Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 155 (2008) 1022–1026
Topology and its Applications
www.elsevier.com/locate/topol

Generic section of a hyperplane arrangement and twisted Hurewicz maps

Masahiko Yoshinaga

Department of Mathematics, Graduate School of Science, Kobe University, 1-1 Rokkodai, Kobe 657-8501, Japan

Received 29 November 2006; received in revised form 7 January 2008; accepted 7 January 2008

Abstract

We consider a twisted version of the Hurewicz map on the complement of a hyperplane arrangement. The purpose of this paper is to prove surjectivity of the twisted Hurewicz map under some genericity conditions. As a corollary, we also prove that a generic section of the complement of a hyperplane arrangement has nontrivial homotopy groups.

Keywords: Hyperplane arrangements; Local system homology groups; Twisted Hurewicz map; Homotopy groups; Lefschetz theorem

1. Twisted Hurewicz map

Let X be a topological space with a base point $x_0 \in X$ and $\mathcal{L}$ a local system of $\mathbb{Z}$-modules on X. Let $f:(S^n,*) \to (X,x_0)$ be a continuous map from the sphere S^n with $n \geqslant 2$. Since S^n is simply connected, the pullback $f^*\mathcal{L}$ turns out to be a trivial local system. Thus given a local section $t \in \mathcal{L}_{x_0}$, $f \otimes t$ determines a twisted cycle with coefficients in $\mathcal{L}$. This induces a twisted version of the Hurewicz map:

$$h:\pi_n(X,x_0)\otimes_{\mathbb{Z}}\mathcal{L}_{x_0} \to H_n(X,\mathcal{L}).$$

The classical Hurewicz map is corresponding to the case of trivial local system $\mathcal{L}=\mathbb{Z}$ with $t=1$.

2. Main result

Let $\mathcal{A}$ be an essential affine hyperplane arrangement in an affine space $V=\mathbb{C}^\ell$, with $\ell \geqslant 3$. Let $M(\mathcal{A})$ denote the complement $V-\bigcup_{H\in\mathcal{A}} H$. A hyperplane $U \subset V$ is said to be *generic to* $\mathcal{A}$ if U is transversal to the stratification induced from $\mathcal{A}$. Let $i:U\cap M(\mathcal{A}) \hookrightarrow M(\mathcal{A})$ denote the inclusion.

In this notation, the main result of this paper is the following:

E-mail address: myoshina@math.kobe-u.ac.jp.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.01.004

Theorem 1. *Let $\mathcal{L}' := i^*\mathcal{L}$ be the restriction of a nonresonant local system $\mathcal{L}$ of arbitrary rank on $M(\mathcal{A})$. Then the twisted Hurewicz map*

$$h : \pi_{\ell-1}\big(U \cap M(\mathcal{A}), x_0\big) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0} \to H_{\ell-1}\big(U \cap M(\mathcal{A}), \mathcal{L}'\big)$$

is surjective.

For the notion of "nonresonant local system," see Theorem 7.

Theorem 1 should be compared with a result proved by Randell in [12]. He proved that the Hurewicz homomorphism $\pi_k(M(\mathcal{A})) \to H_k(M(\mathcal{A}), \mathbb{Z})$ is equal to the zero map when $k \geqslant 2$ for any $\mathcal{A}$. However little is known about twisted Hurewicz maps for other cases.

The key ingredient for our proof of Theorem 1 is an affine Lefschetz theorem of Hamm, which asserts that $M(\mathcal{A})$ has the homotopy type of a finite CW complex whose $(\ell-1)$-skeleton has the homotopy type of $U \cap M(\mathcal{A})$. We obtain $(\ell-1)$-dimensional spheres in $U \cap M(\mathcal{A})$ as boundaries of the ℓ-dimensional top cells. Applying a vanishing theorem for local system homology groups, we show that these spheres generate the twisted homology group $H_{\ell-1}(U \cap M(\mathcal{A}), \mathcal{L})$. We should note that the essentially same arguments are used in [4] to compute the rank of $\pi_{\ell-1}(U \cap M(\mathcal{A}), x_0) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0}$ under a certain asphericity condition on $\mathcal{A}$.

3. Topology of complements

The cell decompositions of affine varieties or hypersurface complements are well studied subjects. Let $f \in \mathbb{C}[x_1, \ldots, x_\ell]$ be a polynomial and $D(f) := \{x \in \mathbb{C}^\ell \mid f(x) \neq 0\}$ be the hypersurface complement defined by f.

