right). \tag{5}$$

Figure 2 shows $E[Q_{(N)}]$ for different values of λ for $L = 100$, and $T = 10$. We see that the increase in bid rate up to $\lambda = 4$ produces rather steep average auction income improvement. There seems to be a critical bid rate at around $\lambda = 6$, above which the improvement in income becomes less pronounced.

Figure 3 shows the rate of income for $\lambda = 1$ and $L = 100$ for different values of the auction duration T. We see that the income rate favours short duration auctions, which tends to stabilise for $T > 10$, beyond which varying the auction

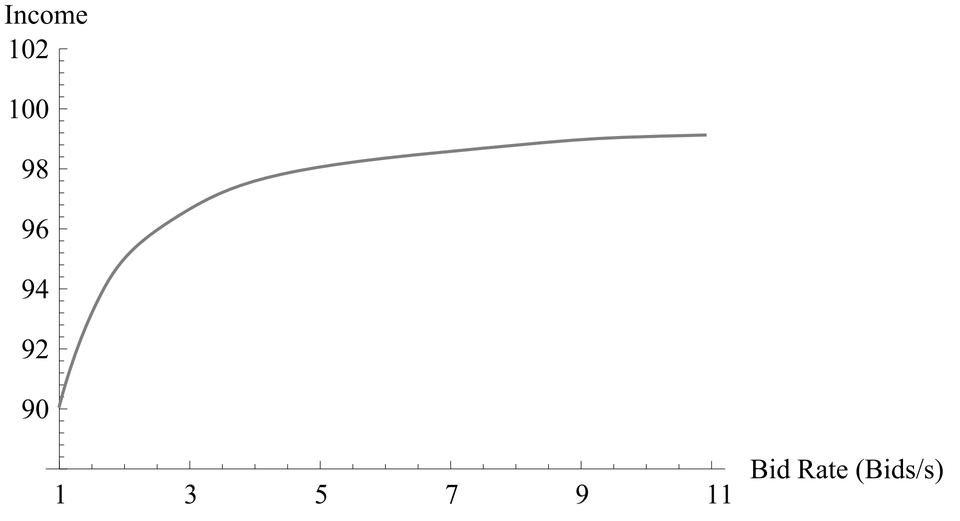

Fig. 2. Auction income as a function of the bid rate.

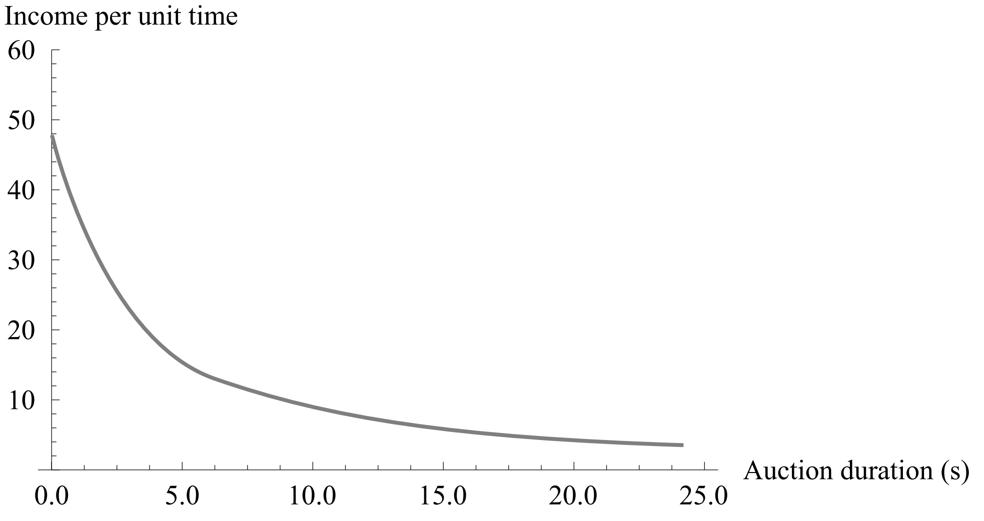

Fig. 3. Auction income per unit time as a function of the auction duration.

duration will not bring about any significant change in the income rate. Figure 4 shows the expected transaction price for $\lambda = 1$, $L = 100$ for different values of the auction duration T. We see that it follows much the same shape as Figure 2.

