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SURJECTIVITY OF OPERATORS IN BANACH SPACES

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Abstract—The surjectivity of operators from a Banach space into its topological conjugate space is important to the study of solutions of integral equations. In this paper, we derive some conditions under which operators will be surjective. In Hilbert space case, weaker conditions are also derived. An application to the coincidence theorem is considered.

1. INTRODUCTION

Let B be a real Banach space, B^* be its topological conjugate space and (u, v) be the paring between $u \in B$ and $v \in B^*$. Let T be an operator from B into B^*. The surjectivity of T plays an important role in the study of some subjects in nonlinear analysis such as weak solutions of differential equations, integral equations, etc. For example, if one is interested in looking for solutions of the Hammerstein integral equation

$$x(t) = \int_0^\infty k(t,s)\, f(s, x(s))\, ds = (KFx)(t),$$

it may be useful to consider $F : B = L^p([0,\infty)) \to B^* = L^q([0,\infty))$ and $K : B^* \to B^{**} = B$ for some $p > 1$ and q such that $p^{-1} + q^{-1} = 1$ depending on the properties of f and k. The aim of this paper is to derive some conditions under which the operator T will be surjective, i.e., $TB = B^*$. Standard results in this direction are, for example, [1, Theorem 4.3], [2, Theorem 12.1 and Corollary 12.1] and [3, Corollary 2]. For related results of accretive operators in Banach spaces, we refer readers to [4] and the references therein.

In Section 2, we state and prove some surjectivity results and an application to the coincidence theorem is considered. In Section 3, we consider the case that B is a Hilbert space. It will be shown that the conditions imposed in Section 2 can be weakened substantially.

2. SURJECTIVITY RESULTS

The operator $T : B \to B^*$ is said to be *continuous on finite-dimensional subspaces* if it is continuous on every finite-dimensional subspace of B. The operator T is said to be *demicontinuous* if it is continuous from the norm topology of B into the weak-star topology of B^*. For any subspace M of B, j_M denotes the injection of M into B and j_M^* be the dual of j_M. We use $B_r(x)$ to denote the closed ball with center x and radius r. For any subset D of B, $\overline{D}$ denote the closure of D.

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We now state and prove the main result of this paper.

THEOREM 2.1. *Let B be a real reflexive Banach space and $T : B \to B^*$ be demicontinuous. Suppose that*

(i) *there exists a function $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$, $\alpha(r) > 0$ for $r > 0$ and $\liminf_{r\to\infty} \alpha(r) > \|Tx_0\|$ for some $x_0 \in B$ such that*

$$|(x-y, Tx-Ty)| \geq \|x-y\|\,\alpha(\|x-y\|) \qquad \text{for all } x,y \in B;$$

(ii) *for any finite-dimensional subspace M, $\|Tx|_M\| \to \infty$ as $\|x\| \to \infty$ and $x \in M$.*

Then T is onto B^*.

PROOF. It suffices to show that $0 \in TB$. Let Γ be the family of all finite-dimensional subspaces of B containing x_0 partially ordered by inclusion. For each $M \in \Gamma$, the operator $T_M : j_M^* T j_M : M \to M^*$ is continuous. Since M is finite-dimensional we may, without loss of generality, assume that M is an Euclidean space $\mathbf{R}^n$ for some n and we can identify M^* with M. For any $x, y \in M$, we have

$$\begin{aligned} |(x-y, T_M\, x - T_M\, y)| &= |(x-y, Tx-Ty)| \\ &\geq \|x-y\|\,\alpha(\|x-y\|). \end{aligned}$$

Hence, T_M is one-to-one and therefore open by [2, Theorem 4.3]. But the set $T_M\, M$ is also closed by condition (ii). Consequently, T_M is onto M^* and hence, there is a unique $x_M \in M$ such that $T_M\, x_M = 0$.

