Entropy of an extremal electrically charged thin shell and the extremal black hole

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A R T I C L E   I N F O

A B S T R A C T

There is a debate as to what is the value of the entropy S of extremal black holes. There are approaches that yield zero entropy $S = 0$, while there are others that yield the Bekenstein–Hawking entropy $S = A_+/4$, in Planck units. There are still other approaches that give that S is proportional to r_+ or even that S is a generic well-behaved function of r_+. Here r_+ is the black hole horizon radius and $A_+ = 4\pi r_+^2$ is its horizon area. Using a spherically symmetric thin matter shell with extremal electric charge, we find the entropy expression for the extremal thin shell spacetime. When the shell's radius approaches its own gravitational radius, and thus turns into an extremal black hole, we encounter that the entropy is $S = S(r_+)$, i.e., the entropy of an extremal black hole is a function of r_+ alone. We speculate that the range of values for an extremal black hole is $0 \leq S(r_+) \leq A_+/4$.

1. Introduction

The entropy S and thermodynamics of black holes have been worked out first by Bekenstein [1] and Hawking and collaborators [2,3]. The Bekenstein–Hawking entropy is given by $S = A_+/4$, where $A_+ = 4\pi r_+^2$, A_+ and r_+ are the horizon area and the horizon radius, respectively, and we are putting all the natural constants equal to one, i.e., we use Planck units. York and collaborators [4–6] (see also [7,8]) have further worked out the black hole thermodynamic properties by using canonical and grand canonical thermodynamic ensembles. There are several other methods that can be used to study black hole thermodynamics, one that suits us here uses matter shells [9–11]. In this method, one studies the generic thermodynamics of the shells at any shell radius, and as one sends the shell to its own gravitational radius one recovers the $S = A_+/4$ Bekenstein–Hawking entropy. This is the quasiblack hole method, the evident power of it was displayed in [12].

A particular class of black holes is the extremal black hole class. Electrically charged black holes in general relativity, the ones we are interested here, have $m \geq Q$, and the extremal black holes

are characterized by having their mass m equal to their electric charge Q, $m = Q$. The extremal black holes seem to have distinct properties. For instance, according to the Hawking temperature formula, extremal black holes have zero temperature. In addition the entropy of an extremal black hole is a subject of a wide debate as there are different reasonings that can be applied which lead to different values for the entropy. Hawking and collaborators [13] and Teitelboim [14] have given topological arguments which point to the conclusion that extremal black holes have zero entropy. Further evidence from other arguments for $S = 0$ for extremal black holes was provided in [15–17], see also [18,19]. One could also argue, naively, that since the Hawking temperature is zero, then according to one of the formulations of the third law of thermodynamics as many textbooks present it should have zero entropy.

However, there remain doubts why the Bekenstein–Hawking formula does not hold. After all, working out the entropy of non-extremal black holes and taking the extremal limit $m = Q$ yields $S = A_+/4$, see, e.g., [2,3,5,10]. In this case, the thermodynamic argument would not hold, the extremal black hole could be a system of minimum energy and degenerate ground state and such systems can have entropy even at zero temperature. Moreover, in string theory, there are arguments, other than geometrical, that make use of a direct counting of string and D-brane states in composite

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systems, which manipulate carefully the turning on of gravity and electricity adiabatically by equal amounts to maintain extremality without changing the counting of states and thus the entropy, that show that the entropy of an extremal black hole is $S = A_+/4$, as first delivered by Strominger and Vafa [20], see [21] for a review. Other methods also indicate the $S = A_+/4$ value. These are methods that use quantum field corrections to the black hole entropy [22], Hamiltonian methods [23,24], and thermodynamics ensemble methods [25–27]. There are works that show that depending on the black hole type, i.e., black holes with scalar fields, one has $S = 0$ or $S = A_+/4$ [28,29].

Given that several different calculations give $S = 0$ or $S = A_+/4$ one might open the box and speculate that other values for the entropy are possible, e.g., any value between 0 and $A_+/4$, or possibly other functions. Indeed, in a semiclassical calculation of the entropy on an extremal black hole background Ghosh and Mitra [30] pointed to an entropy value proportional to r_+ rather than A_+ (i.e., r_+^2). This was followed by some discussion for the exact possible values for the entropy of an extremal black hole [31–33].

