Computers and Mathematics with Applications 59 (2010) 3099–3109

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

The Cournot game under a fuzzy decision environment

Jr-Fong Dang [a], I-Hsuan Hong [b,c,*]

[a] Department of Industrial Engineering & Management, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, Taiwan

[b] Institute of Industrial Engineering, National Taiwan University, 1, Sec. 4, Roosevelt Road, Taipei 106, Taiwan

[c] Department of Mechanical Engineering, National Taiwan University, Taiwan

ARTICLE INFO

Article history:
Received 19 February 2010
Accepted 19 February 2010

Keywords:
Fuzzy set theory
Cournot game
Equilibrium quantity

ABSTRACT

The conventional precision-based decision analysis methodology is invalid for business decision analysis when precise assessment data seldom exist. This paper considers the Cournot game with fuzzy demand and fuzzy costs that are assumed to be triangular fuzzy numbers. Our model utilizes the weighted center of gravity (WCoG) method to defuzzify the fuzzy profit function into a crisp value. We derive the equilibrium Cournot quantity of each firm by simultaneously solving the first-order condition of each firm. Our model explicitly derives the necessary condition to avoid an unreasonable outcome of negative equilibrium quantities and lack of flexibility for modification of the ranking method. In addition, we examine the standard deviation of the fuzzy profit resulting from the fuzziness of each firm's cost and market demand functions. We conduct sensitivity analysis to investigate the effect of parameter perturbations on firms' outcomes. The results indicate that the center of parameter plays an important role in sensitivity analysis and dominates over variations in equilibrium quantity due to a perturbation of fuzzy parameters.

1. Introduction

Game Theory models and analyzes situations in which multiple agents have their own profit functions and independently make their decisions. Often, the decisions of multiple agents affect the payoffs of the others. The classic Cournot model is one of several models commonly applied to analyze such precise scenarios. Liang et al. [1] categorize them as precision-based models since all data and required parameters are precise. However, in the real world, many uncertain factors that exist in the decision environment (e.g., customer demand, production fluctuation, etc.) tend to restrict the usage of the classic Cournot model to real problems. This motivates us to develop the Cournot model with fuzzy parameters instead of precise information, which is usually difficult to obtain, even unavailable, in reality.

The literature has focused on the randomness aspect of uncertainty and many stochastic models have been developed to account for uncertainty in game-theoretical models [2,3] where uncertain parameters are typically modeled by probability distributions. However, the probability distribution may not be available in practice or may be difficult to estimate from limited data points. For instance, it is difficult to provide exact estimates of a manufacturer's variable cost because procurement costs may fluctuate. Under this scenario, the fuzzy set theory is an appropriate modeling tool when uncertain parameters cannot be described in distributions. In addition, the fuzzy set theory provides a mathematical approach to model the intrinsic vagueness and imprecision of human cognitive processes, e.g., the phrase "around x dollars" to describe a cost that can be regarded as a fuzzy number $\widetilde{x}$.

Since Zadeh [4] introduced the concept of fuzzy sets, applications to game theory have been proposed in the literature. In general, there are two streams of fuzzy games: fuzzy matrix games and fuzzy non-cooperative games. Most of them consider

* Corresponding author. Tel.: +886 2 3366 9507; fax: +886 2 2362 5856.
E-mail address: ihong@ntu.edu.tw (I-H. Hong).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.02.031

Table 1
Relative papers of fuzzy games.

Fuzzy parameters	Model type			
	Economic surplus	Revenue	Mutual price	Non-cooperative game
Fuzzy demand		Chang and Yao [11] Yao and Chang [13]	Yao and Wu [12]	Yao and Shih [14] Xu and Zhai [15]
Fuzzy cost				Liang et al. [1] Wang et al. [16]
Fuzzy demand and fuzzy cost	Yao and Wu [8]			

only at zero-sum games. Campos [5] proposes the fuzzy matrix game which is based on the establishment of a fuzzy linear programming for each player, but does not define explicit concepts of the equilibrium strategy. For bi-matrix games with fuzzy payoffs, Maeda [6] defines two types of concepts of equilibrium strategies and investigates their relations. Liu and Kao [7] apply the extension principle and a pair of two-level mathematical programming to construct the membership function of matrix games.

Yao and Wu [8] probably initiated the fuzzy non-cooperative games involving fuzzy data. They apply the ranking method transforming fuzzy numbers into crisp numbers for comparison to defuzzify the demand and supply functions so that consumer surplus and producer surplus can be calculated in a conventional manner. Their concept is also utilized to construct the inventory models by Ouyang and Yao [9] and Wu and Yao [10]. Chang and Yao [11] optimize the revenue of monopoly when the parameters of the demand function are fuzzy numbers. Yao and Wu [12] discuss the best price of two mutual complementary merchandises in fuzzy sense. Yao and Chang [13] obtain the optimal quantity for maximizing the profit function whose parameters are fuzzy numbers. Yao and Shih [14] derive the membership function of the profit function when the optimal quantity occurs. Xu and Zhai [15] develop an optimal technique for dealing with the fuzziness aspect of demand uncertainties for supply chains. Liang et al. [1] propose their duopoly model involving fuzzy cost to obtain the optimal quantity of each firm. Recently, Wang et al. [16] introduce their fuzzy Bayesian game where the decision rules for players are based on the creditability theory. Table 1 shows the classification of fuzzy parameters and model types where fuzzy parameters include the demand, cost or both, and the model types contain economic surplus, revenue, mutual price and non-cooperative games. It should be noted that the approaches described in existing papers may lead to an unreasonable outcome where the model returns a negative optimal quantity in some circumstances due to fuzzy parameters. Further, Liang et al. [1] only account for fuzzy costs without fuzzy demand.

In this paper, we propose the Cournot game with fuzzy demand and cost, which to our knowledge has not appeared in the literature. We highlight two important drawbacks in previous studies: an unclear restriction of occurrence of a negative equilibrium quantity, and limited flexibility for modification of the ranking method in fuzzy modeling. We propose a fuzzy Cournot model with rigorous definitions to ensure a positive equilibrium quantity and with a flexible controlling mechanism for decision-makers. This paper describes a method solving for the equilibrium quantity of each competing firm in a duopoly market with fuzzy parameters that give several important managerial insights by examining the variation of the profit function in the fuzzy environment.

We utilize the weighted center of gravity (WCoG) which is proposed by Bender and Simonovic [17] to defuzzify the profit function of each firm into a crisp value. Each firm desires to make decisions that optimize its particular objective profit functions. In the equilibrium quantity, no firm can be better off by a unilateral change in its solution. Mathematically the equilibrium quantity can be obtained by simultaneously solving the first-order condition of each firm's objective profit function. In addition, we utilize the variation index proposed by Lee and Li [18] to calculate the variation in the firms' profit functions. For simplicity, we assume that the demand function and cost function of the firms exist with the form of linearity and fuzzy parameters. The linearity assumption is commonly used in the literature [11,13] and it assists in obtaining the qualitative managerial insights with less analytical complexity. It also has the desirable properties for approaching the equilibrium quantities.