Let $B_M = \{x_V : M \subset V \in \Gamma\}$ and let $\operatorname{wcl} B_M$ denote the weak closure of B_M. Then the family of sets $\{\operatorname{wcl} B_M : M \in \Gamma\}$ has the finite intersection property. Indeed, for $U, V \in \Gamma$, we can let $M \in \Gamma$ be such that $U \cup V \subset M$. Then $\emptyset \neq \operatorname{wcl} B_M \subset \operatorname{wcl} B_U \cap \operatorname{wcl} B_V$. For each $M \in \Gamma$, since $T_M\, x_M = 0$, we have

$$\begin{aligned} \|x_M - x_0\|\,\alpha(\|x_M - x_0\|) &\leq |(x_M - x_0, Tx_M - Tx_0)| \\ &= |(x_M - x_0, T_M\, x_M - T_M\, x_0)| \\ &= |(x_M - x_0, T_M\, x_0)| \\ &= |(x_M - x_0, Tx_0)| \\ &\leq \|x_M - x_0\|\,\|Tx_0\|. \end{aligned}$$

Therefore, since $\liminf_{r\to\infty} \alpha(r) > \|Tx_0\|$, there exists $r > 0$ such that $\|x_M\| \leq r$ for all $M \in \Gamma$. Consequently, $\operatorname{wcl} B_M \subset \overline{B_r(0)}$ for all $M \in \Gamma$. Since B is reflexive, $\overline{B_r(0)}$ is weakly compact. It follows that $\bigcap_{M\in\Gamma} \operatorname{wcl} B_M \neq \emptyset$.

Let $x \in \bigcap_{M\in\Gamma} \operatorname{wcl} B_M$. For any $y \in B$ let $M \in \Gamma$ be such that $x, y \in M$. Since $x \in \operatorname{wcl} B_M$, by Alaoglu's Theorem there is a sequence $\{x_n\}$ in B_M converging to x weakly. Let $M_n \in \Gamma$ be such that $x_n \in M_n$. Since $T_{M_n} x_n = 0$ for all n, we have

$$\begin{aligned} |(x_n - x, Tx)| &= |(x_n - x, T_{M_n} x)| \\ &= |(x_n - x, T_{M_n}\, x_n - T_{M_n} x)| \\ &= |(x_n - x, Tx_n - Tx)| \\ &\geq \|x_n - x\|\,\alpha(\|x_n - x\|), \end{aligned}$$

from which it follows that $x_n \to x$ since $(x_n - x, Tx) \to 0$ as $n \to \infty$. Now, from the demicontinuity of T and the facts that $T_{M_n}\, x_n = 0$ and $x, y \in M_n$, we have

$$0 = (y - x, T_{M_n}\, x_n) = (y - x, Tx_n) \to (y - x, Tx) \qquad \text{as } n \to \infty.$$

Consequently, $(y - x, Tx) = 0$ for all $y \in B$. Hence, $Tx = 0$ and the result follows.

Recall that an operator $T : B \to B^*$ is *monotone* if $(x-y, Tx-Ty) \geq 0$ for all $x, y \in B$. The operator T is *α-monotone* if there exists $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$, $\alpha(r) > 0$ for $r > 0$ and $\lim_{r\to\infty} \alpha(r) = \infty$ such that

$$(x-y, Tx-Ty) \geq \|x-y\|\,\alpha(\|x-y\|) \qquad \text{for all } x, y \in B.$$

The operator T is *dissipative* if $-T$ is monotone. The operator T is said to be *hemicontinuous* if for any $x, y \in B$ the following function is continuous

$$t \mapsto (x-y, T(tx+(1-t)y)), \qquad 0 \leq t \leq 1.$$

Since any hemicontinuous and monotone operator is demicontinuous [2], the following result is a direct consequence of Theorem 2.1.

COROLLARY 2.2. *Let B be a real reflexive Banach space and $T : B \to B^*$ be hemicontinuous and α-monotone. Then T is onto B^*.*

By Corollary 2.2, we have the following coincidence theorem.

COROLLARY 2.3. *Let B be a real reflexive Banach space and $T, F : B \to B^*$. Suppose that T is hemicontinuous and α-monotone and F is hemicontinuous and dissipative. Then there exists $x \in B$ such that $Tx = Fx$.*

PROOF. Let $G : B \to B^*$ be defined by $Gx = Tx - Fx$ for all $x \in B$. Then G is hemicontinuous and α-monotone. By Corollary 2.2, there exists $x \in B$ such that $Gx = 0$. Therefore, $Tx = Fx$ and the result follows.

By inspecting the proof of Theorem 2.1, it is not difficult to see that the following result is also true.

