Existence for nonoscillatory solutions of higher-order nonlinear neutral difference equations

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Abstract
In this paper, we consider the following forced higher-order nonlinear neutral difference equation

\[ \Delta^m (x_n + c_n x_{n-k}) + \sum_{s=1}^{u} p^{(s)}_n f_s (x_{n-r_s}) = q_n, \quad n \geq n_0, \]

where \( m, u \geq 1, k \geq 0, \) and \( r_s \geq 0 \) are integers, \( \{c_n\}, \{p^{(s)}_n\} (s = 1, 2, \ldots, u) \) and \( \{q_n\} \) are sequence of real numbers and \( f_s \in C(\mathbb{R}, \mathbb{R}) (s = 1, 2, \ldots, u) \). By using Krasnoselskii’s fixed point theorem and some new techniques, we obtain sufficient conditions for the existence of nonoscillatory solutions for general \( \{p^{(s)}_n\} (s = 1, 2, \ldots, u) \) and \( \{q_n\} \) which means that we allow oscillatory \( \{p^{(s)}_n\} (s = 1, 2, \ldots, u) \) and \( \{q_n\} \).

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1. Introduction

Consider the forced higher-order nonlinear neutral delay difference equation

\[ \Delta^m (x_n + c_n x_{n-k}) + \sum_{s=1}^{u} p^{(s)}_n f_s (x_{n-r_s}) = q_n, \quad n \geq n_0 \in \mathbb{N}. \]

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With respect to Eq. (1), throughout we shall assume that \( m, u \geq 1, k \geq 0 \) and \( r \geq 0 \) \((s = 1, 2, \ldots, u)\), and \( \{q_n\}_{n=0}^\infty \) sequence of real numbers which can even be oscillatory, \( f_s \in C(\mathbb{R}, \mathbb{R}) \) and \( x f_s(x) \geq 0 \) for any \( x \neq 0 \) \((s = 1, 2, \ldots, u)\).

The forward difference \( \Delta \) is defined as usual, i.e., \( \Delta x_n = x_{n+1} - x_n \). The higher-order difference for a positive integer \( m \) are defined as \( \Delta^m x_n = \Delta(\Delta^{m-1} x_n), \Delta^0 x_n = x_n \).

Let \( \sigma = \max_{1 \leq s \leq u} \{k, r\} \) and \( N_0 \geq n_0 \) be a fixed nonnegative integer. By a solution of (1), we mean a real sequence \( \{x_n\} \) which is defined for all \( n \geq N_0 - \sigma \) and satisfies (1) for \( n \geq N_0 \).

The neutral delay difference equations arise in a number of important applications including problems in population dynamics when maturation and gestation are included, in “cobweb” models in economics where demand depends on current price but supply depends on the price at an earlier time, and in electrical transmission in lossless transmission lines between circuits in high speed computers.

Oscillation theory of higher-order neutral difference equations has developed very rapidly in recent years. It has concerned itself largely with the oscillatory and nonoscillatory properties of solutions (see, e.g., [1–5, 7, 9–15] and the references cited therein). Agarwal et al. [1], Agarwal and Wong [2], Agarwal et al. [3], Agarwal and Grace [4], Zhang and Yang [10] investigate the oscillatory behavior of solutions of nonlinear neutral difference equation of order \( m (\geq 1) \) of the following form

\[
\Delta^m (x_n + c_n x_{n-k}) + p_n f(x_{n-r}) = 0, \quad n \geq n_0, \tag{2}
\]

where \( r > 0, x f(x) \geq 0 \) for any \( x \neq 0 \). Clearly, Eq. (2) is a special case of Eq. (1).

Recently, Yang and Liu [11] used the Banach contraction mapping principle to obtain a existence criteria for nonoscillatory solutions of (2) with \( c_n \equiv c \in \mathbb{R} \). The following is the main result of [11].

**Theorem A** [11]. Assume that

\( (C_1) \) \( m \geq 2 \) is an even integer;

\( (C_2) \) \( c_n \equiv c \neq \pm 1, p_n \geq 0 (\neq 0) \);

\( (C_3) \) For any \( x, y \in \mathbb{R}, |f(x) - f(y)| \leq L|x - y| \), where \( L \in \mathbb{R}^+ \).