The remainder of this paper is organized as follows. In Section 2, we introduce the concept and definitions. Section 3 addresses the fuzzy Cournot model followed by the proposed method to solve for the equilibrium quantity of each firm in the fuzzy context. Section 4 conducts the sensitivity analysis to investigate the effect of parameter perturbation on firms' outcomes. Section 5 presents the conclusions and suggestions for future research.

2. Definitions

Zadeh [4] introduced fuzzy set theory to analyze and solve problems with sources of vagueness called fuzziness. The word "fuzziness" captures the properties of parameters when in reality decision-makers rarely have sharp boundaries and/or cannot precisely determine them. Below we briefly introduce the definitions and notations used in this paper.

Let X be a universal set. A fuzzy subset $\widetilde{A}$ of X is defined by its membership function $\mu_{\widetilde{A}} : X \to [0, 1]$. We denote by $\widetilde{A}_\alpha = \{x : \mu_{\widetilde{A}}(x) \geq \alpha\}$ the α-level set of $\widetilde{A}$. The fuzzy number, $\widetilde{A}$, is called a normal fuzzy set if there exists x such that

$\mu_{\widetilde{A}}(x) = 1$. In addition, the fuzzy number, $\widetilde{A}$, is called a convex fuzzy set if $\mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y))$ for $\lambda \in [0, 1]$. This paper assumes the universal set X is a real number system, i.e., $X = \Re$. A fuzzy number, $\widetilde{A}$, must satisfy:

(i) $\widetilde{A}$ is a normal and convex fuzzy set,
(ii) its membership function, $\mu_{\widetilde{A}}$, is upper semi-continuous, and
(iii) the α-level set, $\widetilde{A}_\alpha$, is bounded for each $\alpha \in [0, 1]$.

The fuzzy number, $\widetilde{A}$, is called a nonnegative fuzzy number if $\mu_{\widetilde{A}}(x) = 0$ for all $x < 0$, and a nonpositive fuzzy number if $\mu_{\widetilde{A}}(x) = 0$ for all $x > 0$. It is obvious that $\widetilde{A}^L_\alpha$ and $\widetilde{A}^U_\alpha$ are nonnegative real numbers for all $\alpha \in [0, 1]$ if $\widetilde{A}$ is a nonnegative fuzzy number, and $\widetilde{A}^L_\alpha$ and $\widetilde{A}^U_\alpha$ are nonpositive real numbers for all $\alpha \in [0, 1]$ if $\widetilde{A}$ is a nonpositive fuzzy number.

2.1. Fuzzy arithmetic operation

Let "$\odot$" be any binary operation $\oplus$ or $\otimes$ between two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$. The membership function of $\widetilde{a} \odot \widetilde{b}$ is defined by

$$\mu_{\widetilde{a}\odot\widetilde{b}}(z) = \sup_{x\circ y=z} \min\left\{\mu_{\widetilde{a}}(x), \mu_{\widetilde{b}}(y)\right\}$$

using the extension principle in [4], where the operations $\odot, =, \oplus$, and $\otimes$ correspond to the operations $\circ, =, +$, and $\times$. Thus we review the following definition for derivation purposes.

Definition 1 (*[19]*). Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers. Then $\widetilde{a} \oplus \widetilde{b}$ and $\widetilde{a} \otimes \widetilde{b}$ are also fuzzy numbers. Further, we have

$$\left(\widetilde{a} \oplus \widetilde{b}\right)_\alpha = \left[\widetilde{a}^L_\alpha + \widetilde{b}^L_\alpha, \widetilde{a}^U_\alpha + \widetilde{b}^U_\alpha\right]$$

and

$$\left(\widetilde{a} \otimes \widetilde{b}\right)_\alpha = \left[\min\left(\widetilde{a}^L_\alpha \cdot \widetilde{b}^L_\alpha, \widetilde{a}^U_\alpha \cdot \widetilde{b}^U_\alpha, \widetilde{a}^L_\alpha \cdot \widetilde{b}^U_\alpha, \widetilde{a}^U_\alpha \cdot \widetilde{b}^L_\alpha\right), \max\left(\widetilde{a}^L_\alpha \cdot \widetilde{b}^L_\alpha, \widetilde{a}^U_\alpha \cdot \widetilde{b}^U_\alpha, \widetilde{a}^L_\alpha \cdot \widetilde{b}^U_\alpha, \widetilde{a}^U_\alpha \cdot \widetilde{b}^L_\alpha\right)\right].$$

Definition 1 is useful when considering the linear model in this paper.

2.2. Triangular fuzzy number

In practice, the most commonly used fuzzy numbers are triangular fuzzy numbers because they are easy to handle arithmetical operations and they have intuitive interpretations [20]. The triplet, (x, l, r), denotes a triangular fuzzy number with the apex x, left-spread l, and right-spread r. In other words, the lower and upper bounds of x are $x - l$ and $x + r$ respectively. For notational simplicity, we let x^U (x^L) denote the upper (lower) bound of x.

2.3. Weighted center of gravity measure

Many ranking methods have been proposed (see [21,22]). The center of gravity (COG) method, also called the centroid method [23], is widely used because of its straightforward geometrical meaning. However, the COG method is inappropriate to distinguish two fuzzy sets that may have the same centroid, but greatly differ in the degree of fuzziness. In this case, the WCoG method is more useful [17]:

$$\mathrm{WCoG} = \frac{\int g(x)\mu(x)^k \mathrm{d}x}{\int \mu(x)^k \mathrm{d}x}, \tag{1}$$

where $g(x)$ is the horizontal component of the area under scrutiny and $\mu(x)$ is the membership function value. The value of k is a control parameter ranging from $k = 1$ to $k = \infty$. In practice, decision-makers determine $g(x)$ and the magnitude of k. This paper applies the WCoG method to retrieve crisp values. In (1), there are two control parameters: $g(x)$ and k; the final selection of appropriate control parameters depends upon the decision-makers' level of risk tolerance (see [17]).

2.4. Measurement of variation

Lee and Li [18] propose the use of generalized mean and the standard deviation, listed in (2) and (3), based on the probability measure of fuzzy events to rank fuzzy numbers. This method ranks fuzzy numbers on the basis of the fuzzy mean and the spread of fuzzy numbers, and assumes two kinds of the probability distributions of fuzzy numbers: uniform distribution and proportional distribution. We adopt the uniform distribution for calculation simplicity in this paper. The uniform distribution, U, with the spreads $|\widetilde{A}|$ of the fuzzy number has probability density function

$$f(\widetilde{A}) = 1/|\widetilde{A}| \quad \text{and} \quad \widetilde{A} \in U.$$

