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Behavior of entire functions on balls in a Banach space ☆

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ABSTRACT

In this paper we prove that given any two disjoint balls in an infinite dimensional complex Banach space, there exists an entire function which is bounded on one and unbounded on the other.

It is known that in any infinite dimensional complex Banach space X there is an entire function f such that

$$\|f\|_{B(0,1)} := \sup\{|f(x)|:\ x \in X, \|x\| \leqslant 1\} = \infty.$$

One such function is

$$f(x) = \sum_{n=1}^{\infty} [2\varphi_n(x)]^n, \quad x \in X,$$

where (φ_n) is a *Josefson–Nissenzweig* sequence of norm one functionals that tends weak-star to 0 (see, e.g., [6], p. 157). Related to this is the following result in [2]: For every infinite dimensional Banach space X there is an entire function f with

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the following property: for any $\varepsilon > 0$ there is x_ε in the closed unit ball of X such that

$$\|f\|_{B(x_\varepsilon,\varepsilon)} = \infty.$$

This implies that

$$\inf\{r_b(x)\colon\ x \in X\} = 0,$$

where $r_b(x) = \sup\{r\colon\ \|f\|_{B(x,r)} < \infty\}$ is the *radius of boundedness* of f at x. Other related results can be found in [3], [7] and [8].

Our aim here is to prove that given any infinite dimensional complex normed space X and two disjoint balls B_1 and B_2 in X, there is an entire function that is bounded on B_1 and unbounded on B_2. This answer a question posed in [1] and also in the open problems section of the "X Conference on Function Theory on Infinite Dimensional Spaces" held in Madrid in December 2007.

We start with a lemma that is a consequence of the classical Josefson–Nissenzweig theorem (see, e.g., [4], p. 219) and whose proof is based on [2] and [5].

Lemma 1. *Let X be an infinite-dimensional Banach space and $\{\varphi_n\}$ a sequence in X^*, such that $\|\varphi_n\| = 1$ for all n and $\lim_{n\to\infty} \varphi_n(x) = 0$ for every $x \in X$. Then there are $\delta > 0$ and a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$, with $\varphi_{n_1} = \varphi_1$, such that*

$$\operatorname{dist}(\varphi_{n_k}, M_{n_k}) \geqslant \delta \quad \textit{for every } k \geqslant 2,$$

where $M_{n_k} := [\varphi_{n_1}, \varphi_{n_2}, \dots, \varphi_{n_{k-1}}]$ denotes the span of $\{\varphi_{n_1}, \varphi_{n_2}, \dots, \varphi_{n_{k-1}}\}$.

Proof. Note that $M_{n_2} = [\varphi_1]$. Assume that for every $\delta > 0$ there is $n_\delta \in \mathbb{N}, n_\delta \geqslant 2$, such that for all $n > n_\delta$ we have

$$\operatorname{dist}\bigl(\varphi_n, M'_{n_\delta}\bigr) < \frac{\delta}{2},$$

where $M'_{n_\delta} := [\varphi_1, \varphi_2, \dots, \varphi_{n_\delta - 1}]$. It is clear that

$$\operatorname{dist}\bigl(\varphi_n, M'_{n_\delta}\bigr) = 0 < \frac{\delta}{2} \quad \text{for every } n = 1, \dots, n_\delta - 1.$$

Therefore, for every $n \in \mathbb{N}$, $n \neq n_\delta$, there is $\gamma_n \in M'_{n_\delta}$ such that

$$\|\varphi_n - \gamma_n\| < \frac{\delta}{2},$$

which implies that

$$\|\gamma_n\| \leqslant \|\varphi_n\| + \frac{\delta}{2} = 1 + \frac{\delta}{2}.$$

This proves that $\{\gamma_n, n \neq n_\delta\}$ is a bounded subset of the finite-dimensional space M'_{n_δ}, which gives the existence of $\eta_1, \dots, \eta_p \in M'_{n_\delta}$ such that

$$\{\gamma_n, n \neq n_\delta\} \subset \bigcup_{l=1}^{p} B\left(\eta_l, \frac{\delta}{2}\right).$$

So we have

$$\{\varphi_n, n \in \mathbb{N}\} \subset \bigcup_{l=1}^{p} B(\eta_l, \delta) \cup B(\varphi_{n_\delta}, \delta)$$

which yields the impossible conclusion that $\{\varphi_n, n \in \mathbb{N}\}$ is a precompact set.