Further, assume that

\[
\sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \frac{(j - i + m - 2)(m-2)!}{(m-2)!} p_i < \infty. \tag{3}
\]

Then (2) has a bounded nonoscillatory solution.

We note that condition (3) is equivalent to a simple condition

\[
\sum_{i=n_0}^{\infty} i^{m-1} p_i < \infty.
\]

In a recent paper [13], Zhou obtained the following result by using Banach contraction mapping principle.
Theorem B [13]. Assume that

(C4) \( m \geq 1 \) is an odd integer;
(C5) \( f(x) = x \), for any \( x \in \mathbb{R} \).

Further, assume that \( c_n \equiv c \neq -1 \), \( p_n \in \mathbb{R} \) and that

\[
\sum_{i=n_0}^{\infty} i^{m-1} |p_i| < \infty.
\]

Then (2) has a bounded nonoscillatory solution.

In [8], Graef and Thandapani obtained a existence criteria for nonoscillatory solution of forced third-order delay difference equation

\[
\Delta^3 x_n + p_n f(x_{n-k}) = q_n,
\]

where \( \{p_n\} \) and \( \{q_n\} \) are sequences of real numbers. They proved the following result by using Schauder fixed point theorem.

Theorem C [8]. Assume that

(C6) \( f \) is nondecreasing,

and that

\[
\sum_{i=n_0}^{\infty} i^2 |p_i| < \infty \quad \text{and} \quad \sum_{i=n_0}^{\infty} i^2 |q_i| < \infty.
\]

Then (4) has a bounded nonoscillatory solution.

In this paper, using Krasnosel’skii’s fixed point theorem and some new techniques, we obtain some sufficient conditions for the existence of a nonoscillatory solution of forced equation (1) in the case when \( \{p_n^{(s)}\} \) \( (s = 1, 2, \ldots, u) \) and \( \{q_n\} \) can be oscillatory. In particular, our results improve essentially Theorems A, B, and C by removing the restrictive conditions (C1)–(C6).

As is customary, a solution \( \{x_n\} \) of (1) is said to oscillate about zero, or simply to oscillate if the terms \( x_n \) of the sequence \( \{x_n\} \) are neither eventually all positive nor eventually all negative. Otherwise, the solution is called nonoscillatory.

For \( t \in \mathbb{R} \) we define the usual factorial expression \( (t)^{(m)} = \prod_{i=0}^{m-1} (t-i) \) with \( (t)^{(0)} = 1 \).

2. Main results

The space \( l^\infty \) is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under the supremum norm \( l^\infty \) is a Banach space. A subset \( \Omega \) of a Banach space

space $X$ is relatively compact if every sequence in $\Omega$ has a subsequence converging to an element of $X$.

**Definition 1** [5]. A set $S$ of sequences in $l^\infty$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exists an integer $N$ such that

$$|x_i - x_j| < \varepsilon,$$

whenever $i, j > N$ for any $x = \{x_k\}$ in $S$.

**Lemma 1** (Discrete Arzela–Ascoli’s theorem [5]). A bounded, uniformly Cauchy subset $\Omega$ of $l^\infty$ is relatively compact.

**Lemma 2** (Krasnoselskii’s fixed point theorem [6]). Let $X$ be a Banach space, let $\Omega$ be a bounded closed convex subset of $X$ and let $T_1, T_2$ be maps of $\Omega$ into $X$ such that $T_1x + T_2y \in \Omega$ for every pair $x, y \in \Omega$. If $T_1$ is a contraction and $T_2$ is completely continuous, then the equation

$$T_1x + T_2x = x$$

has a solution in $\Omega$.

**Lemma 3** (Schauder’s fixed point theorem [6]). Let $\Omega$ be a closed, convex and nonempty subset of a Banach space $X$. Let $T : \Omega \rightarrow \Omega$ be a continuous mapping such that $T \Omega$ is a relatively compact subset of $X$. Then $T$ has at least one fixed point in $\Omega$. That is, there exists an $x \in \Omega$ such that $Tx = x$.

Our main results are the following five theorems.

**Theorem 1.** Assume that $-1 < c < c_n \leq 0$ and that

$$\sum_{i=n_0}^{\infty} i^{m-1} |p_i^{(1)}| < \infty, \quad s = 1, 2, \ldots, u, \quad (5)$$

and

$$\sum_{i=n_0}^{\infty} i^{m-1} |q_i| < \infty. \quad (6)$$

Then (1) has a bounded nonoscillatory solution.