Indeed, since both λ and T occur together in Equation 4, this is what we would expect. Letting $z = \lambda T$, we have

$$E[Q_N] = \frac{L}{z}\left(z + e^{-z} - 1\right), \tag{6}$$

Differentiating with respect to z, we have

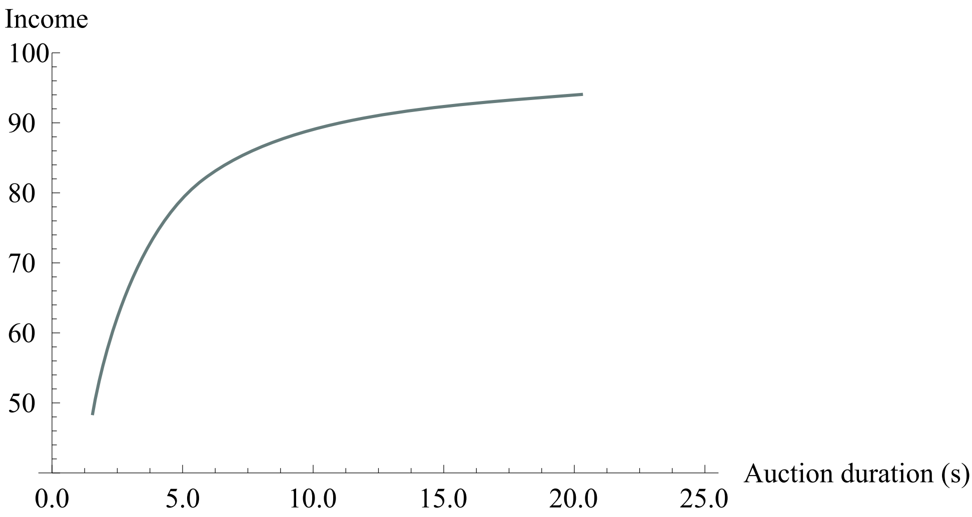

Fig. 4. Auction income as a function of the auction duration.

$$\begin{aligned}\frac{dE\left[Q_{(N)}\right]}{dz} &= \frac{L}{z^2}\left(1 - e^{-z} - ze^{-z}\right) \\ &= \frac{L}{z^2}\left[1 - \frac{1+z}{e^z}\right].\end{aligned}$$

Since $z > 0$, and

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots,$$

we have $(1+z) < e^z$, so that $(1+z)/e^z < 1$, and thus

$$\frac{dE\left[Q_{(N)}\right]}{dz} = \frac{L}{z^2}\left[1 - \frac{1+z}{e^z}\right] > 0. \tag{7}$$

Consequently, we have

$$\begin{aligned}\frac{dE\left[Q_{(N)}\right]}{d\lambda} &= \frac{dE\left[Q_{(N)}\right]}{dz} \times \frac{dz}{d\lambda} \\ &= T \times \frac{dE\left[Q_{(N)}\right]}{dz} > 0.\end{aligned}$$

Likewise, we have

$$\frac{dE\left[Q_{(N)}\right]}{dT} = \lambda \times \frac{dE\left[Q_{(N)}\right]}{dz} > 0. \tag{8}$$

Thus, the transaction price can be increased by either increasing the bid rate or the auction duration. As we shall see later, since the auction duration is controlled by the seller, he/she can use the auction duration as a mechanism for raising his/her surplus.

It is interesting to compare the auction income using an approximation based on Equation 1. If we remove the condition on N by simply replacing N by its average (from the Poisson distribution) of $z = \lambda T$, we have approximately

$$E\left[Q_{(N)}\right] = \frac{Lz}{z+1}. \tag{9}$$

Figure 5 compares the average auction income from Equations 6 and 9 for $L = 100$ for different values of $z = \lambda T$, and we note that for $\lambda = 1$, we have $z = T$. We see that the approximation is quite good for moderate to large values of z. For very large values of $z \gg 1$, the exact formula and the approximation are virtually indistinguishable, and we shall be making use of this approximation to obtain closed-form solutions in the surplus analysis below.

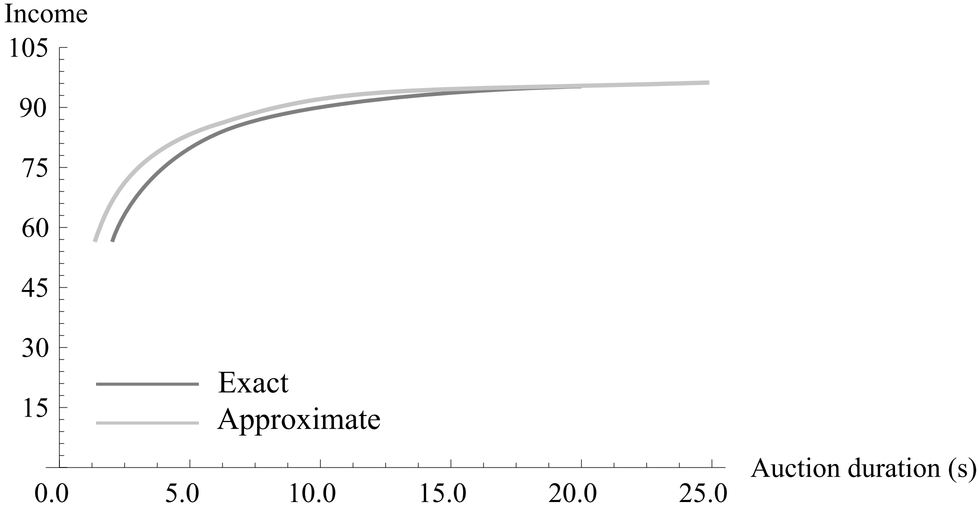

Fig. 5. Auction income as a function of the auction duration with a comparison of exact analysis (Equation 6) and approximation (Equation 9) solutions.