As a consequence, our assumption at the beginning of the proof is false. Thus, there is $\delta > 0$ such that for every $n \in \mathbb{N}$, $n \geqslant 2$, there exists $m > n$ such that $\operatorname{dist}(\varphi_m, M'_n) \geqslant \delta$.

For $n = 2$, it follows that there exists $n_2 > 2$ such that $\operatorname{dist}(\varphi_{n_2}, M'_2) \geqslant \delta$; that is, $\operatorname{dist}(\varphi_{n_2}, [\varphi_1]) \geqslant \delta$. For $n_2 + 1$ there is $n_3 > n_2 + 1$ such that $\operatorname{dist}(\varphi_{n_3}, M'_{n_2+1}) \geqslant \delta$, and then $\operatorname{dist}(\varphi_{n_3}, [\varphi_1, \varphi_{n_2}]) \geqslant \delta$. By continuing this process we get a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ which satisfies the required condition. □

Note that the particular φ_1 that we start with is not relevant in the process but the fact that φ_1 can be fixed will be used later on.

The following theorem is the principal step in the proof of our main result.

Theorem 2. *Let X be a complex Banach space of infinite dimension. Then for every $r < 1$, every $x_1 \in X$ of norm 1 and every $s > 0$, there exists an entire function h such that*

$$\|h\|_{B(0,r)} < \infty \quad \textit{and} \quad \|h\|_{B(x_1,s)} = \infty.$$

Proof. There is no loss of generality if we assume that $s < 1$. According to the above lemma, there is a sequence $\{\varphi_n\}$ in X^* and a $\delta > 0$ such that $\|\varphi_n\| = 1$, $\lim_{n\to\infty} \varphi_n(x) = 0$ for every $x \in X$ and $\operatorname{dist}(\varphi_n, M_n) \geqslant \delta$ for every $n \geqslant 2$, where we choose $\varphi_1 \in X^*$ with norm 1 such that $\varphi_1(x_1) = 1$ and $M_n = [\varphi_1, \varphi_2, \dots, \varphi_{n-1}]$. We note that what we really need will be that $|\varphi_1(x_1)| = 1$, so we can replace x_1 by any λx_1 with $|\lambda| = 1$.

Let us fix some constants that will be used in the proof. First, $t > 1$ is chosen so that $rt < 1$. Next, $\varepsilon > 0$ satisfies $t(1 - \varepsilon) > 1$, and $\gamma \in \mathbb{N}$ is such that

$$[t(1-\varepsilon)]^\gamma > \frac{2}{s\delta}.$$

Finally, $\alpha \in \mathbb{N}$, $\alpha > 2$, is chosen so that

$$\frac{1}{2^{\frac{\alpha-2}{2}}} < \delta, \qquad \frac{1}{2^{\frac{\alpha-2}{2}}} < s, \qquad \frac{1}{2^\alpha} < \varepsilon \quad \text{and} \quad t^{\gamma+1} 2^\gamma \left(\frac{1}{2^{\alpha-1}}\right) < 1.$$

We claim that there is a subsequence $\{n_k\}$ in $\mathbb{N}$ with $n_1 = 1$ and a corresponding sequence $\{x_{n_k}\}$ in X such that $x_{n_1} := x_1$, $\|x_{n_k}\| \leqslant 1$ for every $k \geqslant 1$,

$$|\varphi_{n_j}(x_{n_k})| < \frac{1}{2^\alpha} \quad \text{if } j \neq k \quad \text{and} \quad |\varphi_{n_k}(x_{n_k})| \geqslant \delta - \frac{1}{2^\alpha} \quad \text{for every } k \geqslant 2.$$

Indeed, since $\lim_{n\to\infty} \varphi_n(x_1) = 0$, then there is $n_2 \geqslant 2$ such that $|\varphi_n(x_1)| < \frac{1}{2^\alpha}$ for all $n \geqslant n_2$. By the Hahn–Banach theorem there is $\Psi_1 \in X^{**}$ such that $\|\Psi_1\| = 1$, $\Psi_1(\varphi_{n_2}) = \operatorname{dist}(\varphi_{n_2}, [\varphi_1]) \geqslant \delta$ and $\Psi_1(\varphi_1) = 0$.