The mean, $\widetilde{x}(\widetilde{A})$, and standard deviation, $\widetilde{\sigma}(\widetilde{A})$, of fuzzy number $\widetilde{A}$ are defined as

$$\widetilde{x}(\widetilde{A}) = \int_{S(\widetilde{A})} x\mu_{\widetilde{A}}(x)\mathrm{d}x \Big/ \int_{S(\widetilde{A})} \mu_{\widetilde{A}}(x)\mathrm{d}x, \tag{2}$$

$$\widetilde{\sigma}(\widetilde{A}) = \left[\left(\int_{S(\widetilde{A})} x^2\mu_{\widetilde{A}}(x)\mathrm{d}x \Big/ \int_{S(\widetilde{A})} \mu_{\widetilde{A}}(x)\mathrm{d}x\right) - \left(\widetilde{x}(\widetilde{A})\right)^2\right]^{1/2} \tag{3}$$

where $S(\widetilde{A})$ is the support of fuzzy number $\widetilde{A}$. When fuzzy number $\widetilde{A}$ is a triangular fuzzy number, (2) and (3) can be written as

$$\widetilde{x}(\widetilde{A}) = 1/3\left(x^L + x + x^U\right), \tag{4}$$

$$\widetilde{\sigma}(\widetilde{A}) = 1/18\left(\left(x^L\right)^2 + (x)^2 + \left(x^U\right)^2 - x^Lx - xx^U - x^Lx^U\right). \tag{5}$$

3. The fuzzy Cournot model

As we have noted it is almost impossible to find the exact economic assessment data for parameters' estimation in the real world. Therefore, we apply the fuzzy set theory to solve the fuzzy Cournot game with the fuzzy demand function and fuzzy cost function.

3.1. The Cournot equilibrium quantity in a duopoly market

Let q_i denote the production quantity of firm i, $i = 1, 2$. Consider the general fuzzy profit function of firm i, $\widetilde{\pi}_i$, as

$$\widetilde{\pi}_i = \widetilde{p}(Q) \cdot q_i - \widetilde{TC}(q_i) \tag{6}$$

where $\widetilde{p}(Q)$ is the fuzzy inverse demand function, $\widetilde{TC}(q_i)$ is the fuzzy cost function of firm i and Q is the total quantity in the duopoly market, namely $Q = q_1 + q_2$.

The fuzziness of $\widetilde{p}(Q)$ and $\widetilde{TC}(q_i)$ result from the fuzzy parameters of the inverse demand function and the cost function. From (6) it is clear that if $\widetilde{p}(Q)$ or $\widetilde{TC}(q_i)$ is unbounded, then $\widetilde{\pi}_i$ is unbounded. However, since unboundedness is irrational in the real world, this paper assumes that both $\widetilde{p}(Q)$ and $\widetilde{TC}(q_i)$ are bounded. Thus, $\widetilde{\pi}_i$ is bounded as well. The upper (lower) bound of $\widetilde{\pi}_i$ denoted by π_i^U (π_i^L) can be obtained by the extension principle. According to fuzzy arithmetic operations, the profit function of firm i is a fuzzy number, which may cause difficulty in solving for each firm's equilibrium quantity. In this paper, we utilize (1) to defuzzify the profit function into a crisp value. The center of the fuzzy profit function can be stated as

$$\mathrm{WCoG}(\widetilde{\pi}_i) = \frac{\int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mu(\pi_i)^k\mathrm{d}\pi_i}{\int_{\pi_i^L}^{\pi_i^U} \mu(\pi_i)^k\mathrm{d}\pi_i}. \tag{7}$$

The equilibrium quantity of the conventional Cournot game is directly related to each firm's profit function. Additionally, we imagine that the equilibrium quantity derived by our proposed model is related to the resulting profit function. As expected, if the profit function of each firm is unbounded, the equilibrium quantity is also unbounded. Thus, (7) requires a more rigorous definition so that the estimated profit function of each firm is bounded and the equilibrium quantity indeed exists.

Proposition 1. *If $\int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mathrm{d}\pi_i$ is bounded, the* $\mathrm{WCoG}(\widetilde{\pi}_i)$ *is bounded.*

Proof. Let $\mu(\pi_i)$ be the membership function of the fuzzy profit function. Thus, we observe that the possibility of $\mu(\pi_i)$ is always less than or equal to one. The membership function is contained by a rectangle; the height of the rectangle is equal to one and the base of the rectangle extends between the upper bound of the fuzzy profit function and the lower bound of the fuzzy profit function, $\pi_i^U - \pi_i^L$ respectively. The shape of the membership function $\mu(\pi_i)^k$ differs when the value of k varies, but $\mu(\pi_i)^k$ is still contained by a rectangle whose height and base are 1 and $\pi_i^U - \pi_i^L$. Therefore, one can argue that the integration of membership function $\mu(\pi_i)^k$ has the inequality

$$0 \le \int_{\pi_i^L}^{\pi_i^U} \mu(\pi_i)^k\mathrm{d}\pi_i = w \le \pi_i^U - \pi_i^L, \tag{8}$$

where w is a constant. Thus, (7) can be rewritten as

$$\mathrm{WCoG}(\widetilde{\pi}_i) = 1/w \cdot \int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mu(\pi_i)^k\mathrm{d}\pi_i. \tag{9}$$

Because the maximum value of $\mu(\pi_i)^k$ is 1, (9) can be rearranged as

$$\mathrm{WCoG}(\widetilde{\pi}_i) = 1/w \cdot \int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mu(\pi_i)^k \mathrm{d}\pi_i \le 1/w \cdot \int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mathrm{d}\pi_i. \tag{10}$$

Thus, if $\int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mathrm{d}\pi_i$ is bounded, $\mathrm{WCoG}(\widetilde{\pi}_i)$ is bounded as well. ■

We note that decision-makers who utilize (7) to defuzzify the fuzzy profit function, must check whether $\int_{\pi_i^L}^{\pi_i^U} g(\pi_i)\mathrm{d}\pi_i$ is bounded. The best response function, an important concept in game theory, returns the quantity that yields the optimal profit for firm i given other players' decisions about the quantity. In this paper, we use (7) to defuzzify the fuzzy profit function into a crisp value. Thus, the best response functions of firm i can be obtained by optimizing each firm's profit functions with respect to each firm's decision variable q_i while assuming the competitors' quantity q_j as given. The resulting best response functions are

$$R^i\left(q_j\right) = \arg\max \mathrm{WCoG}\left(\widetilde{\pi}_i\right), \quad i \neq j,\ i, j = 1, 2. \tag{11}$$

We can obtain the equilibrium quantity of each firm by simultaneously solving the first-order condition being obtained by letting the partial derivative of each firm's profit function equal zero. Now that we have crisp values, we can examine the variations of the profit function to provide more information for decision-makers. As discussed earlier, we can express in (3) or (5) the profit variation resulting from the gap of the parameter's spread of both the firm's cost and market demand functions. In addition, the optimistic and pessimistic cases can be realized by membership function $\mu(\pi_i)$, which can be derived by the extension principle in [4]. Moreover, we can adapt our proposed method to fit the different criteria of decision-makers or markets by setting different controlling mechanisms of $g(x)$ and k.