In addition, in a setup using an extremal charged thin shell contracting reversibly and arranged to maintain extremality it was shown in [34] that any value of the entropy of the shell in passing its own gravitational radius could be achieved. In another setting, using also thin shells, it was shown that in the quasiblack hole limit, i.e., when the boundary of the matter is at its own gravitational radius, the entropy of the extremal black hole is a generic function of r_+ [35].

One feature important to note is that a calculation of the stress-energy tensor of quantum fields at the neighborhood of the event horizon excludes the possibility that an extremal black hole can be in thermal equilibrium with radiation at any temperature. An extremal black hole has zero temperature and if the temperature of the surrounding fields is nonzero then the stress-energy diverges strongly [36].

This paper is committed to the study of the entropy of extremally charged spherically symmetric thin shells of any radii, and in particular, to the understanding of the entropy of the system when the radius of the shell is its own gravitational radius, i.e., the extremal shell spacetime turns into an extremal black hole spacetime. This yields an expression for the black hole entropy. We follow the formalism of Martinez [9] developed for electric non-extremal shells in [10] and for rotating BTZ spacetimes in [11]. The thermodynamic analysis of Callen [37] is used, as it was used in [9–11]. The importance of Callen's analysis for black hole thermodynamics was first understood by York [4], see also [5,8]. Here, we restrict ourselves to spherically symmetric systems but, as the results have a rather general character, we believe that, with some minor changes, they are valid for distorted and rotating systems as well (see [11] for a rotating $(2 + 1)$-dimensional spacetime).

The work is organized as follows: In Section 2 we give the mechanical properties of an extremal electrically charged thin shell. Such type of matter is also called electrically counterpoised dust. In Section 3 we will analyze the first law of thermodynamics applied to such a thin shell of any radius and find the entropy of the spacetime. In Section 4 we take the shell to its own gravitational radius and find that the entropy of an extremal black hole is a generic function of r_+. We also speculate on the possible values for the entropy of an extremal black hole. In Section 5 we display another interesting shell that can be taken to its own gravitational radius and find the corresponding entropy. In Section 6 we draw our conclusions.

2. The extremal charged thin shell spacetime

The Einstein–Maxwell equations in four spacetime dimensions are given by

$$G_{\alpha\beta} = 8\pi\, T_{\alpha\beta}\,, \tag{1}$$

$$\nabla_\beta F^{\alpha\beta} = 4\pi\, J^\alpha\,. \tag{2}$$

$G_{\alpha\beta}$ is the Einstein tensor, built from the spacetime metric $g_{\alpha\beta}$ and its first and second derivatives, 8π is the coupling, and we are using units in which the velocity of light is one and the gravitational constant G is also put to one $G = 1$. $T_{\alpha\beta}$ is the energy-momentum tensor. $F_{\alpha\beta}$ is the Faraday–Maxwell tensor, J_α is the electromagnetic four-current and ∇_β denotes covariant derivative. The other Maxwell equation $\nabla_{[\gamma} F_{\alpha\beta]} = 0$, where $[\ldots]$ means antisymmetrization, is automatically satisfied for a properly defined $F_{\alpha\beta}$. Greek indices will be used for spacetime indices and run as $\alpha, \beta = 0, 1, 2, 3$, with 0 being the time index.

The concept of thin shell is associated with the presence of matter in the surface that separates two partitions of spacetime, each with its own metric. We will be considering the case of a four-dimensional spherically symmetric spacetime and a spherical thin shell at some radius R separating an inner region $\mathcal{V}_i$ with flat metric and an outer region $\mathcal{V}_o$ with an extremal Reissner–Nordström line element. Thus, for the inner region the metric is

$$ds_i^2\, g_{\alpha\beta}^i dx^\alpha dx^\beta = -dt_i^2 + dr^2 + r^2\, d\Omega^2\,, \quad r \leq R\,, \tag{3}$$

where $x^\alpha = (t_i, r, \theta, \phi)$ are the inner coordinates, with t_i being the inner time, and (r, θ, ϕ) polar coordinates, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. For the outer region the metric is