3 Surplus Analysis

The concept of allocative efficiency (or sometimes called operational efficiency) is often employed to evaluate how Internet auctions perform [7]. The seller surplus is the difference between the transaction price and the seller's costs (sometimes generically called production cost), while the difference between the buyer's value and the transaction price gives the buyer surplus or consumer surplus (see Figure 6). The total surplus is the seller surplus plus the buyer surplus, and the allocative efficiency is given by the total actual realised surplus expressed as a fraction of the total possible surplus [7]. For simplicity, these quantities are represented in Figure 6 as linear functions, but the ideas remain the same if one or more of these are non-linear.

From the buyers' point of view, their valuation of the auction item is indicated by the maximum price L that they are willing to pay. Thus the buyer surplus β is given by the difference in buyer valuation and the transaction price, which is

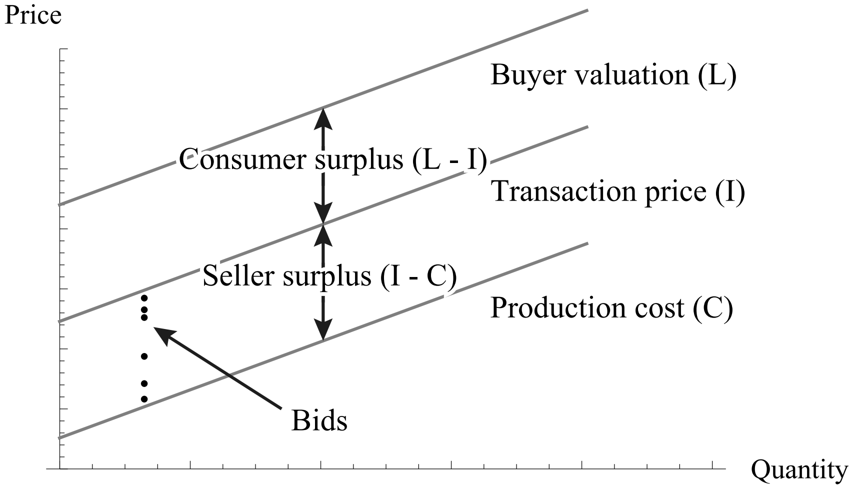

Fig. 6. Measures of auction surpluses.

$$\begin{aligned}\beta &= L - \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right)\\ &= \frac{L}{\lambda T}\left(1 - e^{-\lambda T}\right)\\ &= \frac{L}{z}\left(1 - e^{-z}\right) > 0.\end{aligned}$$

Since $z = \lambda T$ is the average number of bids, the above can be interpreted as follows: the consumer surplus is the private value evenly divided by the number of bids times the probability of having a non-empty auction (i.e. an auction where there is at least one bid). We also see that the higher the value of L, the greater is the buyer surplus. On the other hand, bidder collusion behaviour — where bidders collude in order to lower the transaction price — may be incorporated by having a lower value of L. Shilling behaviour — where the seller artificially inflates the transaction price through disguising as bidders — may be reflected by a higher value for λ, and hence a higher value for $z = \lambda T$, which from Equation 7, will result in a higher transaction price and consequently reducing the buyer surplus while raising the seller surplus.

From the seller's point of view, the aim of auction is to attain improvements in seller surplus through expending more time to achieve a higher price or income. If one simply accepts the first bid that comes along, then its average magnitude is $E\left[Q_i\right] = L/2$. By holding an auction, the average gain in surplus per bid acceptance due to one auction, assuming there are N bids, is

$$\begin{aligned}E\left[Q_{(N)}|N\right] - E\left[Q_i|N\right] &= \frac{LN}{N+1} - \frac{L}{2}\\ &= \frac{(N-1)L}{2(N+1)}.\end{aligned} \tag{10}$$

If there are two or more identical items for sale, to speed things up one might accept the two highest bids, instead of the just the highest one. Accepting more than one bid per auction is quite common in Internet auctions; e.g. Google's ad auctions often accept several bids. From [14], it is shown that the k-th order statistic of N samples from a uniform distribution distributed over a given interval is $k/(N+1)$ of the length of the interval. Thus, in accepting the two highest bids in one auction, the average gain in seller surplus per bid acceptance is

$$\begin{aligned}
&\frac{1}{2}\left\{\left(E\left[Q_{(N)}|N\right]-E\left[Q_i|N\right]\right)+\left(E\left[Q_{(N-1)}|N\right]-E\left[Q_i|N\right]\right)\right\}\\
=&\frac{1}{2}\left(E\left[Q_{(N)}|N\right]-E\left[Q_{(N-1)}|N\right]\right)-E\left[Q_i|N\right]\\
=&\frac{1}{2}\left[\frac{LN}{N+1}+\frac{N-1}{N+1}\right]-\frac{L}{2}\\
=&\frac{(N-2)L}{2(N+1)},
\end{aligned}$$

which is less than the gain in seller surplus in the case where only one bid is accepted, but it takes only one instead of two auction times and associated costs to achieve two acceptances. Correspondingly, the average buyer surplus will increase, since the transaction price of the second item is lower.