3.2. Linear model

Next, we develop the fuzzy Cournot game that includes the parameters with triangular fuzzy numbers, a linear inverse demand and cost functions. Initially, we introduce the conventional Cournot game and generalize it in the fuzzy environment. Given the linear inverse demand function

$$p(Q) = a - bQ, \quad 0 \le Q \le \frac{a}{b} \tag{12}$$

where $a, b > 0$ are given numbers and $p(Q)$ is the unit price, which is a function of the market demand quantity Q. The total cost function of firm i, $i = 1, 2$, denoted by $TC_i(q_i)$, is stated as

$$TC_i(q_i) = c_i + d_i q_i \tag{13}$$

where c_i denotes the production fixed cost of firm i and d_i represents the production variable cost of firm i. Then the profit function of firm i is given by

$$\pi_i = p(Q) \cdot q_i - TC_i(q_i) = (a - bQ)q_i - TC_i(q_i). \tag{14}$$

Without loss of generality, we assume that all parameters are fuzzy numbers since crisp values can be treated as degenerated fuzzy numbers. In other words, the last two elements in the triplet of the fuzzy number are equal to zero for a crisp value. Particularly in the linear model, we let fuzzy sets $\widetilde{p}(Q) = \widetilde{a} - \widetilde{b}Q$, $\widetilde{TC}_i(q_i) = \widetilde{c}_i + \widetilde{d}_i q_i$, and $\widetilde{\pi}_i = \widetilde{p}(Q) \cdot q_i - \widetilde{TC}_i(q_i)$ where $\widetilde{p}(Q)$, $\widetilde{TC}_i(q_i)$, and $\widetilde{\pi}_i$ represent the fuzzy price, fuzzy cost function of firm i and fuzzy profit function of firm i, $i = 1, 2$, respectively. Triangular fuzzy numbers are adopted in this paper because they are considered the most suitable for modeling the market demand; see [24,25]. We assume that all parameters are nonnegative triangular fuzzy numbers; in other words, all elements in the following triplets of fuzzy numbers are nonnegative.

$$\begin{aligned} &\widetilde{a} = (a, l_a, r_a)\,, \\ &\widetilde{b} = (b, l_b, r_b)\,, \\ &\widetilde{c}_i = \left(c_i, l_{c_i}, r_{c_i}\right), \\ &\widetilde{d}_i = \left(d_i, l_{d_i}, r_{d_i}\right). \end{aligned} \tag{15}$$

By Definition 1, when $\widetilde{a}$ and $\widetilde{b}$ are triangular fuzzy numbers, we have

$$\widetilde{p}(Q) = (a - bQ, l_a + r_b Q, r_a + l_b Q)\,. \tag{16}$$

To derive the lower bound of price, we must substitute the lower bound of a, namely $a - l_a$, and the upper bound of b, namely $b + r_b$, into (12). Also, the upper bound of price can by derived by substituting the upper bound of a, namely $a + r_a$, and the lower bound of b, namely $b - l_b$, into (12). Similarly, $\widetilde{TC}_i(q_i)$ and $\widetilde{\pi}_i$ are given by

$$\begin{aligned} &\widetilde{TC}_i(q_i) = \left(TC_i, l_{c_i} + l_{d_i} q_i, r_{c_i} + r_{d_i} q_i\right), \\ &\widetilde{\pi}_i = \left(\pi_i, (l_a + r_b Q)\, q_i + \left(r_{c_i} + r_{d_i} q_i\right), (r_a + l_b Q)\, q_i + \left(l_{c_i} + l_{d_i} q_i\right)\right). \end{aligned} \tag{17}$$

Note that $\widetilde{p}(Q)$, $\widetilde{TC}_i(q_i)$, and $\widetilde{\pi}_i$ are triangular fuzzy numbers as well due to the extension principle.

The controlling mechanism should be clearly defined and follows Proposition 1 before applying (1) to defuzzify the fuzzy profit function into a crisp value. For illustrative purposes, this paper assumes $k = 1$ and $g(x) = x$. Simply put, it is the same as the centroid method and it weights all of the values with different possibility to form a single value to represent a fuzzy number. Since each firm's fuzzy profit function is a triangular fuzzy number, $\mathrm{WCoG}(\widetilde{\pi}_i)$ can be easily calculated as

$$\mathrm{WCoG}\left(\widetilde{\pi}_i\right) = \frac{\int_{-\infty}^{\infty} \pi_i \mu_{\widetilde{\pi}_i} \mathrm{d}\pi_i}{\int_{-\infty}^{\infty} \mu_{\widetilde{\pi}_i} \mathrm{d}\pi_i} = \frac{1}{3}\left(\pi_i^L + \pi_i + \pi_i^U\right). \tag{18}$$

Thus, the partial derivative of $\mathrm{WCoG}(\widetilde{\pi}_i)$ can be stated as

$$\frac{\partial \mathrm{WCoG}\left(\widetilde{\pi}_i\right)}{\partial q_i} = a - 2bq_i - bq_j - d_i + \frac{1}{3}[(r_a - l_a) + 2(l_b - r_b)q_i + (l_b - r_b)q_j + (l_{d_i} - r_{d_i})], \quad i, j = 1, 2,\ i \neq j. \tag{19}$$

By letting (19) equal zero, the first-order condition of firm i, $i = 1, 2$, can be obtained. The equilibrium quantity of firm i follows by simultaneously solving the first-order conditions of firms 1 and 2. Thus, we have

$$q_i = \frac{\left[a + \left(d_j - 2d_i\right) - \frac{1}{3}\left(l_a - r_a\right) + \frac{2}{3}\left(l_{d_i} - r_{d_i}\right) - \frac{1}{3}\left(l_{d_j} - r_{d_j}\right)\right]}{\left[3b - (l_b - r_b)\right]}, \quad i, j = 1, 2,\ i \neq j. \tag{20}$$

The denominator of (20), $3b - (l_b - r_b)$, can be decomposed into three terms

$$3b - (l_b - r_b) = (b - l_b) + b + (b + r_b). \tag{21}$$

Because $b - l_b$ is nonnegative, (21) is obviously greater than zero. To ensure a nonnegative equilibrium quantity of firm i, we impose the condition such that $q_i \geq 0$, $i = 1, 2$. From this condition follows Assumption 1.

Assumption 1. $a + (d_j - 2d_i) - \frac{1}{3}(l_a - r_a) + \frac{2}{3}(l_{d_i} - r_{d_i}) - \frac{1}{3}(l_{d_j} - r_{d_j}) \geq 0$.

In addition, (20) is the same as the equilibrium quantity of the conventional Cournot game when the spreads of all parameters are equal to zero. Next, we calculate the variance of each firm's profit by the Lee and Li [18] index and we suppose the distribution of the fuzzy profit function is the uniform distribution. The standard deviation of the fuzzy profit function can be defined as

$$\sigma\left(\widetilde{\pi}_i\right) = \left[\int_{S(\widetilde{\pi}_i)} (\pi_i^2 \mu_{\widetilde{\pi}_i} \mathrm{d}\pi_i) \Big/ \int_{S(\widetilde{\pi}_i)} (\mu_{\widetilde{\pi}_i} \mathrm{d}\pi_i)^2 - (\mathrm{WCoG}(\widetilde{\pi}_i))^2\right]^{1/2}. \tag{22}$$

By substituting (18) into (22), when the fuzzy profit function is a triangular fuzzy number, (22) can be rewritten as

$$\widetilde{\sigma}\left(\widetilde{\pi}_i\right) = 1/18\left(\left(\pi_i^L\right)^2 + (\pi_i)^2 + \left(\pi_i^U\right)^2 - \pi_i^L\pi_i^U - \pi_i\pi_i^U - \pi_i^L\pi_i\right), \tag{23}$$

where π_i^U, π_i, and π_i^L can be obtained by (17). Using the solution procedure proposed in this section, we now have both the equilibrium quantity and the standard deviation of the fuzzy profit function of firms 1 and 2.