$$ds_o^2 = g_{\alpha\beta}^o dx^\alpha dx^\beta$$
$$= -\left(1 - \frac{m}{r}\right)^2 dt_o^2 + \frac{dr^2}{\left(1 - \frac{m}{r}\right)^2} + r^2 d\Omega^2\,, \quad r \geq R\,, \tag{4}$$

where $x^\alpha = (t_o, r, \theta, \phi)$ are the outer coordinates, with t_o being the outer time, and (r, θ, ϕ) polar coordinates. In addition, m is the ADM mass, and Q is the electric charge. In the extremal case they are related by

$$m = Q\,. \tag{5}$$

On the hypersurface itself, $r = R$, the metric is that of a 2-sphere with an additional time dimension, such that the line element is

$$ds_\Sigma^2 = h_{ab} dy^a dy^b = -d\tau^2 + R^2(\tau) d\Omega^2\,, \quad r = R\,, \tag{6}$$

where we have chosen $y^a = (\tau, \theta, \phi)$ as the time and spatial coordinates on the shell. Latin indices apply for the components on the hypersurface. The time coordinate τ is the proper time for an observer located at the shell. The shell radius is given by the parametric equation $R = R(\tau)$ for an observer on the shell. We consider a static shell so that $R(\tau) =$ constant. On each side of the hypersurface, the parametric equations for the time and radial coordinates are denoted by $t_i = T_i(\tau)$, $r_i = R_i(\tau)$, and $t_o = T_o(\tau)$, $r_o = R_o(\tau)$.

Imposing that the fluid in the shell is a perfect fluid with stress-energy tensor $S^a{}_b$ given by

$$S^a{}_b = (\sigma + p)u^a u_b + p h^a{}_b\,, \tag{7}$$

where u^a is the 3-velocity of a shell element, one finds through the junction conditions (see, e.g., [10])

$$\sigma = \frac{m}{4\pi R^2}\,, \tag{8}$$

$$p = 0\,. \tag{9}$$

Matter for which $p = 0$ and totally supported by electric forces against gravitational collapse is called extremal matter or, sometimes, electrically counterpoised dust. The rest mass of the shell M is defined as

$$\sigma = \frac{M}{4\pi R^2},\tag{10}$$

and so in the extremal case

$$M = m.\tag{11}$$

The gravitational radius r_+ of the shell is given by the zero of the g_{00}^o in Eq. (4). It is actually a double zero, one gives the gravitational radius r_+, the other the Cauchy horizon r_- of the shell. The double zero means that for the extremal spacetime the two radii coincide,

$$r_+ = r_-,\tag{12}$$

and we call it r_+ from now on. The zero of the g_{00}^o in Eq. (4) then gives

$$r_+ = m,\tag{13}$$

and so

$$r_+ = r_- = m = Q = M.\tag{14}$$

The gravitational radius r_+ is also the horizon radius when the shell radius R is inside r_+, i.e., when the spacetime contains a black hole. Clearly, gravitational and horizon radii have the same expression. Nevertheless, they are distinct. The gravitational radius is a property of the matter and the spacetime generated by it, independently of whether there is a black hole or not, whereas there is a horizon radius in the case there is a black hole.

One can define the gravitational area A_+ as the area spanned by the gravitational radius, namely,

$$A_+ = 4\pi r_+^2.\tag{15}$$

This is also the event horizon area when there is a black hole. It is useful to define the shell's redshift function k as

$$k = 1 - \frac{r_+}{R}.\tag{16}$$

The area A of the shell, an important quantity, is given by

$$A = 4\pi R^2.\tag{17}$$

Also for a shell with electric charge density σ_e one finds (see, e.g., [10])

$$\frac{Q}{R^2} = 4\pi \sigma_e.\tag{18}$$

This equation relates the total charge Q, the charge density σ_e, and the shell's radius R.

An obvious inequality, for the shell is that it should be always outside its own gravitational radius, so

$$R \geq r_+.\tag{19}$$

Then the physical allowed values for k in Eq. (16) are in the interval $[0, 1]$. Since the pressure of the matter in the shell is zero and the energy density is considered positive, see Eq. (9), the energy conditions, weak, strong, and dominant, are always obeyed for $R \geq r_+$.