**Proof.** By (5) and (6), we choose a $n_1 > n_0$ sufficiently large such that

$$\frac{1}{(m-1)!} \sum_{i=n_1}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| M_1 + |q_i| \right) \leq \frac{1 + c}{3},$$

where $M_1 = \max_{2\leq i \leq 4/3} \{|f_\xi(x) : 1 \leq s \leq u\}$.
Let $l_{n_0}^{\infty}$ be the set of all real sequence $x = \{x_n\}_{n=n_0}^{\infty}$ with the norm $\|x\| = \sup_{n \geq n_0} |x_n| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $l_{n_0}^{\infty}$ as follows:

$$\Omega = \left\{ x = \{x_n\} \in l_{n_0}^{\infty}, \frac{2(1+c)}{3} \leq x_n \leq \frac{4}{3}, \ n \geq n_0 \right\}.$$ 

Define two maps $T_1$ and $T_2 : \Omega \rightarrow l_{n_0}^{\infty}$ as follows:

$$T_1x_n = \begin{cases} 1 + c - c_n x_{n-k}, & n \geq n_1, \\ \frac{2(1+c)}{3} \leq x_n \leq \frac{4}{3}, \ n \geq n_0 \end{cases},$$

$$T_2x_n = \begin{cases} \frac{(-1)^{m+1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n + m-1)^{(m-1)} (\sum_{s=1}^{u} p_i^{(s)} f(x_{i-r_s}) - q_i), & n \geq n_1, \\ T_2x_n, & n_0 \leq n \leq n_1. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega$, $T_1x + T_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $n \geq n_1$, we get

$$T_1x_n + T_2y_n \leq 1 + c - c_n x_{n-k} + \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n + m-1)^{(m-1)} \left( \sum_{s=1}^{u} p_i^{(s)} f_s(y_{i-r_s}) + |q_i| \right)$$

$$\leq 1 + c - \frac{4}{3} c + \frac{1}{(m-1)!} \sum_{i=n_1}^{\infty} (i-n + m-1)^{(m-1)} \left( \sum_{s=1}^{u} p_i^{(s)} |M_1| + |q_i| \right)$$

$$\leq 1 + c - \frac{4}{3} c + \frac{1+c}{3} = \frac{4}{3}.$$ 

Furthermore, we have

$$T_1x_n + T_2y_n \geq 1 + c - c_n x_{n-k} - \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n + m-1)^{(m-1)} \left( \sum_{s=1}^{u} p_i^{(s)} f_s(y_{i-r_s}) + |q_i| \right)$$

$$\geq 1 + c - \frac{1}{(m-1)!} \sum_{i=n_1}^{\infty} (i-n + m-1)^{(m-1)} \left( \sum_{s=1}^{u} p_i^{(s)} |M_1| + |q_i| \right)$$

$$\geq 1 + c - \frac{1+c}{3} = \frac{2(1+c)}{3}.$$

Hence, $\frac{2(1+c)}{3} \leq T_1x_n + T_2y_n \leq \frac{4}{3}$ for $n \geq n_0$.

Thus we have proved that $T_1x + T_2y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that $T_1$ is a contraction mapping on $\Omega$.

In fact, for $x, y \in \Omega$ and $n \geq n_1$, we have

$$|T_1x_n - T_1y_n| \leq c_n |x_{n-k} - y_{n-k}| \leq c\|x - y\|.$
This implies that

$$\|T_1x - T_1y\| \leq -c\|x - y\|.$$  

Since \(0 < -c < 1\), we conclude that \(T_1\) is a contraction operator on \(\Omega\).

(iii) We now show that \(T_2\) is completely continuous.