3.2.1. Uniqueness and existence of equilibrium quantity

The equilibrium quantity (20) is a solution to a system of two first-order conditions. Non-existence of an equilibrium quantity is a potential outcome depending on the value of the parameters. In the linear model proposed in this paper, the equilibrium quantity does exist under the condition we discuss later and there is a simple way to show that the equilibrium quantity is unique.

Proposition 2. *If the lower bound of parameter b is greater than zero, namely $b - l_b > 0$, there exists a unique equilibrium quantity of firms* 1 *and* 2.

Proof. By letting (19) equal zero, the equilibrium quantities of firms 1 and 2 are obtained by simultaneously solving their two first-order conditions. Obviously, (19) is a linear system when the demand and cost functions are linear. We can solve a linear system by Cramer's Rule (see [26]).

Let Δ be a 2×2 matrix defined as

$$\Delta = \begin{bmatrix} 2B & B \\ B & 2B \end{bmatrix}, \quad \text{where } B = b - \frac{1}{3}(l_b - r_b). \tag{24}$$

The determinant of Δ, denoted by $\det(\Delta)$, is $4B^2 - B^2 = 3B^2$. We can decompose entry B into three terms as

$$B = \frac{1}{3}(b - l_b) + \frac{1}{3}b + \frac{1}{3}(b + r_b). \tag{25}$$

The first term in (25), $(b - l_b)$, is the lower bound of parameter b. If the lower bound of parameter b is greater than zero, we have $B > 0$. By Cramer's Rule, we show that the equilibrium quantity of firms 1 and 2 is unique if and only if $\det(\Delta) \neq 0$. This completes the proof. ■

3.2.2. Illustrative examples

The solution procedure that we have proposed is to find the equilibrium quantity in a duopoly market. To demonstrate the proposed method, we consider the Cournot game with the following problem setups.

Example 1. Suppose $p(Q) = 12 - Q, 0 \leq Q \leq 12$, then $a = 12$, $b = 1$ and cost function of firms 1 and 2, $TC_1(q_1) = 2 + 4q_1$, $TC_2(q_2) = 1 + 5q_2$ in the crisp case. The equilibrium quantity of the conventional Cournot game can be obtained by substituting zero spreads in our model. Hence, the equilibrium quantity of firm 1 is 3 units ($q_1^* = 3$), and the equilibrium quantity of firm 2 is 2 units ($q_2^* = 2$).

Example 2. We consider the case with fuzzy parameters. Let $\widetilde{a} = (12, 1, 2)$, $\widetilde{b} = (1, 1, 1)$, $\widetilde{c}_1 = (2, 1, 1)$, $\widetilde{d}_1 = (4, 1, 2)$, $\widetilde{c}_2 = (1, 1, 1)$, and $\widetilde{d}_2 = (5, 2, 1)$. We can obtain the equilibrium quantity of firms 1 and 2 as

$$q_1 = \frac{12 + (5-8) - \frac{1}{3}(1-2) + \frac{2}{3}(1-2) - \frac{1}{3}(2-1)}{3 \times 1} = \frac{25}{9},$$
$$q_2 = \frac{12 + (4-10) - \frac{1}{3}(1-2) - \frac{1}{3}(1-2) + \frac{2}{3}(2-1)}{3 \times 1} = \frac{22}{9}.$$

Given the equilibrium quantity of each firm, we calculate the standard deviation of the fuzzy profit function of each firm as

$$\widetilde{\sigma}\left(\widetilde{\pi}_i\right) = 1/18\left(\left(\pi_i^L\right)^2 + (\pi_i)^2 + \left(\pi_i^U\right)^2 - \pi_i^L\pi_i^U - \pi_i\pi_i^U - \pi_i^L\pi_i\right), \quad i = 1, 2 \tag{26}$$

where

$$\begin{array}{ll} \pi_i^L = ((a - l_a) - (b + r_b)\,Q)\,q_i - \left(c_i + r_{c_i}\right) - \left(d_i + r_{d_i}\right) q_i & \\ \pi_i = (a - bQ)\,q_i - c_i - d_i q_i & i = 1, 2. \\ \pi_i^U = ((a + r_a) - (b - l_b)\,Q)\,q_i - \left(c_i - l_{c_i}\right) - \left(d_i - l_{d_i}\right) q_i & \end{array}$$

Substituting the q_i, Q, and parameters into (26), the standard deviation of the profit of firms 1 and 2 is

$$\widetilde{\sigma}\left(\widetilde{\pi}_1\right) = 94.72,$$
$$\widetilde{\sigma}\left(\widetilde{\pi}_2\right) = 74.52.$$

4. Sensitivity analysis

The goal of sensitivity analysis is to investigate the effect of parameter perturbations on the resulting outcome such as the equilibrium quantity and firms' profits so that we gain several managerial insights from the proposed model. As we can see, (20) indicates that the terms, $l_a - r_a$ and $l_{d_j} - r_{d_j}$, have negative coefficients. Parameters l_a and r_a can be interpreted as the left- and right-spreads of fuzzy parameter $\widetilde{a}$ in the demand function. An increase in $l_a - r_a$ (the difference between left- and right-spreads in a) would result in a decrease in each firm's equilibrium quantity. Similarly, from (20), an increase in $l_{d_j} - r_{d_j}$ (the difference between left- and right-spreads of the competitor's production variable cost d_j) would lead to a decrease in the equilibrium quantity of the firm as well. In (20), the term, $l_{d_i} - r_{d_i}$, is with a positive coefficient. As a result, an increase in $l_{d_i} - r_{d_i}$ (the difference between the left- and right-spreads of its own production variable cost d_i) would lead to an increase in the equilibrium quantity. It is interesting to note that the coefficient magnitude of $l_{d_i} - r_{d_i}$ is greater than that of $l_{d_j} - r_{d_j}$; in other words, the equilibrium quantity decreases in $l_{d_j} - r_{d_j}$, but increases in $l_{d_i} - r_{d_i}$ at a faster rate. However, there are other factors affecting the change in the equilibrium quantity rather than the differences between the left- and right-spreads. From (20), decision-makers cannot examine how a total spread of a fuzzy parameter could affect the equilibrium quantity; for example, it is not obvious how the change in $l_a + r_a$ affects q_i for firm i. The next section describes our study of parameter centers in sensitivity analysis.

4.1. Effect of parameter centers

As discussed earlier, fuzzy parameters can be defuzzified into a crisp value representing the center of the associated fuzzy parameter. We now look at the impacts of parameter centers.