In general, the average of the highest K bids, given there are N bids, is

$$\begin{aligned}
\frac{1}{K}\sum_{j=0}^{K-1}E\left[Q_{(N-j)}|N\right]&=\frac{L}{K}\sum_{j=0}^{K-1}\frac{N-j}{N+1}\\
&=\frac{(2N-K+1)L}{2(N+1)}.
\end{aligned} \tag{11}$$

Thus, the conditional average gain in surplus per acceptance is

$$\begin{aligned}
&\frac{1}{K}\sum_{j=0}^{K-1}\left\{E\left[Q_{(N-j)}|N\right]-E\left[Q_i|N\right]\right\}\\
=&\frac{1}{K}\left\{\sum_{j=0}^{K-1}E\left[Q_{(N-j)}|N\right]\right\}-E\left[Q_i|N\right]\\
=&\frac{1}{K}\sum_{j=0}^{K-1}E\left[Q_{(j)}|N\right]-\frac{L}{2}\\
=&\frac{(2N-K+1)L}{2(N+1)}-\frac{L}{2}\\
=&\frac{(N-K)L}{2(N+1)},
\end{aligned}$$

which from Equation 10 is always below the gain in surplus resulting from accepting a single bid per auction. The average total income in accepting the top K bids, from Equation 11, is

$$\sum_{j=0}^{K-1}E\left[Q_{(N-j)}|N\right]=\frac{(2N-K+1)LK}{2(N+1)}. \tag{12}$$

Consider a variation of the basic model, in which the K highest bids are accepted in one auction. Note that accepting K highest bids requires that there are at least K arrivals (and of course at least K items for sale), and for meaningful operation, this requires that $T \gg K \times$ Mean Inter-arrival Time or $\lambda T \gg K$. Removing the condition on N, and noting that $N \geq K$, we have, for the average total income in accepting the top K bids,

$$\begin{aligned}
&\sum_{j=0}^{K-1} E\left[Q_{(N-j)}|N\right] \\
&= \sum_{N=K}^{\infty} \frac{LK(2N-K+1)}{2(N+1)} \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= \frac{LK}{2} \sum_{N=K}^{\infty} \left[2 - \frac{K+1}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^N}{N!} \\
&= LK \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \frac{LK}{2\lambda T} \sum_{N=K}^{\infty} \left[\frac{K+1}{N+1}\right] \times \frac{e^{-\lambda T}(\lambda T)^{N+1}}{N!} \\
&= LK \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^N}{N!} - \frac{LK(K+1)}{2\lambda T} \sum_{N=K}^{\infty} \frac{e^{-\lambda T}(\lambda T)^{N+1}}{N!} \\
&= LK \left\{1 - \sum_{j=0}^{K-1} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\} - \frac{LK(K+1)}{2\lambda T} \left\{1 - \sum_{j=0}^{K} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\}.
\end{aligned}$$

That is, we have for the expected income I_K when we choose to accept K top bids in a single auction

$$I_K = LK \left\{1 - \sum_{j=0}^{K-1} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\} - \frac{LK(K+1)}{2\lambda T} \left\{1 - \sum_{j=0}^{K} \frac{e^{-\lambda T}(\lambda T)^j}{j!}\right\}.$$

We see that for $K = 1$, the above reduces to Equation 5, and for the important special case $K = 2$, we have

$$I_2 = \frac{L}{\lambda T}\left(2\lambda T + \lambda T e^{-\lambda T} - \frac{(\lambda T)^2 e^{-\lambda T}}{2} + 3e^{-\lambda T} - 3\right). \tag{13}$$

Consider the cost Ω of holding an auction, which may be related to the auction time and associated costs such as fees paid to the auction site, and payments to financial intermediaries. We assume that these costs are otherwise not incurred if the item is sold through other channels. Let the price of a unit of the good be C. If no auctions are held, then the expected seller surplus would simply be $(L/2 - C)$, where, as indicated from the arguments above, $L/2$ represents the average value of the first offer, and we assume that it will be accepted. By holding an auction, the seller surplus S becomes

$$S = \frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - (C + \Omega).$$

Thus, the break-even point of holding an auction is given by the improvement in surplus offset by the auction cost

$$\frac{L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - \frac{L}{2} = \Omega.$$

As shown in Equation 8 above, the longer the auction, the higher the expected income. Supposing one wishes to attain a certain level of seller surplus S_o, then the minimal auction duration T^* is given by the solution to the following equation

$$S_0 = \frac{L}{\lambda T^*}\left(\lambda T^* + e^{-\lambda T^*} - 1\right) - (C + \Omega). \tag{14}$$