THEOREM 2.4. *Let B be a real reflexive Banach space and $T : B \to B^*$. Suppose that the following conditions are satisfied:*

(i) *T is continuous on finite-dimensional subspaces;*
(ii) *for each $\{x_n\}$ converging weakly to x,*

$$\liminf_{n\to\infty}(y, Tx_n) \leq (y, Tx) \qquad \text{for each } y \in B;$$

(iii) *there exists $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$, $\alpha(r) > 0$ for $r > 0$ and $\lim_{r\to\infty} \alpha(r) > \|Tx_0\|$ for some $x_0 \in B$ such that*

$$|(x-y, Tx-Ty)| \geq \|x-y\|\,\alpha(\|x-y\|) \qquad \text{for all } x, y \in B;$$

(iv) *for any finite-dimensional subspace M, $\|Tx|_M\| \to \infty$ as $\|x\| \to \infty$ and $x \in M$.*

Then T is onto B^*.

PROOF. Again, it suffices to show that $0 \in TB$. By employing the same argument as that of Theorem 2.1, it can be shown that there exists $x \in B$ with the property that for each $y \in B$ there exists a sequence $\{x_n\}$ weakly convergent to x such that $(y-x, Tx_n) = 0$ for all n. Then by condition (ii), we have

$$\begin{aligned} 0 &= \liminf_{n\to\infty}(y-x, Tx_n) \\ &\leq (y-x, Tx). \end{aligned}$$

Consequently, $(y-x, Tx) \geq 0$ for all $y \in B$. Therefore, $Tx = 0$ and the result follows.

COROLLARY 2.5. *Let B be a real reflexive Banach space and $T : B \to B^*$. Suppose that the following conditions are satisfied:*

(i) *the function $x \mapsto (x, Tx)$ is sequentially weakly lower semi-continuous on B;*

(ii) *for each* $\{x_n\}$ *converging weakly to* x,

$$\liminf_{n\to\infty}(y, Tx_n) \leq (y, Tx) \qquad \text{for each } y \in B;$$

(iii) *there exists* $\alpha : [0,\infty) \to [0,\infty)$ *with* $\alpha(0) = 0$, $\alpha(r) > 0$ *for* $r > 0$ *and* $\lim_{r\to\infty} \alpha(r) > \|Tx_0\|$ *for some* $x_0 \in B$ *such that*

$$|(x-y, Tx-Ty)| \geq \|x-y\|\,\alpha(\|x-y\|) \qquad \text{for all } x,y \in B;$$

(iv) *for any finite-dimensional subspace* M, $\|Tx|_M\| \to \infty$ *as* $\|x\| \to \infty$ *and* $x \in M$.

Then T *is onto* B^*.

PROOF. As the proof of [5, Theorem 2] shows, any operator satisfying conditions (i) and (ii) must be necessarily continuous on finite-dimensional subspaces. The result then is a direct consequence of Theorem 2.4.

3. HILBERT SPACE CASE

When B is real Hilbert space, assumptions of Theorems 2.1 and 2.4 can be weakened substantially. As the following result shows, the demicontinuity condition of Theorem 2.1 can be replaced by the condition that T is continuous on finite-dimensional subspaces and the condition (ii) of Theorem 2.4 is unnecessary.

THEOREM 3.1. *Let* H *be a real Hilbert space whose inner product is also denoted as* $(\cdot,\cdot)$ *and let* $T : H \to H$ *be continuous on finite-dimensional subspaces. Suppose that*

(i) *there exists* $\alpha : [0,\infty) \to [0,\infty)$ *with* $\alpha(0) = 0$, $\alpha(r) > 0$ *for* $r > 0$ *and* $\lim_{r\to\infty} \alpha(r) > \|Tx_0\|$ *for some* $x_0 \in H$ *such that*

$$|(x-y, Tx-Ty)| \geq \|x-y\|\,\alpha(\|x-y\|) \qquad \text{for all } x,y \in H;$$

(ii) *for any finite-dimensional subspace* M, $\|Tx|_M\| \to \infty$ *as* $\|x\| \to \infty$ *and* $x \in M$.

Then T *is onto* H.

PROOF. Essentially, we follow the proof of [2, Theorem 11.6]. It is again enough to show that $0 \in TH$. Let Γ be the family of all finite-dimensional subspaces of H containing x_0 partially ordered by inclusion. For each $M \in \Gamma$, let P_M be the orthogonal projection of H onto M. Since $P_M^* = P_M$, for any $x, y \in M$ we have

$$\begin{aligned} |(x-y, P_M Tx - P_M Ty)| &= |(x-y, Tx-Ty)| \\ &\geq \|x-y\|\,\alpha(\|x-y\|). \end{aligned}$$

Hence, $P_M T|_M$ is one-to-one and therefore open by [2, Theorem 4.3]. But $P_M TM$ is also closed by condition (ii). Consequently, $P_M T$ is onto M and hence there is a unique $x_M \in M$ such that $P_M Tx_M = 0$.