It is worth noting that in the limit $R \to r_+$ there are subtleties connected with the behavior of the boundary's geometry. Indeed, there is a discontinuity because of the timelike character of the boundary from the inside and the lightlike character of the boundary from the outside (see [12] for details). However, here they are essentially irrelevant since in what follows we consider the external region only.

3. Entropy of an extremal charged thin shell

3.1. Entropy and the first law of thermodynamics for an extremal charged thin shell

In the study of the thermodynamics and entropy of a thin shell we use units in which the Boltzmann constant is one. We assume that the shell in static equilibrium at radius R has a well defined local temperature T and an entropy S. The entropy S is a function of the shell's rest mass M, area A, and charge Q, i.e.,

$$S = S(M, A, Q).\tag{20}$$

The first law of thermodynamics can be then written as

$$TdS = dM + pdA - \Phi dQ,\tag{21}$$

or, defining the inverse temperature β,

$$\beta \equiv \frac{1}{T}\tag{22}$$

one has

$$dS = \beta(dM + pdA - \Phi dQ),\tag{23}$$

where dS, dM, dA, and dQ are the differentials of the entropy, rest mass, area, and electric charge of the shell, respectively, whereas T, p and Φ are its temperature, pressure, and thermodynamic electric potential, respectively. To obtain the entropy S of the shell, one thus needs in general to specify three equations of state, namely, $p = p(M, A, Q)$, $\beta = \beta(M, A, Q)$, and $\Phi = \Phi(M, A, Q)$.

The extremal case is a special case, the extremality condition will constraint the possible configurations. From Eq. (14) we have for an extremal shell

$$dQ = dM.\tag{24}$$

Thus, the number of independent variables reduces to two, namely, M and A, and so, $p = p(M, A)$, $\beta = \beta(M, A)$, and $\Phi = \Phi(M, A)$. It is more convenient to work out with the shell's radius R than its area A, which can be done from Eq. (17), so the equations of state are of the form

$$p = p(M, R),\ \beta = \beta(M, R),\ \Phi = \Phi(M, R).\tag{25}$$

Now, from Eq. (9), one has that the equation of state for the pressure is

$$p(M, R) = 0.\tag{26}$$

Thus, the first law (23) is now

$$dS = \beta(1 - \Phi)dM,\tag{27}$$

and since from Eq. (14) $M = r_+$ and $dM = dr_+$, one can write the first law as

$$dS = \beta(1 - \Phi)dr_+,\tag{28}$$

where now

$$\beta = \beta(r_+, R),\ \Phi = \Phi(r_+, R).\tag{29}$$

The integrability condition for Eq. (28) reduces to a simple equation, namely,

$$\beta(1 - \Phi) = s(r_+),\tag{30}$$

where s is a function of r_+ alone and is arbitrary as long as it gives a positive meaningful entropy. Since $\beta \geq 0$ and $s \geq 0$ we have the following constraint on Φ,

$$\Phi \leq 1. \tag{31}$$

The result given in Eq. (30), that the most general function of the product of two functions (namely, β and $1 - \Phi$) of r_+ and R is a function of r_+ alone, is new and interesting. Then Eq. (28) together with Eq. (30) yields

$$dS = s(r_+)dr_+. \tag{32}$$

The function $s(r_+)$ is thus a kind of entropy density. Integrating Eq. (32), we conclude that the entropy of the extremal shell is given by

$$S = S(r_+), \quad R \geq r_+, \tag{33}$$

where we have assumed that the constant of integration is zero. Thus the entropy of an extremal charged thin shell is a function of r_+ alone. Depending on the choice of $s(r_+)$ we can obtain a wide range of values for the entropy $S(r_+)$ of the shell. Since $\beta(r_+, R)$ and $\Phi(r_+, R)$ are arbitrary as long as they obey the constraint (30), this shows that the extremal case is indeed quite special. Such a result does not appear in the non-extremal case at all, see [10].

3.2. Choices for the matter equations of state of an extremal charged thin shell

The functions $\beta(r_+, R)$ and $\Phi(r_+, R)$ being arbitrary can be chosen at our will, bearing in mind Eq. (30).