While we may solve for the above using numerical methods, we may obtain closed-form solutions by using the approximation from the previous section. Using this approximation, the above becomes

$$S_0 = \frac{Lz'}{z'+1} - (C + \Omega),$$

giving

$$z' = \frac{S_0 + C + \Omega}{L - (S_0 + C + \Omega)},$$

and this will provide a reasonable approximation for $z' \gg 1$. Thus, the approximate optimal auction duration T' is

$$T' = \frac{S_0 + C + \Omega}{\lambda\left[L - (S_0 + C + \Omega)\right]},$$

Letting $S_0 + C + \Omega = 90$, $L = 100$, and $\lambda = 1$, and numerically solving Equation 14 provides the exact minimum T^* in order to achieve a minimum surplus of S_0 which in this case is found to be $T^* = 10$ (see Figure 7). As can be seen from Figure 7, any value of $T > 10$ will yield at least a surplus of S_0. The corresponding approximate solution gives $T' = 90/(100-90) = 9$, which yields an error of just under 10%. From the sellers point of view, in order to quickly determine the optimal T^*, while avoiding the elaborate procedure of finding a numerical solution to Equation 14, one can simply first solve for T', and then add a safety factor to ensure that the resultant surplus $\geq S_0$, which in the present case may be 10%. A higher safety factor may be used to ensure greater certainty of achieving the required level of minimum surplus.

Next, the overall surplus in accepting K bids per auction is

$$I_K - KC - \Omega$$

While the overall surplus in selling K items through K separate auctions would be

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - K(C + \Omega).$$

Thus, it would be more profitable to sell K items in K separate auctions instead of selling them in a single auction if the expected surplus of the latter is higher, i.e.

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - K(C + \Omega) > I_K - KC - \Omega$$

or

$$\frac{LK}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - (K - 1)\Omega > I_K$$

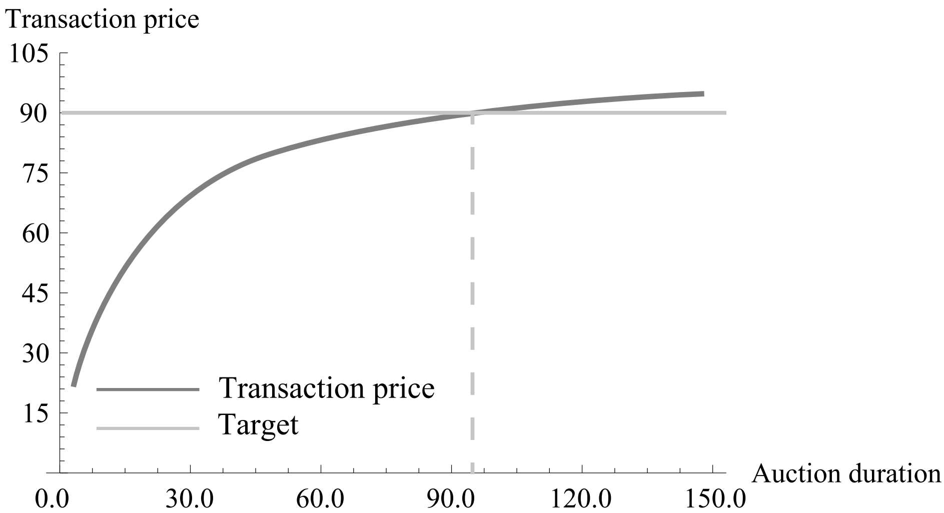

Fig. 7. Numerical determination of optimal auction duration.

For the important special case $K = 2$, the above becomes

$$\frac{2L}{\lambda T}\left(\lambda T + e^{-\lambda T} - 1\right) - \Omega > I_2$$

and making use of Equation 13, this condition simplifies to

$$e^z\left(1 - \frac{z\Omega}{L}\right) > 1 + z - \frac{z^2}{2}.$$

We see that in terms of magnitude, the left hand side increases exponentially in z, while the right hand side increases quadratically. Thus, for sufficiently large z, the left hand side will go negative with a large magnitude, while the right hand side will also go negative with a comparatively smaller magnitude; consequently the above inequality will not hold for large z. Thus, when the number of bids is large, it is always preferable to sell the items in single auctions.