By Goldstine's theorem (see, e.g., [4], p. 13), there is $x_{n_2} \in \overline{B}(0, 1)$ such that

$$|\Psi_1(\varphi_j) - \varphi_j(x_{n_2})| < \frac{1}{2^\alpha} \quad \text{for } j = n_1 \text{ and } n_2.$$

Bearing in mind that $\Psi_1(\varphi_1) = 0$ we get $|\Psi_1(\varphi_{n_2}) - \varphi_{n_2}(x_{n_2})| < \frac{1}{2^\alpha}$ and $|\varphi_1(x_{n_2})| < \frac{1}{2^\alpha}$, which implies that

$$|\varphi_{n_2}(x_{n_2})| \geqslant |\Psi_1(\varphi_{n_2})| - \frac{1}{2^\alpha} \geqslant \delta - \frac{1}{2^\alpha}.$$

Since $\lim_{n\to\infty} \varphi_n(x_{n_2}) = 0$ there is $n_3 > n_2$ such that $|\varphi_n(x_{n_2})| < \frac{1}{2^\alpha}$ for all $n \geqslant n_3$ (and also $|\varphi_{n_3}(x_1)| < \frac{1}{2^\alpha}$). As $\operatorname{dist}(\varphi_{n_3}, M_{n_3}) \geqslant \delta$, there is $\Psi_2 \in X^{**}$ such that

$$\|\Psi_2\| = 1, \qquad \Psi_2(\varphi_{n_3}) = \operatorname{dist}(\varphi_{n_3}, [\varphi_1, \varphi_{n_2}]) \geqslant \operatorname{dist}(\varphi_{n_3}, M_{n_3}) \geqslant \delta$$

and

$$\Psi_2(\varphi_1) = 0 \quad \text{and} \quad \Psi_2(\varphi_{n_2}) = 0.$$

Again Goldstine's theorem gives the existence of $x_{n_3} \in \overline{B}(0, 1)$ such that $|\varphi_1(x_{n_3})| < \frac{1}{2^\alpha}$, $|\varphi_{n_2}(x_{n_3})| < \frac{1}{2^\alpha}$ and $|\Psi_2(\varphi_{n_3}) - \varphi_{n_3}(x_{n_3})| < \frac{1}{2^\alpha}$. Therefore,

$$|\varphi_{n_3}(x_{n_3})| \geqslant |\Psi_2(\varphi_{n_3})| - \frac{1}{2^\alpha} \geqslant \delta - \frac{1}{2^\alpha}.$$

The claim follows by a straightforward induction argument.

Let us check that the function $h : X \to \mathbb{C}$ given by

$$h(x) = \sum_{k=2}^{\infty} \big([t\varphi_1(x)]^\gamma t\varphi_{n_k}(x)\big)^k$$

satisfies the required conditions.

Since $\{\varphi_n\}$ tends to 0 weak-star and $\|\varphi_n\| = 1$ for all n, we see that $\{\varphi_n\}$ tends to 0 uniformly on the compact subsets of X. Let $K \subset X$ be an arbitrary compact set. For every $x \in K$,

$$\sum_{k=2}^{\infty} \big|[t\varphi_1(x)]^\gamma t\varphi_{n_k}(x)\big|^k \leqslant \sum_{k=2}^{\infty} \big([t\|\varphi_1\|_K]^\gamma t\|\varphi_{n_k}\|_K\big)^k.$$

As

$$\lim_{k\to\infty}(t\|\varphi_1\|_K)^\gamma t\|\varphi_{n_k}\|_K = 0$$

the series which defines h is uniformly convergent on K, and then h is an entire function on X.