Theorem 2 *(Affine Lefschetz Theorem). (See [5].) Let U be a sufficiently generic hyperplane in $\mathbb{C}^\ell$. Then,*

(a) *The space $D(f)$ has the homotopy type of a space obtained from $D(f) \cap U$ by attaching ℓ-dimensional cells.*
(b) *Let $i_p : H_p(D(f) \cap U, \mathbb{Z}) \to H_p(D(f), \mathbb{Z})$ denote the homomorphism induced by the natural inclusion $i : D(f) \cap U \hookrightarrow D(f)$. Then*

$$i_p \text{ is } \begin{cases} \text{isomorphic} & \text{for } p = 0, 1, \ldots, \ell-2, \\ \text{surjective} & \text{for } p = \ell-1. \end{cases}$$

Suppose $i_{\ell-1}$ is also isomorphic. Then as noted by Dimca and Papadima [3] (see also Randell [13]), the number of ℓ-dimensional cells attached would be equal to the Betti number $b_\ell(D(f))$ and the chain boundary map $\partial : C_\ell(D(f), \mathbb{Z}) \to C_{\ell-1}(D(f), \mathbb{Z})$ of the cellular chain complex associated to the cell decomposition is equal to zero. Otherwise $i_{\ell-1} : H_{\ell-1}(D(f) \cap U, \mathbb{Z}) \to H_{\ell-1}(D(f), \mathbb{Z})$ has a nontrivial kernel $\partial(C_\ell(D(f), \mathbb{Z}))$.

In the case of hyperplane arrangements, homology groups and homomorphisms i_p are described combinatorially in terms of the intersection poset [9]. Let us recall some notation. Let $\mathcal{A}$ be a finite set of affine hyperplanes in $\mathbb{C}^\ell$,

$$L(\mathcal{A}) = \left\{ X = \bigcap_{H \in I} H \;\middle|\; I \subset \mathcal{A} \right\}$$

be the set of nonempty intersections of elements of $\mathcal{A}$ with reverse inclusion $X < Y \Leftrightarrow X \supset Y$, for $X, Y \in L(\mathcal{A})$. Define a rank function on $L(\mathcal{A})$ by

$$r : L(\mathcal{A}) \to \mathbb{Z}_{\geqslant 0}, \qquad X \mapsto \operatorname{codim} X,$$

the Möbius function $\mu : L(\mathcal{A}) \to \mathbb{Z}$ by

$$\mu(X) = \begin{cases} 1 & \text{for } X = V, \\ -\sum_{Y < X} \mu(Y) & \text{for } X > V, \end{cases}$$

and the characteristic polynomial $\chi(\mathcal{A}, t)$ by

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

Let $E^1 = \bigoplus_{H\in\mathcal{A}} \mathbb{C}e_H$ and $E = \bigwedge E^1$ be the exterior algebra of E^1, with pth graded term $E^p = \bigwedge^p E^1$. Define a $\mathbb{C}$-linear map $\partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geqslant 2$

$$\partial(e_{H_1} \cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1} \cdots \widehat{e_{H_k}} \cdots e_{H_p}$$

for all $H_1, \ldots, H_p \in \mathcal{A}$. A subset $S \subset \mathcal{A}$ is said to be dependent if $r(\bigcap S) < |S|$, where $\bigcap S = \bigcap_{H\in S} H$. For $S = \{H_1, \ldots, H_p\}$, we write $e_S := e_{H_1} \cdots e_{H_p}$.

Definition 3. Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by

$$\left\{ e_S \;\middle|\; \bigcap S = \phi \right\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

The Orlik–Solomon algebra $A(\mathcal{A})$ is defined by $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$.

Theorem 4. *(See Orlik–Solomon [8].) Fix a defining linear form α_H for each $H \in \mathcal{A}$. Then the correspondence $e_H \mapsto d\log\alpha_H$ induces an isomorphism of graded algebras*:

$$A(\mathcal{A}) \xrightarrow{\cong} H^*\big(M(\mathcal{A}), \mathbb{C}\big).$$

The Betti numbers of $M(\mathcal{A})$ are given by

$$\chi(\mathcal{A}, t) = \sum_{k=0}^{\ell} (-1)^k b_k\big(M(\mathcal{A})\big) t^{\ell-k}.$$

From the above description of cohomology ring of $M(\mathcal{A})$, we have:

Theorem 5. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$ and U be a hyperplane generic to $\mathcal{A}$. Then $i : U \cap M(\mathcal{A}) \hookrightarrow M(\mathcal{A})$ induces isomorphisms $i_p : H_p(M(\mathcal{A}) \cap U, \mathbb{Z}) \xrightarrow{\cong} H_p(M(\mathcal{A}), \mathbb{Z})$ for $p = 0, \ldots, \ell - 1$.*

Proof. It is easily seen from the genericity that

$$L(\mathcal{A} \cap U) \cong L_{\leqslant \ell-1}(\mathcal{A}) := \big\{ X \in L(\mathcal{A}) \;\big|\; r(X) \leqslant \ell - 1 \big\}. \tag{1}$$

In particular a generic intersection preserves the part of rank $\leqslant \ell - 1$. Hence $A(\mathcal{A} \cap U) \cong A^{\leqslant \ell-1}(\mathcal{A})$. This induces isomorphisms $H^{\leqslant \ell-1}(M(\mathcal{A})) \cong H^{\leqslant \ell-1}(M(\mathcal{A}) \cap U)$. Since homology groups $H_*(M(\mathcal{A}), \mathbb{Z})$ are torsion free, the theorem is the dual of these isomorphisms. □

Using these results inductively, the complement $M(\mathcal{A})$ of the hyperplane arrangement $\mathcal{A}$ has a minimal cell decomposition.

Theorem 6. *(See [13,3,11].) The complement $M(\mathcal{A})$ is homotopic to a minimal CW cell complex, i.e., the number of k-dimensional cells is equal to the Betti number $b_k(M(\mathcal{A}))$ for each $k = 0, \ldots, \ell$.*

4. Proof of the main theorem

First we recall the vanishing theorem of homology groups for a "generic" or nonresonant local system $\mathcal{L}$ of complex rank r.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, let U be a hyperplane generic to $\mathcal{A}$ and let $i : M(\mathcal{A}) \cap U \hookrightarrow M(\mathcal{A})$ be the inclusion. Now we assume that $\mathcal{A}$ is essential, i.e., $\mathcal{A}$ contains ℓ linearly independent hyperplanes $H_1, \ldots, H_\ell \in \mathcal{A}$. Let $\mathbb{P}^\ell$ be the projective space, which is a compactification of our vector space V. The projective closure of $\mathcal{A}$ is defined as $\mathcal{A}_\infty := \{\bar{H} \mid H \in \mathcal{A}\} \cup \{H_\infty\}$, where $\mathbb{P}^\ell = V \cup H_\infty$. A nonempty intersection $X \in L(\mathcal{A}_\infty)$ defines the subarrangement $(\mathcal{A}_\infty)_X = \{H \in \mathcal{A}_\infty \mid X \subset H\}$ of $\mathcal{A}_\infty$. A subspace $X \in \mathcal{A}_\infty$ is called dense if $(\mathcal{A}_\infty)_X$ is indecomposable, that is, not the product of two nonempty arrangements. Let $\rho : \pi_1(M(\mathcal{A}), x_0) \to GL_r(\mathbb{C})$ be the monodromy representation associated to $\mathcal{L}$. Choose a point $p \in X \setminus \bigcup_{H \in \mathcal{A}_\infty \setminus (\mathcal{A}_\infty)_X} H$ and a generic line L passing through p. Then a small loop γ in L around $p \in L$ determines a total turn monodromy $\rho(\gamma) \in GL_r(\mathbb{C})$. The conjugacy class of $\rho(\gamma)$ in $GL_r(\mathbb{C})$ depends only on $X \in L(\mathcal{A}_\infty)$, and is denoted by T_X.

The following vanishing theorem of local system cohomology groups is obtained in [2]. (See also [1,7,10].)

Theorem 7. *Let $\mathcal{L}$ be a nonresonant local system on $M(\mathcal{A})$ of rank r, that is, for each dense subspace $X \subset H_\infty$ the corresponding monodromy operator T_X does not admit 1 as an eigenvalue. Then*

$$\dim H^k\big(M(\mathcal{A}), \mathcal{L}\big) = \begin{cases} (-1)^\ell r \cdot \chi(M(\mathcal{A})) & \text{for } k = \ell, \\ 0 & \text{for } k \neq \ell, \end{cases}$$

where $\chi(M(\mathcal{A}))$ is the Euler characteristic of the space $M(\mathcal{A})$.