Sometimes, the auction fee structure is such that the auction website would charge for a certain percentage of the income payment, which for instance is the common practice of eBay. Denoting by ξ such a percentage, then the overall surplus in accepting K bids per auction is

$$(1 - \xi)I_K - KC.$$

In adopting the same approximation as before by suitably replacing N by λT, then from Equation 12, we have

$$I_K \simeq \frac{(2\lambda T - K + 1)LK}{2(\lambda T + 1)}.$$

Using this approximation for the special case $K = 2$, we have

$$I_2 \simeq \frac{(2\lambda T - 1)L}{\lambda T + 1},$$

so that the overall surplus is approximately

$$\frac{L(1-\xi)(2\lambda T-1)}{\lambda T+1}-2C.$$

Thus it would be preferable to accept two bids per auction rather than to accept a single bid in two separate auctions if

$$2(1-\xi)I_1-2C<(1-\xi)I_2-2C,$$

or,

$$\frac{2\lambda TL}{\lambda T+1}<\frac{2\lambda TL-L}{\lambda T+1},$$

which is never the case for $L>0$. Thus, for this particular auction fee structure, unlike the previous case, it would always be preferable to run two separate auctions rather than a single auction given that the number of bids is large. In fact, even when the number of bids is not large, the general validity of this choice can be seen from Equation 12, where the total income from running a single auction is

$$(1-\xi)I_{K|N}=\frac{LK(1-\xi)(2N-K+1)}{2(N+1)},$$

where $I_{K|N}$ signifies the total income conditioning on N. The corresponding quantity in running K separate auctions is

$$KI_{1|N}=\frac{(1-\xi)NLK}{N+1}$$

Thus, it is preferable to run separate auctions if

$$\frac{N}{N+1}>\frac{2N-K+1}{2(N+1)}$$

which will be valid whenever $K>1$. Thus, for this particular auction fee structure, unlike the previous one, it is always more advantageous for the seller to sell the items in separate auctions.

4 Simulation

To enable the comparison between observed and theoretical values and to validate the mathematical models, an auction process simulator that implements the pseudocode for the auction algorithm, has been constructed in C++. In order to sample values from the uniform and exponential distributions for the private value of bidders and the rate of bids respectively, the Boost C++ Library is used. In particular, we use the `variate_generator` with the `uniform_01` and `exponential_distribution` headers, which is implemented on top of the `mersenne_twister` psuedo-random number generator. The result is outputted as a space-delimited text string that states lambda, which is the incoming rate of bids and usually the variable we change, the duration of that auction, and the revenue generated from that auction. Ten thousand trials are run for each arrival rate, which is sampled in 0.01 intervals in the units concerned over the desired interval.

Figures 8, 9 and 10 superimpose simulation data on Figures 2, 3 and 4, respectively. The simulation is a validation of the theoretical results and from the close alignment of the theoretical and simulated curves, the simulation seems to corroborate the above theory.

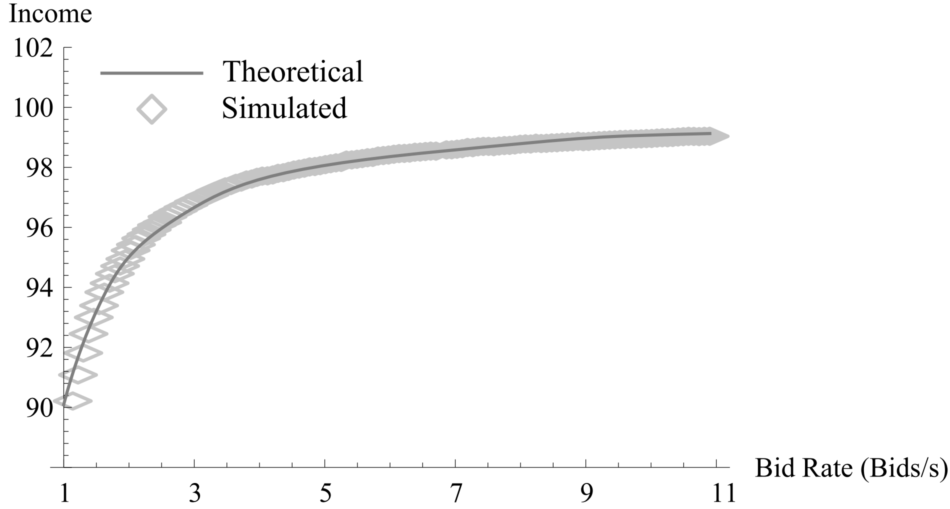

Fig. 8. Simulated and theoretical auction income as a function of the bid rate.

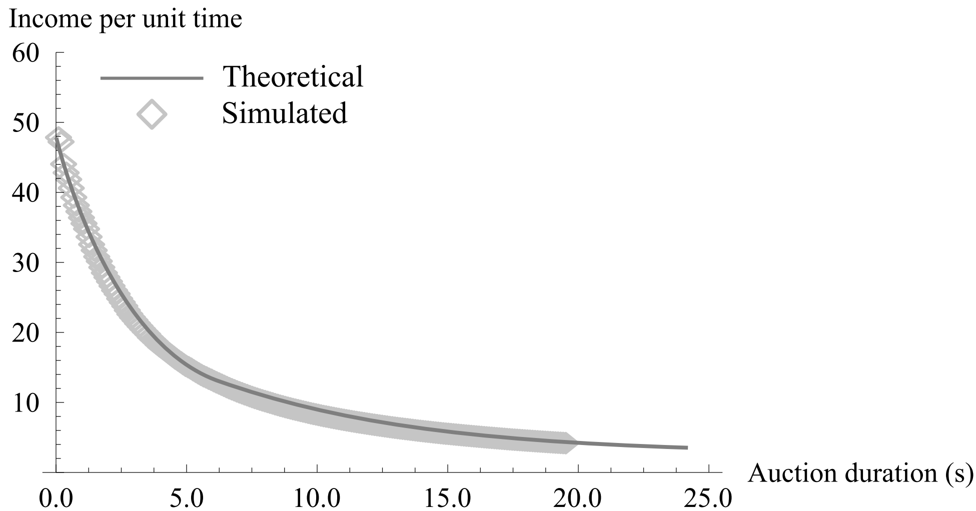

Fig. 9. Simulated and theoretical auction income per unit time as a function of the auction duration.