Now, in gravitational thermodynamics, there is the well-known Tolman formula for static spacetimes. The formula relates the temperature at r of something, i.e., $T(r)$ with the temperature of that something at infinity T_∞. In terms of the inverse temperatures $\beta(r) = 1/T(r)$ and $b = 1/T_\infty$ the Tolman formula is given by $\beta(r) = b\sqrt{-g_{00}(r)}$, where $g_{00}(r)$ is the zero component of the static metric. The quantity b here is a number, a constant. Thus, the inverse temperature $\beta(r)$ is redshifted from b, i.e., conversely, the temperature at r, $T(r)$, is blueshifted from the temperature at infinity, T_∞.

Remarkably, for the thin shell, the Tolman formula can be derived directly through the integrability conditions for non-extremal thin shells (see [9] for neutral shells, and [10] for electrically charged non-extremal shells). This is an important particular case of the Tolman formula for which $r = R$. For generic non-extremal shells the integrability conditions give $\beta(r_+, r_-, R) = b(r_+, r_-) k(r_+, r_-, R)$, where, $k(r_+, r_-, R) = \sqrt{-g_{00}(R)}$, i.e., $k(r_+, r_-, R) = \sqrt{\left(1 - \frac{r_+}{R}\right)\left(1 - \frac{r_-}{R}\right)}$, see [10] for details. This is also the expression for β obtained for a nonextremal Reissner–Nordström black hole, see [5]. Taking the limit to extremal shells, i.e., $r_+ = r_-$, we find $\beta(r_+, R) = b(r_+) k(r_+, R)$, where $k(r_+, R)$ is given in Eq. (16).

Now, in the extremal case, the only integrability condition is Eq. (30), and it has nothing to do with the Tolman formula. However, among all other possible choices, Tolman formula $\beta(r_+, r_-, R) = b(r_+, r_-) k(r_+, r_-, R)$, allows for a nontrivial generalization. It is the Toman formula for extremal shells and there is an important difference between the Tolman formula for nonextremal and extremal shells. For nonextremal shells, one finds from the integrability conditions that $b = b(r_+, r_-)$, i.e., b cannot depend on R, see [10]. For extremal shells, on the other hand, nothing prevents us from including in b a dependence not only on $r_+ = r_-$,

but also on R, $b = b(r_+, R)$. As a result the generic Tolman formula in the extremal case is then

$$\beta(r_+, R) = b(r_+, R) k, \tag{34}$$

where $k = k(r_+, R)$ as given in Eq. (16). The function $b(r_+, R \to \infty)$ represents the inverse of the temperature of the shell if it were located at infinity. Of course, we can invoke the Tolman formula directly from general thermodynamic argumentation.

With the choice for β given in Eq. (34), one finds that Eq. (30) yields $\Phi(r_+, R) = \frac{\left(1 - \frac{s(r_+)}{b(r_+, R)}\right) - \frac{r_+}{R}}{k}$. This can be put in the form

$$\Phi(r_+, R) = \frac{\phi(r_+, R) - \frac{r_+}{R}}{k}, \tag{35}$$

where we have defined $\phi(r_+, R) \equiv 1 - \frac{s(r_+)}{b(r_+, R)}$, i.e., ϕ is such that

$$b(1 - \phi) = s(r_+). \tag{36}$$

From Eq. (35) one sees that $\phi(r_+, R \to \infty)$ represents the electric potential of the shell if it were located at infinity.

We could proceed and give specific equations for $b(r_+, R)$ and $\phi(r_+, R)$, and determine the thermodynamic properties of the shell including its thermodynamic stability. We will not do it here, see [9,10] for some cases dealing with uncharged shells and non-extremal charged shells, respectively. We could also give other equations of state, as long as they obey Eq. (30).

We stick to the above, namely, Eqs. (34) and (35), and study some particular instances that allow us to take the gravitational radius, i.e., the black hole, limit.

4. Entropy of an extremal black hole

We want now to study the extremal black hole limit, i.e., the case where the extremal shell is taken to its own gravitational radius, $R = r_+$.