Proposition 3. *The equilibrium quantity of each firm is only affected by the centers of the parameters.*

Proof. Considering the center of triangular fuzzy parameter a, it is trivial having the center of a in (27).

$$a^c = \frac{1}{3}\left(a - l_a + a + a + r_a\right). \tag{27}$$

Table 2
Partial derivatives of outcomes with respect to different centers.

	a^c	b^c	d_i^c	d_j^c
Q	$2/3b^c$	$-3(2a^c - d_j^c - d_i^c)/(3b^c)^2$	$-1/3b^c$	$-1/3b^c$
q_i	$1/3b^c$	$-3(a^c + d_j^c - 2d_i^c)/(3b^c)^2$	$-2/3b^c$	$1/3b^c$
$\text{WCoG}(\widetilde{\pi}_i)$	$\frac{2(a^c+d_j^c-2d_i^c)}{9b^c}$	$\frac{-3(2a^c-d_j^c-d_i^c)(a^c+d_j^c+d_i^c)}{27(b^c)^2}$	$\frac{-4(a^c+d_j^c-2d_i^c)}{3b^c}$	$\frac{2a^c+2d_j-2d_i}{9b^c}$

Similarly, the centers of the other fuzzy parameters can be written as

$$\begin{aligned} b^c &= \frac{1}{3}\left(b - l_b + b + b + r_b\right), \\ d_i^c &= \frac{1}{3}\left(d_i - l_{d_i} + d_i + d_i + r_{d_i}\right), \\ d_j^c &= \frac{1}{3}\left(d_j - l_{d_j} + d_j + d_j + r_{d_j}\right). \end{aligned} \tag{28}$$

Substituting (27) into (20), (20) can be rewritten as

$$q_i = \frac{a^c + d_j^c - 2d_i^c}{3b^c}. \tag{29}$$

Although both (20) and (29) represent the equilibrium quantity of each firm, (29) is much more convenient for illustrative purposes in sensitivity analysis and it completes the proof. ■

All possible perturbation in parameters can be represented in the right-hand side of (27) or (28) and translated into the change in the center of the associated fuzzy parameter. From Proposition 3, the equilibrium quantity is a function of the center of fuzzy parameters a^c, b^c, d_i^c, and d_j^c as shown in (29). Therefore, the change in the equilibrium quantity due to any perturbation in fuzzy parameters can be explored by (29). Suppose that the right-spread of fuzzy parameter a, namely r_a, increases. It is clear that a^c would be larger and it is with a positive coefficient in (29). As the result, the equilibrium quantity increases in a^c.

From (29), the total market demand is written as

$$Q = q_1 + q_2 = \frac{2a^c - d_i^c - d_j^c}{3b^c}. \tag{30}$$

We can now conduct the sensitivity analysis of the total market demand and equilibrium quantity of firms. Taking the partial derivative of (29) and (30) with respect to a^c, we have

$$\frac{\partial q_i}{\partial a^c} = \frac{1}{3b^c}, \tag{31}$$

$$\frac{\partial Q}{\partial a^c} = \frac{2}{3b^c}. \tag{32}$$

Similarly, we also take the partial derivatives of (29) and (30) with respect to other centers of fuzzy parameters. Following Assumption 1, the resulting equilibrium quantity is positive, and implies that the numerator of (29), $a^c + d_j^c - 2d_i^c$, is positive as well. The total quantity of the market demand is positive simply because of the positive quantities of both individual firms; therefore, the numerator of (30), $2a^c - d_j^c - d_i^c$, is positive. Table 2 summarizes these results (for example, $\frac{\partial Q}{\partial a^c} = \frac{2}{3b^c}$ is listed in the upper-left cell in Table 2). From Table 2, we have the following statements:

- the total market demand, Q, increases in a^c, but decreases in b^c, d_i^c, and d_j^c, and
- the equilibrium quantity of firm i, q_i, increases in a^c and d_j^c, but decreases in b^c and d_i^c.

Note that the equilibrium quantity of firm i increases in d_j^c, but the equilibrium quantity of firm j decreases in d_j^c at a faster rate so that an increase in d_j^c would lead to a decrease in Q.

Next, we consider the weighted center of the fuzzy profit function, $\text{WCoG}(\widetilde{\pi}_i)$, as perturbations of the center of fuzzy parameters occur. After some algebraic manipulations, (18) can be rewritten as

$$\text{WCoG}\left(\widetilde{\pi}_i\right) = \left(a^c - b^c Q\right) q_i - \left(c_i^c + d_i^c q_i\right). \tag{33}$$

Again, we take partial derivative of (33) with respect to a^c and we have

$$\frac{\partial \text{WCoG}(\widetilde{\pi}_i)}{\partial a^c} = q_i + a^c \frac{\partial q_i}{\partial a^c} - b^c Q \frac{\partial q_i}{\partial a^c} - b^c \frac{\partial Q}{\partial a^c} q_i - d_i^c \frac{\partial q_i}{\partial a^c}. \tag{34}$$

Using the results of the first two rows in Table 2, (34) can be rewritten as

$$\frac{\partial \mathrm{WCoG}(\widetilde{\pi}_i)}{\partial a^c} = q_i + a^c \frac{1}{3b^c} - \frac{1}{3}Q - \frac{2}{3}q_i - d_i^c \frac{1}{3b^c} = a^c \frac{1}{3b^c} - \frac{1}{3}q_j - d_i^c \frac{1}{3b^c} = \frac{2}{3}q_i. \tag{35}$$

Similarly, we can also take the partial derivative of (33) with respect to the other centers of fuzzy parameters. From earlier discussion, we know $a^c + d_j^c - 2d_i^c > 0$ and $2a^c - d_j^c - d_i^c > 0$. Trivially, $2a^c + 2d_j^c - 2d_i^c$ is positive since $2a^c + 2d_j^c - 2d_i^c$ is greater than $a^c + d_j^c - 2d_i^c$. Therefore, the sign of the last row in Table 2 can be determined by these inequalities. We summarize these results indicating the change rate in $\mathrm{WCoG}(\widetilde{\pi}_i)$ with respect to the centers of fuzzy parameters. From the last row in Table 2, we have the following statement:

- the weighted center of the fuzzy profit function increases in a^c and d_j^c, but decreases in b^c and d_i^c.