Figure 11, plots Equation 13 and its theoretical counterpart, i.e. both the theoretical and simulated auction income for auctions where two bids are accepted. The theoretical result is shown by the solid line, while the sample point in the simulation are plotted using grey diamonds. Also included in the graph is a simulation of the auction income derived from two auctions. This is used for comparison and the trade-off between additional seller surplus and the auction taken twice as long is up to the seller. The reason why the income derived from the theoretical result is greater than that of the simulation is that the former assumes that there are always two bids present, while in the simulation, this is not always the case (at $\lambda = 1$,

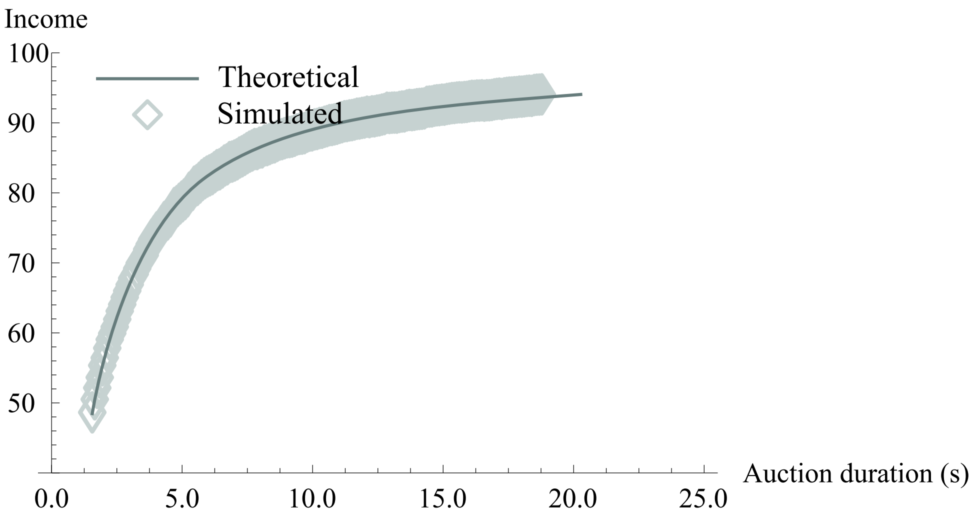

Fig. 10. Simulated and theoretical auction income as a function of the auction duration.

and with the auction lasting 4 time units, there are an average of 4 bids arriving in a given auction; the probability of 0 or 1 bid arriving during the entirety of the auction is 0.09158 using the Poisson probabilities, a figure that is non-negligble). Also of note is that if the seller were to sell directly to buyers, he would, on average, receive an income of 50 per item or an income of 100 for two items; all the simulated data points for accepting two bids per auction lie above this threshold.

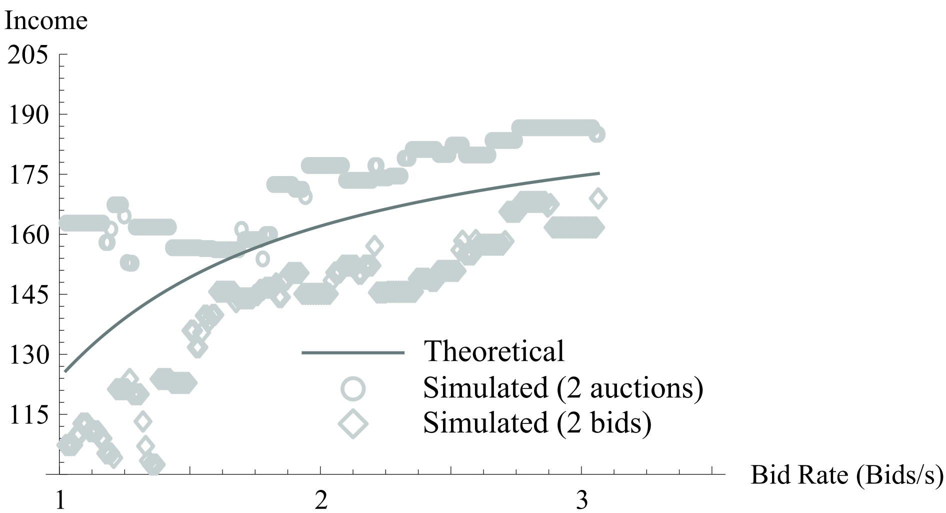

Fig. 11. Simulated and theoretical auction income for auctions that accept two bids per auction compared with income gained from two separate auctions.