First, we will show that \(T_2\) is continuous. Let \(x^{(v)} = \{x_n^{(v)}\} \in \Omega\) be such that \(x_n^{(v)} \to x_n\) as \(v \to \infty\). Because \(\Omega\) is closed, \(x = \{x_n\} \in \Omega\). For \(n \geq n_1\), we have

$$|T_2x_n^{(v)} - T_2x_n| \leq \frac{1}{(m - 1)!} \sum_{i=n_1}^{\infty} (i - n + m - 1)^{(m-1)}$$

$$\times \left( \sum_{s=1}^{u} |p_i^{(s)}| \left| f_s(x_{i-r_s}) - f_s(x_{i-r_s}) \right| \right).$$

Since

\[
(i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| \left| f_s(x_{i-r_s}) - f_s(x_{i-r_s}) \right| \right) 
\leq (i - n + m - 1)^{(m-1)} \sum_{s=1}^{u} |p_i^{(s)}| \left( |f_s(x_{i-r_s})| + |f_s(x_{i-r_s})| \right) 
\leq 2M_1 (i - n + m - 1)^{(m-1)} \sum_{s=1}^{u} |p_i^{(s)}| \leq 2M_1 \sum_{s=1}^{u} i^{m-1} |p_i^{(s)}| 
\]

and \(|f_s(x_{i-r_s}) - f_s(x_{i-r_s})| \to 0\) as \(v \to \infty\) for \(s = 1, 2, \ldots, u\), in view of (5) and applying the Lebesgue dominated convergence theorem, we conclude that \(\lim_{v \to \infty} \|T_2x^{(v)} - T_2x\| = 0\). This means that \(T_2\) is continuous.

Next, we will show that \(T_2\) is relatively compact. For any given \(\varepsilon > 0\), by (5), there exists \(N \geq n_1\) such that

$$\frac{1}{(m - 1)!} \sum_{i=N}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| |M_1 + |q_s|| \right) < \frac{\varepsilon}{2}.$$ 

Then for any \(x = \{x_n\} \in \Omega\) and \(t, n \geq N\),

\[
|T_2x_t - T_2x_n| \leq \frac{1}{(m - 1)!} \sum_{i=t}^{\infty} (i - t + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| |M_1 + |q_s|| \right) 
\]

$$+ \frac{1}{(m - 1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| |M_1 + |q_s|| \right)$$

$$\leq \frac{1}{(m - 1)!} \sum_{i=t}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| M_1 + |q_s|| \right)$$

$$+ \frac{1}{(m - 1)!} \sum_{i=n}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| M_1 + |q_s|| \right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
This means that $T_2\Omega$ is uniformly Cauchy. Hence, by Lemma 1, $T_2\Omega$ is relatively compact. By Lemma 2, there is $x = \{x_n\} \in \Omega$ such that $T_1x + T_2x = x$. Clearly, $x = \{x_n\}$ is a bounded positive solution of Eq. (1). This completes the proof of Theorem 1.

\[ \blacksquare \]

**Theorem 2.** Assume that $-\infty < c_n \equiv c < -1$ and that (5) and (6) hold. Then (1) has a bounded nonoscillatory solution.

**Proof.** By (5) and (6), we choose a $n_1 > n_0$ sufficiently large such that

\[
-\frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} p_i(s) \left| M_2 + |q_i| \right| \right) \leq -\frac{c+1}{2},
\]

where $M_2 = \max_{-c+1/2 \leq x \leq -2c} \{ f_s(x) : 1 \leq s \leq u \}$.

Let $l_\infty^{\infty}$ be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset $\Omega$ of $l_\infty^{\infty}$ as follows:

\[
\Omega = \left\{ x = \{x_n\} \in l_\infty^{\infty} : -\frac{c+1}{2} \leq x_n \leq -2c, \ n \geq n_0 \right\}.
\]

Define two maps $T_1$ and $T_2 : \Omega \to l_\infty^{\infty}$ as follows:

\[
T_1x_n = \begin{cases} 
-\frac{c-1}{c} - \frac{1}{c_n} x_{n+k}, & n \geq n_1, \\
T_1x_{n_1}, & n_0 \leq n \leq n_1.
\end{cases}
\]

\[
T_2x_n = \begin{cases} 
\frac{(-1)^{m+1}}{c_n (m-1)!} \sum_{i=n+k}^{\infty} (i - k + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} p_i(s) f_s(x_{i-r_s}) \right), & n \geq n_1, \\
T_2x_{n_1}, & n_0 \leq n \leq n_1.
\end{cases}
\]