If we take the extremal shell to its gravitational radius $R = r_+$, i.e., when taking the black hole limit, we have to make several choices. First we fix the shell at some radius $R > r_+$ and choose the functions β and Φ, or b and ϕ, appropriately, only afterwards we send the shell to $R = r_+$. Second, knowing that the Hawking temperature measured at infinity for an extremal black hole is $T_H = 0$, we choose the temperature at infinity T_∞ as $T_\infty = 0$, i.e., $b = \infty$. Thus from Eq. (34) we find $\beta = \infty$ and so the temperature on the shell is zero, $T = 0$. Third we find ϕ and Φ. From Eq. (36) we find that $\phi = 1$, such that $1 - \phi = s/b$, for some well specified s. Then also $\Phi = 1$. In other words, we can choose ϕ as close to 1 as we like and T_∞ as small as we want (or b as large as we want). Afterwards, keeping the product fixed (for a given r_+), we can send ϕ to 1 and T_∞ to zero (or b to infinity). We see that the shell at $R > r_+$ has been prepared with $T_\infty = 0$ and $\phi = 1$, such that $b(1 - \phi) = s$ and so $\beta(1 - \Phi) = s$. The shell is now correctly prepared. Having correctly prepared the shell we can take it to its gravitational radius if we wish.

Now, indeed take the shell to its own gravitational radius $R = r_+$, i.e., take the black hole limit. Since the entropy differential for the shell depends only on r_+ through the function $s(r_+)$ that is arbitrary, see Eq. (32), we conclude that the entropy of the extremal shell in the extremal black hole limit is given by

$$S = S(r_+), \quad R = r_+. \tag{37}$$

This is the extremal black hole limit of an extremal shell. Such a configuration is a quasiblack hole.

Our approach implies that the entropy of an extremal black hole can assume any well-behaved function of r_+. The precise

Table 1

Comparison between a special extremal shell at its own gravitational radius and an extremal black hole both with $T_\infty = 0$.

Case	T at ∞	Local T on r_+	Backreaction on r_+	Φ	Entropy
Special shell	0	0	Finite	1 on R	Well-defined $S(r_+)$
Black hole	0	0	Finite	1 on r_+	In debate, $S = 0$, $\frac{A_+}{4}$, $S(r_+)$

Table 2

Comparison between a generic extremal shell at its own gravitational radius and an extremal black hole with infinite temperature at the horizon.

Case	T at ∞	Local T on r_+	Backreaction on r_+	Φ	Entropy
Generic shell	0	Finite, not 0	Finite	< 1 on R	Well-defined $S(r_+)$
Black hole	$T_\infty \neq 0$	∞	Infinite	1	Not known or undefined

function of the entropy depends on the constitution of the matter that collapsed to form the black hole. Depending on the choice of s that, in turn, depends on the choices for β and Φ, we can obtain any function of r_+ for the entropy S of the extremal black hole. The fact that the entropy in the extremal case is model-dependent agrees with the previous work [35] and more early studies [34]. Of course, a particular class of entropies for the extremal black hole would be the Bekenstein–Hawking entropy $S(r_+) = \frac{1}{4} A_+ = \pi r_+^2$ in units where Planck's constant, along with the velocity of light and G, is also put to one, i.e., in Planck units. In summary, our result is quite different from the non-extremal case, where the entropy can only have the Bekenstein–Hawking functional dependence $S(r_+) = \frac{1}{4} A_+$ [10].

Note, that, although the importance of the product $\beta (1 - \Phi)$ has been raised in the extremal black hole context in [30] (see also [31]), the result that the most general function of the product $\beta (1 - \Phi)$ is a well-behaved, but otherwise arbitrary, function of r_+ is new. There are additional differences between [30,31] and our work. In [30,31], the product $\beta (1 - \Phi)$ enters the path integral over fluctuating geometries, so it appears in a quantum context. In doing so, finite nonzero β are not forbidden. However, for such β quantum backreaction destroys the extremal horizon [36]. In our approach, we consider a shell, not a black hole, and can adjust β and Φ at the shell radius in such a way that for any R close to r_+ backreaction remains finite [35].