Note that $\mathcal{L}$ is nonresonant if and only if the dual local system $\mathcal{L}^\vee$ is nonresonant. From the universal coefficient theorem

$$H^k\big(M(\mathcal{A}), \mathcal{L}\big) \cong \mathrm{Hom}_{\mathbb{C}}\big(H_k\big(M(\mathcal{A}), \mathcal{L}^\vee\big), \mathbb{C}\big),$$

we also have the similar vanishing theorem for local system homology groups $H_k(M(\mathcal{A}), \mathcal{L})$.

From Theorem 2(a) we may identify, up to homotopy equivalence, $M(\mathcal{A})$ with a finite ℓ-dimensional CW complex for which the

$$(\ell - 1)\text{-skeleton has the homotopy type of } M(\mathcal{A}) \cap U. \tag{2}$$

We denote the attaching maps of ℓ-cells by $\phi_k : \partial c_k \cong \mathrm{S}^{\ell-1} \to M(\mathcal{A}) \cap U$ ($k = 1, \ldots, b = b_\ell(M(\mathcal{A}))$), where $c_k \cong \mathrm{D}^\ell$ is the ℓ-dimensional unit disk. Hence $\phi = \{\phi_k\}_{k=1,\ldots,b}$ satisfies

$$\Big(\big(M(\mathcal{A}) \cap U\big) \cup_\phi \bigcup_k c_k\Big) \quad \text{is homotopic to } M(\mathcal{A}).$$

Let $\mathcal{L}$ be a rank r local system over $M = M(\mathcal{A})$. For our purposes, it suffices to prove that $h(\phi_k)$ ($k = 1, \ldots, b$) generate $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L})$. Let

$$0 \to C_\ell \xrightarrow{\partial_{\mathcal{L}}} C_{\ell-1} \xrightarrow{\partial_{\mathcal{L}}} \cdots \xrightarrow{\partial_{\mathcal{L}}} C_0 \to 0 \tag{3}$$

be the twisted cellular chain complex associated with the CW decomposition for $M(\mathcal{A})$. Then from (2), the twisted chain complex for $M(\mathcal{A}) \cap U$ is obtained by truncating (3) as

$$0 \to C_{\ell-1} \xrightarrow{\partial_{\mathcal{L}}} \cdots \xrightarrow{\partial_{\mathcal{L}}} C_0 \to 0. \tag{4}$$

It is easily seen that if $\mathcal{L}$ is generic in the sense of Theorem 7, then the restriction $i^*\mathcal{L}$ is also generic. Applying Theorem 7 to (3), only the ℓth homology survives. Similarly, only the $(\ell - 1)$st homology survives in (4). Note that $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L}) = \mathrm{Ker}(\partial_{\mathcal{L}} : C_{\ell-1} \to C_{\ell-2})$. Thus we conclude that

$$\partial_{\mathcal{L}} : C_\ell \to H_{\ell-1}\big(M(\mathcal{A}) \cap U, i^*\mathcal{L}\big) \tag{5}$$

is surjective. Since the map (5) is determined by

$$C_\ell \ni [c_k] \mapsto [\partial c_k] = h(\phi_k),$$

$\{h(\phi_k)\}_{k=1,\ldots,b}$ generate $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L})$. This completes the proof of Theorem 1.

Lemma 8. *The Euler characteristic of $M(\mathcal{A}) \cap U$ is not equal to zero, more precisely,*

$$(-1)^{\ell-1} \chi\big(M(\mathcal{A}) \cap U\big) > 0.$$

Given a hyperplane $H \in \mathcal{A}$, we define $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}' \cap H$. Then characteristic polynomials for these arrangements satisfy an inductive formula:

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t).$$

By Theorem 4, the Euler characteristic $\chi(M(\mathcal{A}))$ of the complement is equal to $\chi(\mathcal{A}, 1)$.