(i) We shall show that for any $x, y \in \Omega$, $T_1x + T_2y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $n \geq n_1$, we get

\[
T_1x_n + T_2y_n \\
\leq -c - 1 - \frac{1}{c_n} \sum_{i=n+k}^{\infty} (i - k + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} p_i(s) f_s(x_{i-r_s}) + |q_i| \right) \\
\leq -c - 1 + 2 \frac{1}{c (m-1)!} \sum_{i=n+k}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} p_i(s) \left| M_2 + |q_i| \right| \right) \\
\leq -c + 1 - \frac{c+1}{2} \leq -2c.
\]

Furthermore, we have
\[
T_1 x_n + T_2 y_n \\
\geq -c - 1 - \frac{1}{c_n} x_{n+k} \\
+ \frac{1}{c_n (m-1)!} \sum_{i=n+k}^{\infty} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{n} |p_i^{(s)}| f_s(y_i - r_s) + |q_i| \right) \\
\geq -c - 1 + \frac{1}{c (m-1)!} \sum_{i=n_1}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{n} |p_i^{(s)}| M_2 + |q_i| \right) \\
\geq -c - 1 + \frac{c + 1}{2} = -\frac{c + 1}{2}.
\]

Hence,
\[
-\frac{c + 1}{2} \leq T_1 x_n + T_2 y_n \leq -2c, \quad \text{for } n \geq n_0.
\]

Thus we have proved that \( T_1 x + T_2 y \in \Omega \) for any \( x, y \in \Omega \).

(ii) We shall show that \( T_1 \) is a contraction mapping on \( \Omega \).
In fact, for \( x, y \in \Omega \) and \( n \geq n_1 \), we have
\[
|T_1 x_n - T_1 y_n| \leq -\frac{1}{c_n} |x_{n-k} - y_{n-k}| \leq -\frac{1}{c} \|x - y\|.
\]

This implies that
\[
\|T_1 x - T_1 y\| \leq -\frac{1}{c} \|x - y\|.
\]

Since \( 0 < -1/c < 1 \), we conclude that \( T_1 \) is a contraction mapping on \( \Omega \).

(iii) We now show that \( T_2 \) is completely continuous.
First, we will show that \( T_2 \) is continuous. Let \( x^{(v)} = \{x_n^{(v)}\} \in \Omega \) be such that \( x_n^{(v)} \rightarrow x_n \) as \( v \rightarrow \infty \). Since \( \Omega \) is closed, \( x = \{x_n\} \in \Omega \). For \( n \geq n_1 \), we have
\[
|T_2 x_n^{(v)} - T_2 x_n| \leq \frac{1}{c (m-1)!} \sum_{i=n_1+k}^{\infty} (i - n + k + m - 1)^{(m-1)} \\
\times \left( \sum_{s=1}^{n} |p_i^{(s)}| f_s(x_i^{(v)} - x_i - r_s) - f_s(x_i^{(v)} - r_s) \right).
\]

Since
\[
(i - n + k + m - 1)^{(m-1)} \left( \sum_{s=1}^{n} |p_i^{(s)}| f_s(x_i^{(v)} - r_s) \right) \\
\leq (i - n + k + m - 1)^{(m-1)} \sum_{s=1}^{n} |p_i^{(s)}| \left( |f_s(x_i^{(v)}) - f_s(x_i - r_s)| \right) \\
\leq 2M_2 (i - n + m - 1)^{(m-1)} \sum_{s=1}^{n} |p_i^{(s)}| \leq 2M_2 \sum_{s=1}^{n} (m-1) |p_i^{(s)}|
\]

Therefore, \( T_2 \) is continuous.
and \( |f_s(x_{v_{i-1}}^{(v)}) - f_s(x_{v_i})| \to 0 \) as \( v \to \infty \) for \( s = 1, 2, \ldots, u \), in view of (5) and applying the Lebesgue dominated convergence theorem, we conclude that \( \lim_{v \to \infty} \|T_{2x(v)} - T_{2x}\| = 0 \). This means that \( T_2 \) is continuous.