Our preceding calculations and discussion were rigorous. Now, we speculate on ways to constrain the entropy function $S(r_+)$ for the extremal black hole. For instance, the initial Bekenstein arguments for black holes [1], non-extremal ones, proved that an entropy proportional to $A_+^{1/2}$ should be discarded on the basis of the second law of thermodynamics. However, since extremal black holes have a different character from non-extremal ones, these arguments do not hold here. Another possible constraint is the following. For the usual, non-extremal, black holes the entropy is $S(r_+) = \frac{1}{4} A_+$. In this case, when one takes the shell to its own gravitational radius the pressure at the shell blows up, $p \to \infty$ [10], and the spacetime is assumed to take the Hawking temperature. In a sense this means that all possible degrees of freedom are excited and the black hole takes the Bekenstein–Hawking entropy, the maximum possible entropy. Taking the extremal limit from a non-extremal black hole, one finds that in this particular limit the extremal black hole entropy is the Bekenstein–Hawking entropy. Thus, this suggests that the maximum entropy that an extremal black hole can take is the Bekenstein–Hawking entropy. Therefore, in this sense, the range of values for the entropy of an extremal black hole is

$$0 \leq S(r_+) \leq \frac{1}{4} A_+, \tag{38}$$

or $0 \leq S(r_+) \leq \pi r_+^2$. The case studied by Ghosh and Mitra [30,31] has $S \propto r_+$ and so is within our limits.

Table 1 summarizes the comparison between an extremal shell at its own gravitational radius with $T_\infty = 0$, which we have called a special shell, and an extremal black hole with $T_\infty = 0$.

5. Another interesting shell at the gravitational radius limit: a generic shell

There is a more generic shell that has interesting properties and can be taken also to the gravitational radius limit with no unbound back reaction.

We now suppose that the shell has a small nonzero local temperature (i.e., finite large β), rather than zero. We follow Eq. (34) and Eq. (35) and keep in mind the constraints (31) and (36).

From Eq. (34) we see that the product bk is the important quantity. At any R prepare the shell so that $b = \bar{b}/k$, i.e., $b = \bar{b}/(1 - r_+/R)$, for some $\bar{b}$ finite. Then $\beta = \bar{b}$ and is finite, and holds for any $R > r_+$. Note that the temperature measured at infinity $T_\infty = 1/b$ is finite. Prepare also the shell so that $(1 - \phi) = (1 - \bar{\phi})k$, so that $(1 - \Phi) = (1 - \bar{\phi})$ for any $R > r_+$. Note that the potential measured at infinity ϕ is less than one.

Now, take the gravitational radius limit, $R = r_+$. In this limit k goes to zero, but we have prepared the shell to keep β bounded even in this black hole limit, as a small k has been compensated by a large b. In this limit the temperature measured at infinity $T_\infty = 1/b$ is zero and thus coincides with the Hawking temperature, $T_H = 0$. The temperature $T = 1/\beta$ at the shell is nonzero and finite. Thus in this case, quantum backreaction also remains bounded even for $R = r_+$. Since β is finite and we have $\beta(1 - \Phi) = s(r_+)$ we have $\Phi < 1$. The entropy of the shell at $R = r_+$ is then also $S = S(r_+)$. It is this way of reasoning that was used in [12] in a general discussion of the entropy for the extremal case. In doing so, any β and $\Phi < 1$ obeying Eq. (30) are suitable, and an entropy as an arbitrary (within limits of physical reasonability) function of r_+, $S = S(r_+)$ was obtained. This case sharply contrasts with the extremal black hole case where any $T_\infty = b^{-1} \neq 0$ leads to $\beta \to 0$ and infinite local temperature β^{-1} on the horizon with divergent backreaction that destroys the horizon (for this extremal black hole case see [13] where nothing is said about quantum backreaction and it is argued that the entropy is zero, however results in [36] show that the backreaction grows unbound if T_H is not zero).

Table 2 summarizes the comparison between an extremal shell at its own gravitational radius with $T_\infty = 0$ and β finite nonzero, which we have called a generic shell, and an extremal black hole with T_∞ not zero.

6. Conclusions

Upon consideration of spherically symmetric systems and through the formalism of thin matter shells and their thermodynamics properties, we have shown our solution for the ongoing debate concerning the entropy of an extremal black hole. Although

it would be necessary a full quantum theory of gravity to fully understand the result obtained, it is nonetheless interesting to see that the use of the junction conditions for Einstein equation leads inevitably to the suggestion that extremal black holes are a different class of objects than non-extremal black holes, due to the fact that their entropy depends on the particularities of the matter distribution which originated the black hole.

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