Let $B_M = \{x_V : M \subset V \in \Gamma\}$ and let $\operatorname{wcl} B_M$ denote the weak closure of B_M. Then the family of sets $\{\operatorname{wcl} B_M : M \in \Gamma\}$ has the finite intersection property. Since $\|P_M\| = 1$, we have

$$\begin{aligned} \|x_M - x_0\|\,\alpha(\|x_M - x_0\|) &\leq |(x_M - x_0, Tx_M - Tx_0)| \\ &= |(x_M - x_0, P_M Tx_M - P_M Tx_0)| \\ &= |(x_M - x_0, P_M Tx_0)| \\ &\leq \|x_M - x_0\|\,\|Tx_0\|. \end{aligned}$$

Therefore, since $\liminf_{r\to\infty} \alpha(r) > \|Tx_0\|$, there exists $r > 0$ such that $\|x_M\| \leq r$ for all $M \in \Gamma$. Consequently, $\operatorname{wcl} B_M \subset \overline{B_r(0)}$ for all $M \in \Gamma$. Since H is a Hilbert space, $\overline{B_r(0)}$ is weakly compact. It follows that $\bigcap_{M\in\Gamma} \operatorname{wcl} B_M \neq \emptyset$.

Let $x \in \bigcap_{M \in \Gamma} \operatorname{wcl} B_M$ and fix $M \in \Gamma$ such that $x, Tx \in M$. Since $x \in \operatorname{wcl} B_M$, by Alaoglu's Theorem there is a sequence $\{x_n\}$ in B_M converging to x weakly. Let $M_n \in \Gamma$ be such that $x_n \in M_n$. Since $P_{M_n} T x_n = 0$ for all n, we have

$$\begin{aligned}
|(x_n - x, Tx)| &= |(x_n - x, P_{M_n} Tx)| \\
&= |(x_n - x, P_{M_n} Tx_n - P_{M_n} Tx)| \\
&= |(x_n - x, Tx_n - Tx)| \\
&\geq \|x_n - x\| \, \alpha(\|x_n - x\|),
\end{aligned}$$

from which it follows that $x_n \to x$. Now, as $(x_M, Tx_M) = (x_M, P_M Tx_M) = 0$, we have

$$\begin{aligned}
0 &= (x_n - x_M, P_{M_n} T x_n) \\
&= (x_n - x_M, P_{M_n} Tx_n - P_{M_n} Tx_M) + (x_n - x_M, P_{M_n} Tx_M) \\
&= (x_n - x_M, Tx_n - Tx_M) + (x_n, Tx_M) - (x_M, Tx_M) \\
&= (x_n - x_M, Tx_n - Tx_M) + (x_n, Tx_M).
\end{aligned}$$

Therefore,

$$|(x_n - x_M, Tx_n - Tx_M)| = |(x_n, Tx_M)|. \tag{1}$$

Hence, it follows from (1) and (i) that

$$\begin{aligned}
0 &= (x, P_M Tx_M) \\
&= |(x, Tx_M)| \\
&= \lim_{n \to \infty} |(x_n, Tx_M)| \\
&= \lim_{n \to \infty} |(x_n - x_M, Tx_n - Tx_M)| \\
&\geq \lim_{n \to \infty} \|x_n - x_M\| \, \alpha(\|x_n - x_M\|) \\
&= \|x - x_M\| \, \alpha(\|x - x_M\|).
\end{aligned}$$

Consequently, $x = x_M$. Since $Tx \in M$, we finally have

$$\|Tx\|^2 = (Tx, Tx) = (Tx, P_M Tx_M) = 0.$$

Hence, $Tx = 0$ and the result follows.

We note that Theorem 3.1 generalizes [2, Theorem 11.6].

References

1. V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach Space*, Sijthoff and Noordhoff, Bucarest, Romania, (1978).
2. K. Demling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, (1985).
3. J.S. Guo and J.C. Yao, Zeros of operators in Banach spaces, *Appl. Math. Lett.* **5**, 55–57 (1992).
4. K. Demling, Zeros of accretive operators, *Manuscripta Mathematica* **13**, 365–374 (1974).
5. M. Théra, A note on the Hartman-Stampacchia theorem, In *Nonlinear Analysis and Applications*, (Edited by Lakshmikantham), pp. 573–577, Dekker, New York, (1987).