Next, we will show that \( T_2 \Omega \) is relatively compact. For any given \( \varepsilon > 0 \), by (5), there exists \( N \geq n_1 \) such that

\[
\frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} |q_i| (i + m - 1)^{m-1} \left( \sum_{s=1}^{u} |p_s^{(s)}| M_2 + |q_i| \right) < \frac{\varepsilon}{2}.
\]

Then for any \( x = \{x_n\} \in \Omega \) and \( n \geq N \),

\[
|T_{2x_1} - T_{2x_2}| \leq \frac{1}{c(m-1)!} \sum_{i=n+k}^{\infty} (i - n - k + m - 1)^{m-1} \left( \sum_{s=1}^{u} |p_s^{(s)}| M_2 + |q_i| \right)
\]

This means that \( T_2 \Omega \) is uniformly Cauchy. Hence, by Lemma 1, \( T_2 \Omega \) is relatively compact.

By Lemma 2, there is \( x = \{x_n\} \in \Omega \) such that \( T_1x + T_2x = x \). Clearly, \( x = \{x_n\} \) is a bounded positive solution of Eq. (1). This completes the proof of Theorem 2. \( \square \)

**Theorem 3.** Assume that \( 0 \leq c_n < c < 1 \) and that (5) and (6) hold. Then (1) has a bounded nonoscillatory solution.

**Proof.** By (5) and (6), we choose a \( n_1 > n_0 \) sufficiently large such that

\[
\frac{1}{(m-1)!} \sum_{i=n_1}^{\infty} (i + m - 1)^{m-1} \left( \sum_{s=1}^{u} |p_s^{(s)}| M_3 + |q_i| \right) \leq 1 - c,
\]

where \( M_3 = \max_{2(1-c) \leq s \leq u} f_s(x_1) \).

Let \( l_{n_0}^\infty \) be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset \( \Omega \) of \( l_{n_0}^\infty \) as follows:

\[
\Omega = \{ x = \{x_n\} \in l_{n_0}^\infty : 2(1-c) \leq x_n \leq 4, \ n \geq n_0 \}.
\]
Define two maps $T_1$ and $T_2 : \Omega \rightarrow l^\infty_{\mathbb{N}}$ as follows:

\[
T_1 x_n = \begin{cases} 
3 + c - c_n x_{n-k}, & n \geq n_1, \\
T_1 x_{n_1}, & n_0 \leq n \leq n_1, 
\end{cases}
\]

\[
T_2 x_n = \begin{cases} 
\sum_{i=n}^{\infty} (i - n + m - 1)(m-1)(\sum_{s=1}^{u} p^{(s)} f_s(x_{i-r_s}) - q_i), & n \geq n_1, \\
T_2 x_{n_1}, & n_0 \leq n \leq n_1. 
\end{cases}
\]

(i) We shall show that for any $x, y \in \Omega$, $T_1 x + T_2 y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $n \geq n_1$, we get

\[
T_1 x_n + T_2 y_n \leq 3 + c - c_n x_{n-k} + \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)(m-1)(\sum_{s=1}^{u} p^{(s)} f_s(y_{i-r_s}) + |q_i|) 
\leq 3 + c + \frac{1}{(m-1)!} \sum_{i=n_1}^{\infty} (i + m - 1)(m-1)(\sum_{s=1}^{u} p^{(s)} |M_3| + |q_i|) 
\leq 3 + c + 4 - c = 4.
\]

Furthermore, we have

\[
T_1 x_n + T_2 y_n \geq 3 + c - c_n x_{n-k} - \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i - n + m - 1)(m-1)(\sum_{s=1}^{u} p^{(s)} f_s(y_{i-r_s}) + |q_i|) 
\geq 3 + c - 4 - c - (1 - c) = 2(1 - c).
\]

Hence,

\[2(1 - c) \leq T_1 x_n + T_2 y_n \leq 4, \quad \text{for } n \geq n_0.
\]

Thus we have proved that $T_1 x + T_2 y \in \Omega$ for any $x, y \in \Omega$.

(ii) We shall show that $T_1$ is a contraction mapping on $\Omega$.

In fact, for $x, y \in \Omega$ and $n \geq n_1$, we have

\[
|T_1 x_n - T_1 y_n| \leq c_n |x_{n-k} - y_{n-k}| \leq c \|x - y\|.
\]

This implies that

\[
\|T_1 x - T_1 y\| \leq c \|x - y\|.
\]

Since $0 < c < 1$, we conclude that $T_1$ is a contraction mapping on $\Omega$.