Moreover,

$$\|h\|_{B(0,r)} \leqslant \sum_{k=2}^{\infty}\left((t\|\varphi_1\|_{B(0,r)})^\gamma t\|\varphi_{n_k}\|_{B(0,r)}\right)^k \leqslant \sum_{k=2}^{\infty}\left[(tr)^{\gamma+1}\right]^k < \infty.$$

Now, let us fix $j \geqslant 2$ and estimate $|h(x_1 + sx_{n_j})|$. On the one hand,

$$\begin{aligned}|t\varphi_1(x_1 + sx_{n_j})|^\gamma &\geqslant t^\gamma\left(|\varphi_1(x_1)| - s|\varphi_1(x_{n_j})|\right)^\gamma\\ &> t^\gamma\left(1 - \frac{s}{2^\alpha}\right)^\gamma > t^\gamma\left(1 - \frac{1}{2^\alpha}\right)^\gamma,\end{aligned}$$

while on the other hand,

$$\begin{aligned}|t\varphi_{n_j}(x_1 + sx_{n_j})| &\geqslant t\left(s|\varphi_{n_j}(x_{n_j})| - |\varphi_{n_j}(x_1)|\right) > t\left(s\left(\delta - \frac{1}{2^\alpha}\right) - \frac{1}{2^\alpha}\right)\\ &> t\left(s\delta - \frac{1}{2^{\alpha-1}}\right) > t\frac{s\delta}{2}.\end{aligned}$$

Therefore the jth summand of $h(x_1 + sx_{n_j})$ satisfies:

$$\begin{aligned}&\left(|t\varphi_1(x_1 + sx_{n_j})|^\gamma |t\varphi_{n_j}(x_1 + sx_{n_j})|\right)^j\\ &\quad > \left[t^{\gamma+1}\left(1 - \frac{1}{2^\alpha}\right)^\gamma \frac{s\delta}{2}\right]^j > \left[t[t(1-\varepsilon)]^\gamma \frac{s\delta}{2}\right]^j.\end{aligned}$$

Let us see how we can control the sum of the other terms by a constant independent of j. By our choice of t, γ, and α, given $k \geqslant 2, k \neq j$,

$$\begin{aligned}&|t\varphi_1(x_1 + sx_{n_j})|^\gamma |t\varphi_{n_k}(x_1 + sx_{n_j})|\\ &\quad \leqslant [t(1+s)]^\gamma t\left(\frac{1}{2^\alpha} + s\frac{1}{2^\alpha}\right) \leqslant t^{\gamma+1}2^\gamma\left(\frac{1}{2^{\alpha-1}}\right) < 1.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{k=2,k\neq j}^{\infty} \left|t[\varphi_1(x_1 + sx_{n_j})]^\gamma t\varphi_{n_k}(x_1 + sx_{n_j})\right|^k\\ &\quad \leqslant \sum_{k=2}^{\infty}\left(t^{\gamma+1}2^\gamma\left(\frac{1}{2^{\alpha-1}}\right)\right)^k = M, \quad \text{say.}\end{aligned}$$

As a consequence,

$$
\begin{aligned}
|h(x_1 + sx_{n_j})| &\geqslant \left|[t\varphi_1(x_1 + sx_{n_j})]^\gamma t\varphi_{n_j}(x_1 + sx_{n_j})\right|^j \\
&\qquad - \sum_{k=2, k\neq j}^{\infty} \left|[t\varphi_1(x_1 + sx_{n_j})]^\gamma t\varphi_{n_k}(x_1 + sx_{n_j})\right|^k \\
&\geqslant \left[t[t(1-\varepsilon)]^\gamma \frac{s\delta}{2}\right]^j - M.
\end{aligned}
$$

Since $t[t(1-\varepsilon)]^\gamma \frac{s\delta}{2} > 1$, we get

$$
\sup_{j \geqslant 2} |h(x_1 + sx_{n_j})| = \infty
$$

and thus $\|h\|_{B(x_1,s)} = \|h\|_{\overline{B}(x_1,s)} = \infty$. $\square$

Our main result, below, is now seen to be an easy corollary of our previous theorem.

Corollary 3. *Let X be a complex Banach space. Then, for every $B(x_1, r_1)$ in X, every point $x_2 \in X \backslash \overline{B}(x_1, r_1)$ and every $r_2 > 0$ there is an entire function f such that*

$$
\|f\|_{B(x_1,r_1)} < \infty
$$

and

$$
\|f\|_{B(x_2,r_2)} = \infty.
$$

By replacing f by an appropriate multiple we can get the supremum on $B(x_1, r_1)$ to be as small as we want.

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