4.2. Analysis of variation of the fuzzy profit function

It is interesting to discuss the effect of the perturbation of fuzzy parameter centers on the standard deviation of firms' profits. Taking the partial derivative of (23) with respect to a^c, we have

$$\frac{\partial \sigma\left(\widetilde{\pi}_i\right)}{\partial a^c} = \mathrm{WCoG}\left(\widetilde{\pi}_i\right)\frac{2}{3}q_i - \frac{1}{6}\frac{\partial\left(\pi_i^L\cdot\pi_i^U + \pi_i^L\cdot\pi_i + \pi_i\cdot\pi_i^U\right)}{\partial a^c}. \tag{36}$$

Our purpose is to show the mathematical relation between a^c and $\sigma(\widetilde{\pi}_i)$. Considering the second term of (36), $\frac{\partial(\pi_i^L\cdot\pi_i^U+\pi_i^L\cdot\pi_i+\pi_i\cdot\pi_i^U)}{\partial a^c}$, it can be decomposed into three terms of $\partial(\pi_i^L\cdot\pi_i^U)/\partial a^c$, $\partial(\pi_i^L\cdot\pi_i)/\partial a^c$ and $\partial(\pi_i\cdot\pi_i^U)/\partial a^c$. The first term can be further simplified as

$$\begin{aligned}
\frac{\partial\left(\pi_i^L\cdot\pi_i^U\right)}{\partial a^c} &= \frac{\partial \pi_i^L}{\partial a^c}\pi_i^U + \frac{\partial \pi_i^U}{\partial a^c}\pi_i^L \\
&= \left\{3q_i + \frac{b^*\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + a^L - d_i^U}{3b^c}\right\}\pi_i^U + \left\{3q_i + \frac{b^{**}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + (a+r_a) - \left(d_i - l_{d_i}\right)}{3b^c}\right\}\pi_i^L \\
&= 3q_i\pi_i^U + \theta\pi_i^U + 3q_i\pi_i^L + \delta\pi_i^L,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
b^* &= \frac{b+r_b}{b^c}, \\
b^{**} &= \frac{b-l_b}{b^c} \\
\theta &= \frac{b^*\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + (a-l_a) - \left(d_i + r_{d_i}\right)}{3b^c}, \\
\delta &= \frac{b^{**}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + (a+r_a) - \left(d_i - l_{d_i}\right)}{3b^c}.
\end{aligned}$$

Similarly, the second and third terms are

$$\begin{aligned}
\frac{\partial\left(\pi_i^L\cdot\pi_i\right)}{\partial a^c} &= \frac{\partial \pi_i^L}{\partial a^c}\pi_i + \frac{\partial \pi_i}{\partial a^c}\pi_i^L \\
&= \left\{3q_i + \frac{b^*\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + (a-l_a) - \left(d_i + r_{d_i}\right)}{3b^c}\right\}\pi_i + \left\{3q_i + \frac{b^{***}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + a - d_i}{3b^c}\right\}\pi_i^L \\
&= 3q_i\pi_i + \theta\pi_i + 3q_i\pi_i^L + \eta\pi_i^L, \\
\frac{\partial\left(\pi_i\cdot\pi_i^U\right)}{\partial a^c} &= \frac{\partial \pi_i}{\partial a^c}\pi_i^U + \frac{\partial \pi_i^U}{\partial a^c}\pi_i \\
&= \left\{3q_i + \frac{b^{***}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + a - d_i}{3b^c}\right\}\pi_i^U + \left\{3q_i + \frac{b^{**}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + (a+r_a) - \left(d_i - l_{d_i}\right)}{3b^c}\right\}\pi_i \\
&= 3q_i\pi_i^U + \eta\pi_i^U + 3q_i\pi_i + \delta\pi_i
\end{aligned} \tag{38}$$

where

$$\begin{aligned}
b^{***} &= \frac{b}{b^c}, \\
\eta &= \frac{b^{***}\left(\frac{5}{3}d_i^c - \frac{1}{3}d_j^c\right) + a - d_i}{3b^c}.
\end{aligned}$$

Substituting (37) and (38) into (36), we obtain

$$\frac{\partial\sigma\left(\widetilde{\pi}_i\right)}{\partial a^c}=-\frac{7}{3}\mathrm{WCoG}\left(\widetilde{\pi}_i\right)q_i-\frac{1}{6}\left\{\left(\theta+\eta\right)\pi_i^U+\left(\theta+\delta\right)\pi_i+\left(\delta+\eta\right)\pi_i^L\right\}$$

where $\mathrm{WCoG}(\widetilde{\pi}_i)$, q_i, π_i^U, π_i, and π_i^L can be found in (26). Similarly, taking the partial derivative of (23) with respect to d_i^c, d_j^c, and b^c, respectively, we have

$$\begin{aligned}
\frac{\partial\sigma\left(\widetilde{\pi}_i\right)}{\partial d_i^c}&=\mathrm{WCoG}\left(\widetilde{\pi}_i\right)\frac{5}{3}q_i-\frac{1}{6}\left\{\left(\theta'+\eta'\right)\pi_i^U+\left(\theta'+\delta'\right)\pi_i+\left(\delta'+\eta'\right)\pi_i^L\right\},\\
\frac{\partial\sigma\left(\widetilde{\pi}_i\right)}{\partial d_j^c}&=\mathrm{WCoG}\left(\widetilde{\pi}_i\right)\frac{2a^c+2d_j^c-2_i^c}{9b^c}-\frac{1}{6}\left\{\left(\theta''+\eta''\right)\pi_i^U+\left(\theta''+\delta''\right)\pi_i+\left(\delta''+\eta''\right)\pi_i^L\right\},\\
\frac{\partial\sigma\left(\widetilde{\pi}_i\right)}{\partial b^c}&=\mathrm{WCoG}\left(\widetilde{\pi}_i\right)\frac{-3\left(2a^c-d_j^c-d_i^c\right)\left(a^c+d_j^c+d_i^c\right)}{27\left(b^c\right)^2}-\frac{1}{6}\frac{\partial\left(\pi_i^L\cdot\pi_i^U+\pi_i^L\cdot\pi_i+\pi_i\cdot\pi_i^U\right)}{\partial b^c}
\end{aligned}\tag{39}$$

where

$$\begin{aligned}
\theta'&=\frac{\frac{1}{3}b^*\left(5a^c-d_j^c-4d_i^c\right)-2\left(a-l_a\right)+2\left(d_i+r_{d_i}\right)}{3b^c},\\
\delta'&=\frac{\frac{1}{3}b^{**}\left(5a^c-d_j^c-4d_i^c\right)-2\left(a+r_a\right)+2\left(d_i-l_{d_i}\right)}{3b^c},\\
\eta'&=\frac{\frac{1}{3}b^{***}\left(5a^c-d_j^c-4d_i^c\right)-2a+2d_i}{3b^c},\\
\theta''&=\frac{b^*\left(a^c-d_i^c\right)+\left(a-l_a\right)-\left(d_i+r_{d_i}\right)}{3b^c},\\
\delta''&=\frac{b^{**}\left(a^c-d_i^c\right)+\left(a+r_a\right)-\left(d_i-l_{d_i}\right)}{3b^c},\\
\eta''&=\frac{b^{***}\left(a^c-d_i^c\right)+a-d_i}{3b^c},\\
\frac{\partial\left(\pi_i^L\cdot\pi_i^U\right)}{\partial b^c}&=\left\{\left(b^*-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i^U}{3b^c}-\pi_i^L\right)\right\}\pi_i^U+\left\{\left(b^{**}-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i^L}{3b^c}-\pi_i^U\right)\right\}\pi_i^L,\\
\frac{\partial\left(\pi_i\cdot\pi_i^U\right)}{\partial b^c}&=\left\{\left(b^{***}-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i^U}{3b^c}-\pi_i\right)\right\}\pi_i^U+\left\{\left(b^{**}-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i}{3b^c}-\pi_i^U\right)\right\}\pi_i,\\
\frac{\partial\left(\pi_i^L\cdot\pi_i\right)}{\partial b^c}&=\left\{\left(b^*-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i^U}{3b^c}-\pi_i^L\right)\right\}\pi_i+\left\{\left(b^{***}-3\right)Qq_i+\frac{1}{b^c}q_i\left(\frac{d_i^U}{3b^c}-\pi_i\right)\right\}\pi_i^L.
\end{aligned}$$

Hence, (38) and (39) indicate the change rate in the standard deviation with respect to fuzzy parameter centers. One can easily conduct sensitivity analysis of the firm's profit variation by examining the above results.