Similar to the proof of Theorem 1, we can show that $T_2$ is completely continuous. By Lemma 2, there is $x = \{x_n\} \in \Omega$ such that $T_1 x + T_2 x = x$. Clearly, $x = \{x_n\}$ is a bounded positive solution of Eq. (1). This completes the proof of Theorem 3. \qed
Theorem 4. Assume that $1 < c \equiv c_n < \infty$ and that (5) and (6) hold. Then (1) has a bounded nonoscillatory solution.

Proof. By (5) and (6), we choose a $n_1 > n_0$ sufficiently large such that

$$\frac{1}{c(m-1)!} \sum_{i=n_1+k}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| M_4 + |q_i| \right) \leq c - 1,$$

where $M_4 = \max_{2(c-1) \leq s \leq u} \{ f_s(x) : 1 \leq s \leq u \}$.

Let $l_{n_1}^\infty$ be the set as in the proof of Theorem 1. We define a closed, bounded and convex subset $\Omega$ of $l_{n_1}^\infty$ as follows:

$$\Omega = \{ x = \{ x_n \} \in l_{n_1}^\infty : 2(c-1) \leq x_n \leq 4c, \ n \geq n_0 \}.$$

Define two maps $T_1$ and $T_2 : \Omega \rightarrow l_{n_1}^\infty$ as follows:

$$T_1 x_n = \begin{cases} 3c + 1 - \frac{1}{c_n} x_{n+k}, & n \geq n_1, \\ T_1 x_{n_1}, & n_0 \leq n < n_1. \end{cases}$$

$$T_2 x_n = \begin{cases} \left( \frac{1}{c_n (m-1)!} \sum_{i=n_1+k}^{\infty} (i - n - k + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| f_s(x_{i-\tau_i}) + |q_i| \right) \right), & n \geq n_1, \\ T_2 x_{n_1}, & n_0 \leq n < n_1. \end{cases}$$

(i) We shall show that for any $x, y \in \Omega$, $T_1 x + T_2 y \in \Omega$.

In fact, for every $x, y \in \Omega$ and $n \geq n_1$, we get

$$T_1 x_n + T_2 y_n$$

$$\leq 3c + 1 - \frac{1}{c_n} x_{n+k}$$

$$+ \frac{1}{c_n (m-1)!} \sum_{i=n_1+k}^{\infty} (i - n - k + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| f_s(x_{i-\tau_i}) + |q_i| \right)$$

$$\leq 3c + 1 + \frac{1}{c (m-1)!} \sum_{i=n_1+k}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| M_4 + |q_i| \right)$$

$$\leq 3c + 1 + (c - 1) = 4c.$$

Furthermore, we have

$$T_1 x_n + T_2 y_n$$

$$\geq 3c + 1 - \frac{1}{c_n} x_{n+k}$$

$$- \frac{1}{c_n (m-1)!} \sum_{i=n_1+k}^{\infty} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| f_s(x_{i-\tau_i}) + |q_i| \right)$$

$$\geq 3c + 1 - 4 \frac{1}{c (m-1)!} \sum_{i=n_1}^{\infty} (i + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |f_i^{(s)}| M_4 + |q_i| \right)$$

$$\geq 3c - 3 - (c - 1) = 2(c - 1).$$
Hence,
\[ 2(c - 1) \leq T_1 x_n + T_2 y_n \leq 4c, \quad \text{for } n \geq n_0. \]

Thus we have proved that \( T_1 x + T_2 y \in \Omega \) for any \( x, y \in \Omega \).

(ii) We shall show that \( T_1 \) is a contraction mapping on \( \Omega \).
In fact, for \( x, y \in \Omega \) and \( n \geq n_1 \), we have
\[ |T_1 x_n - T_1 y_n| \leq \frac{1}{c} |x_{n-k} - y_{n-k}| \leq \frac{1}{c} \|x - y\|. \]

This implies that
\[ \|T_1 x - T_1 y\| \leq \frac{1}{c} \|x - y\|. \]

Since \( 0 < 1/c < 1 \), we conclude that \( T_1 \) is a contraction mapping on \( \Omega \).

Similar to the proof of Theorem 2, we can show that \( T_2 \) is completely continuous. By Lemma 2, there is \( x = \{x_n\} \in \Omega \) such that \( T_1 x + T_2 x = x \). Clearly, \( x = \{x_n\} \) is a bounded positive solution of Eq. (1). This completes the proof of Theorem 4. \( \square \)

**Theorem 5.** Assume that \( c_n \equiv 1 \) and that (5) and (6) hold. Then (1) has a bounded non-oscillatory solution.