Proof of Lemma 8. From (1) and definition of the characteristic polynomial, we have

$$\chi(\mathcal{A} \cap U, t) = \frac{\chi(\mathcal{A}, t) - \chi(\mathcal{A}, 0)}{t}.$$

The proof of the lemma is by induction on the number of hyperplanes. If $|\mathcal{A}| = \ell$, $\mathcal{A}$ is linearly isomorphic to the Boolean arrangement, i.e. one defined by $\{x_1 \cdot x_2 \cdots x_\ell = 0\}$, for a certain coordinate system $(x_1, \dots, x_\ell)$. In this case, $\chi(\mathcal{A}, t) = (t-1)^\ell$, and we have $(-1)^{\ell-1}\chi(M(\mathcal{A}) \cap U) = 1$. Assume that $\mathcal{A}$ contains more than ℓ hyperplanes. We can choose a hyperplane $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is essential. Then $\mathcal{A}'' = \mathcal{A}' \cap H$ is also essential, and obviously U is generic to $\mathcal{A}'$ and $\mathcal{A}''$. Thus we have

$$\begin{aligned}(-1)^{\ell-1}\chi(\mathcal{A} \cap U) = (-1)^{\ell-1}\chi(\mathcal{A} \cap U, 1) &= (-1)^{\ell-1}\big(\chi(\mathcal{A}' \cap U, 1) - \chi(\mathcal{A}'' \cap U, 1)\big) \\ &= (-1)^{\ell-1}\chi(\mathcal{A}' \cap U, 1) + (-1)^{\ell-2}\chi(\mathcal{A}'' \cap U, 1) > 0. \qquad \square\end{aligned}$$

Using Lemma 8, we have the following nonvanishing of the homotopy group, which generalizes a classical result of Hattori [6].

Corollary 9. *Let $2 \leqslant k \leqslant \ell - 1$ and $F^k \subset V$ be a k-dimensional subspace generic to $\mathcal{A}$. Then $\pi_k(M(\mathcal{A}) \cap F^k) \neq 0$.*

Remark 10. We can also prove Corollary 9 directly in the following way. Suppose $\pi_{\ell-1}(M(\mathcal{A}) \cap U) = 0$. Then the attaching maps $\{\phi_k : \partial c_k = S^{\ell-1} \to M(\mathcal{A}) \cap U\}$ of the top cells are homotopic to the constant map. Hence we have a homotopy equivalence

$$M(\mathcal{A}) \text{ is homotopic to } \big(M(\mathcal{A}) \cap U\big) \vee \bigvee_k S^\ell.$$

However this contradicts to the fact that cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is generated by degree one elements (Theorem 4). Hence we have $\pi_{\ell-1}(M(\mathcal{A}) \cap U) \neq 0$.

Remark 11. We should also note that other results on the nonvanishing of higher homotopy groups of generic sections are found in Randell [12] (for generic sections of aspherical arrangements), in Papadima–Suciu [11] (for hypersolvable arrangements) and in Dimca–Papadima [3] (for iterated generic hyperplane sections).

Acknowledgements

The author expresses deep gratitude to Professor M. Falk and Professor S. Papadima, for indicating Remark 10 to him. The author also thanks to the referee for a lot of suggestions which improve the paper. The author was supported by the JSPS Postdoctoral Fellowships for Research Abroad.

References

[1] K. Cho, A generalization of Kita and Noumi's vanishing theorems of cohomology groups of local system, Nagoya Math. J. 147 (1997) 63–69.
[2] D.C. Cohen, A. Dimca, P. Orlik, Nonresonance conditions for arrangements, Ann. Inst. Fourier (Grenoble) 53 (6) (2003) 1883–1896.
[3] A. Dimca, S. Papadima, Hypersurface complements Milnor fibers and minimality of arrangements, Ann. of Math. 158 (2003) 473–507.
[4] A. Dimca, S. Papadima, Equivariant chain complexes, twisted homology and relative minimality of arrangements, Ann. Sci. École Norm. Sup. (4) 37 (3) (2004) 449–467.
[5] H. Hamm, Lefschetz Theorems for Singular Varieties, Singularities, Proc. Sympos. Pure Math., vol. 40, 1983, pp. 547–557.
[6] A. Hattori, Topology of C^n minus a finite number of affine hyperplanes in general position, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (2) (1975) 205–219.
[7] T. Kohno, Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. Ser. A 62 (1986) 144–147.
[8] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (1980) 167–189.
[9] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer-Verlag, 1992.
[10] P. Orlik, H. Terao, Arrangements and Hypergeometric Integrals, MSJ Mem. 9 (2001).
[11] S. Papadima, A. Suciu, Higher homotopy groups of complements of complex hyperplane arrangements, Adv. Math. 165 (1) (2002) 71–100.
[12] R. Randell, Homotopy and group cohomology of arrangements, Topology Appl. 78 (1997) 201–213.
[13] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements, Proc. Amer. Math. Soc. 130 (2002) 2737–2743.