5 Summary and Conclusions

Closed-form expressions are obtained for the stochastic analysis of Internet auction process presented in this paper, and key performance metrics of transaction price and auction income per unit time are derived. It is found that the auction income critically depends on the number of competing bids, so that as the number of bids increases, the auction income increases, and it climbs relatively sharply at the beginning but gradually does so slowly, while the income rate tends to favour shorter auctions. Both exact expressions and approximate formulae are given, with the latter providing a good estimate when the number of bids is large.

Analysis of surplus for buyers and sellers in Internet auction processes are given, and by suitably adjusting the different parameters, bidder collusion and seller shilling behaviours may be represented. Compared with other means and channels for the exchange of goods, the aim of the sellers selling items through auctions is to increase the transaction price. Appropriately controlling the auction process would be effective in raising the surplus for the seller, who can optimise this by adjusting the auction duration and the number of bids accepted per auction. The exact optimal auction duration may be obtained through numerical methods, while closed-form results are obtained for an approximate solution. Depending on the auction fee structure and seller utility, it may sometimes be advantageous to accept two or more bids per auction. Throughout the analysis, it is found that the average number of bids is key determinant of performance. Simulations have been performed and close agreement with theoretical analysis is observed.

In future work, rather than simply validating the fee structure presented in this paper against theoretical results, the model will be compared with real-world data scraped from auction sites. This will also be conducted alongside other more complex auction fee structures such as eBay's auction listings which require both an insertion fee for the listing according to the price bracket that the starting price falls in, as well as final value fees which subtract 10% from the winning bid with the exception of mobile phones on contract and property. The ratings of sellers — which relate to such factors as seller reputation, reliability, readiness to resolve disputes and provide refunds, delivery efficiency as well as seller surplus — are key considerations in Internet auctions and may also be incorporated into future optimisation models.

References

[1] E. Kaghashvili, "Online Timeshift Auctions," Sep. 22 2009, uS Patent App. 20,100/076,860.

[2] A. Ockenfels and A. Roth, "Late and multiple bidding in second price Internet auctions: Theory and evidence concerning different rules for ending an auction," *Games and Economic Behavior*, vol. 55, no. 2, pp. 297–320, 2006.

[3] ——, "The timing of bids in internet auctions: Market design, bidder behavior, and artificial agents," *AI Magazine*, vol. 23, no. 3, p. 79, 2002.

[4] P. Bajari and A. Hortacsu, "The winner's curse, reserve prices, and endogenous entry: empirical insights from eBay auctions," *RAND Journal of Economics*, vol. 34, no. 2, pp. 329–355, 2003.

[5] S. Parsons, J. A. Rodriguez-Aguilar, and M. Klein, "Auctions and bidding: A guide for computer scientists," *ACM Comput. Surv.*, vol. 43, pp. 10:1–10:59, February 2011. [Online]. Available: http://doi.acm.org/10.1145/1883612.1883617

[6] D. Lucking-Reiley, "Using field experiments to test equivalence between auction formats: Magic on the Internet," *American Economic Review*, vol. 89, no. 5, pp. 1063–1080, 1999.

[7] R. Vragov, "Operational efficiency of decentralized Internet auction mechanisms," *Electronic Commerce Research and Applications*, vol. 9, no. 2, pp. 111–125, 2010.

[8] E. Katok and A. Kwasnica, "Time is Money: The effect of clock speed on sellers revenue in Dutch auctions," *Experimental Economics*, vol. 11, no. 4, pp. 344–357, 2008.

[9] E. Gelenbe, "Analysis of single and networked auctions," *ACM Transactions on Internet Technology (TOIT)*, vol. 9, no. 2, pp. 1–24, 2009.

[10] R. Bapna, W. Jank, G. Shmueli, and G. Hyderabad, "Consumer surplus in online auctions," *Information Systems Research*, vol. 19, pp. 400–416, 2008.

[11] W. Jank, G. Shmueli, and S. Wang, "Forecasting Online Auctions using Dynamic Models," in *Workshop on Data Mining for Business Applications*. Citeseer, 2006, p. 1.

[12] V. Hyde, W. Jank, and G. Shmueli, "A family of growth models for representing the price process in online auctions," in *Proceedings of the ninth international conference on Electronic commerce*. ACM, 2007, p. 408.

[13] D. Lucking-Reiley, D. Bryan, N. Prasad, and D. Reeves, "Pennies from Ebay: The Determinants of Price in Online Auctions," *The Journal of Industrial Economics*, vol. 55, no. 2, pp. 223–233, 2007.

[14] B. Barry C. Arnold, N. Balakrishnan, and H. Nagaraja, *A First Course in Order Statistics (Classics in Applied Mathematics)*. SIAM, 2008.