**Proof.** By (5) and (6), we choose a \( n_1 > n_0 \) sufficiently large such that
\[ \frac{1}{(m-1)!} \sum_{i=n+k}^{\infty} (i+m-1)^{m-1} \left( \sum_{s=1}^{u} p_i^{(s)} |f_s(x)| + |q_i| \right) \leq 1, \]

where \( M_5 = \max_{2 \leq s \leq 4} \{f_s(x)\}: 1 \leq s \leq u \).

We define a closed, bounded and convex subset \( \Omega \) of \( l^\infty_{n_0} \) as follows:
\[ \Omega = \{ x = \{x_n\} \in l^\infty_{n_0}: 2 \leq x_n \leq 4, \ n \geq n_0 \}. \]

Define a map \( T: \Omega \to l^\infty_{n_0} \) as follows:
\[ T x_n = \begin{cases} 3 + \frac{(-1)^{n+1}}{(m-1)!} \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i-n+m-1)^{m-1} \left( \sum_{s=1}^{u} p_i^{(s)} f_s(x_{i-r}) - q_i \right), & n \geq n_1, \\ T x_{n_1}, & n_0 \leq n \leq n_1. \end{cases} \]

(i) We shall show that \( T \Omega \subseteq \Omega \).

In fact, for every \( x \in \Omega \) and \( n \geq n_1 \), we get
\[ T x_n \leq 3 + \frac{1}{(m-1)!} \sum_{j=1}^{u+n} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i-n+m-1)^{m-1} \left( \sum_{s=1}^{u} p_i^{(s)} f_s(x_{i-r}) + |q_i| \right) \]
\[ \leq 3 + \frac{1}{(m - 1)!} \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| M_5 + |q_i| \right) \]
\[ \leq 4. \]

Furthermore, we have
\[ T x_n \geq 3 - \frac{1}{(m - 1)!} \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| f_s(x_{i-r_s}) + |q_i| \right) \]
\[ \geq 3 - \frac{1}{(m - 1)!} \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} |p_i^{(s)}| M_5 + |q_i| \right) \]
\[ \geq 2. \]

Hence, \( T \Omega \subset \Omega \).

Similar to the proof of Theorem 1, we can show that \( T_2 \) is completely continuous. By Lemma 3, there is a \( x = \{x_n\} \in \Omega \) such that \( Tx = x \), that is
\[ x_n = \begin{cases} 3 + \frac{(-1)^{m+1}}{(m - 1)!} \sum_{j=1}^{\infty} \sum_{i=n+(2j-1)k}^{n+2jk-1} (i - n + m - 1)^{(m-1)} \\ \times \left( \sum_{s=1}^{u} p_i^{(s)} f_s(x_{i-r_s}) - q_i \right), & n \geq n_1, \\ x_{n_1}, & n_0 \leq n \leq n_1. \end{cases} \]

It follows that
\[ x_n + x_{n-k} = 3 + \frac{(-1)^{m+1}}{(m - 1)!} \sum_{i=n}^{\infty} (i - n + m - 1)^{(m-1)} \left( \sum_{s=1}^{u} p_i^{(s)} f_s(x_{i-r_s}) - q_i \right) , \]
\[ n \geq n_1. \]

Clearly, \( x = \{x_n\} \) is a bounded positive solution of Eq. (1). This completes the proof of Theorem 5. \( \Box \)

**Remark 1.** For the special case where \( u = 1, c_n \equiv c \neq -1 \), Theorems 1–5 improve essentially Theorems A, B, and C of [8,11,13] by removing the conditions (C 1 )–(C 6 ).

**Remark 2.** For the critical case \( c_n \equiv -1 \), it is also possible that Eq. (1) has no nonoscillatory solution in spite of the fact that (5) and (6) hold. For example, we consider neutral difference equation
\[ \Delta^m (x_n - x_{n-k}) + \frac{1}{\mu^\alpha} x_{n-r} = 0, \]  
\[ n \geq n_1. \]  
where \( k > 0, r \geq 0, m < \alpha < m + 1 \). Clearly, (5) and (6) hold. But, by Theorem 1 in [14], Eq. (7) has no nonoscillatory solution.
References