5. Conclusion

The real world is with many uncertainties in the decision-making environment so that the conventional Cournot model is inadequate to explain or describe the real situation. Many stochastic models have been developed to handle uncertainty in game-theoretical models where the uncertain parameters are typically modeled by probability distributions. However, probability distributions may not be available in practice or may be difficult to estimate from limited or absent data points. The fuzzy set theory is an appropriate tool to model such a situation when uncertain parameters cannot be described in probability distributions. In addition, the fuzzy set theory provides a mathematical approach to model intrinsic vagueness and the imprecision of human cognitive processes.

In this paper, we propose the fuzzy Cournot game for resolving fuzziness aspect of demand and cost uncertainty. We present a solving procedure for the equilibrium quantity of a general fuzzy Cournot model and indicate the condition to ensure the existence and uniqueness of the equilibrium quantity. For simplicity, we assume that the demand and cost functions of firms exist with the form of linearity and fuzzy parameters. The linearity assumption is commonly used in the literature and it helps us obtain the qualitative managerial insights without computation complexity. The equilibrium quantities of firms can be obtained by our proposed method. We further investigate the standard deviation of the fuzzy profit function of each firm to provide decision-makers with more information about the variations in profits.

In addition, we conduct sensitivity analysis to examine the effect of parameter perturbation on firms' outcomes including the equilibrium quantity, total market demand and weighted center of the fuzzy profit function. It is worth mentioning that the resulting equilibrium quantity of each firm varies with its own parameter at a faster rate than with its competitor's parameters. A second finding indicates that the center of parameter plays an important role in sensitivity analysis and it dominates how the equilibrium quantity varies due to a perturbation of fuzzy parameters. Further research may include the issue of fuzzy parameters with probability distributions, and our proposed procedure requires additional refinement for this series of research questions.

Acknowledgement

This research was supported in part by the National Science Council of Taiwan under Grant NSC96-2221-E-009-085.

References

[1] G.-S. Liang, L.-Y. Lin, C.-F. Liu, The optimum output quantity of a duopoly market under a fuzzy decision environment, Computers and Mathematics with Applications 56 (2008) 1176–1187.
[2] J. Dshalalow, W. Huang, On noncooperative hybrid stochastic games, Nonlinear Analysis. Hybrid Systems 2 (2008) 803–811.
[3] M. Sagara, H. Mukaidani, T. Yamamoto, Numerical solution of stochastic Nash games with state-dependent noise for weakly coupled large-scale systems, Applied Mathematics and Computation 197 (2008) 844–857.
[4] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[5] L. Campos, Fuzzy linear programming models to solve fuzzy matrix games, Fuzzy Sets and Systems 32 (1989) 275–289.
[6] T. Maeda, Characterization of the equilibrium strategy of the bimatrix game with fuzzy payoff, Journal of Mathematical Analysis and Applications 251 (2000) 885–896.
[7] S.-T. Liu, C. Kao, Solution of fuzzy matrix games: An application of the extension principle, International Journal of Intelligent Systems 22 (2007) 891–903.
[8] J.-S. Yao, K. Wu, Consumer surplus and producer surplus for fuzzy demand and fuzzy supply, Fuzzy Sets and Systems 103 (1999) 421–426.
[9] L.-Y. Ouyang, J.-S. Yao, A minimax distribution free procedure for mixed inventory model involving variable lead time with fuzzy demand, Computers and Operations Research 29 (2002) 471–487.
[10] K. Wu, J.-S. Yao, Fuzzy inventory with backorder for fuzzy order quantity and fuzzy shortage quantity, European Journal of Operational Research 150 (2003) 320–352.
[11] S.-C. Chang, J.-S. Yao, Optimal revenue for demand function in fuzzy sense, Fuzzy Sets and Systems 111 (2000) 419–431.
[12] J.-S. Yao, K. Wu, The best prices of two mutual complements in the fuzzy sense, Fuzzy Sets and Systems 111 (2000) 433–454.
[13] J.-S. Yao, S.-C. Chang, Economic principle on profit in the fuzzy sense, Fuzzy Sets and Systems 117 (2001) 113–127.
[14] J.-S. Yao, T.-S. Shih, Fuzzy revenue for fuzzy demand quantity based on interval-valued fuzzy sets, Computers and Operations Research 29 (2002) 1495–1535.
[15] R. Xu, X. Zhai, Optimal model for single-period supply chain problems with fuzzy demand, Information Sciences 178 (2008) 3374–3381.
[16] C. Wang, W. Tang, R. Zhao, Static Bayesian games with finite fuzzy types and the existence of equilibrium, Information Sciences 178 (2008) 4688–4698.
[17] M.J. Bender, S.P. Simonovic, A fuzzy compromise approach to water resource systems planning under uncertainty, Fuzzy Sets and Systems 115 (2000) 35–44.
[18] E.S. Lee, R.L. Li, Comparison of fuzzy numbers based on the probability measure of fuzzy events, Computers and Mathematics with Applications 15 (1988) 887–896.
[19] H.-C. Wu, Linear regression analysis for fuzzy input and output data using the extension principle, Computers and Mathematics with Applications 45 (2003) 1849–1859.
[20] J. De Andrés Sánchez, A.T. Gómez, Estimating a fuzzy term structure of interest rates using fuzzy regression techniques, European Journal of Operational Research 154 (2004) 804–818.
[21] S.-J. Chen, C.-L. Hwang, Fuzzy Multiple Attribute Decision Making: Method and Application, Springer, New York, 1992.
[22] T.J. Ross, Fuzzy Logic with Engineering Application, McGraw-Hill, Singapore, 1997.
[23] R.R. Yager, On a general class of fuzzy connectives, Fuzzy Sets and Systems 4 (1980) 235–242.
[24] D. Petrovic, R. Roy, R. Petrovic, Supply chain modeling using fuzzy sets, International Journal of Production Economics 59 (1999) 443–453.
[25] I. Giannoccaro, P. Pontrandolfo, B. Scozzi, A fuzzy echelon approach for inventory management in supply chains, European Journal of Operational Research 149 (2003) 185–196.
[26] R. Larson, B.H. Edwards, D.C. Falvo, Elementary Linear Algebra, fifth ed., Houghton Mifflin Company, Boston, 2004.