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Generalized group actions in a global setting

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Abstract

We study generalized group actions on differentiable manifolds in the Colombeau framework, extending previous work on flows of generalized vector fields and symmetry group analysis of generalized solutions. As an application, we analyze group invariant generalized functions in this setting.

Keywords: Generalized group actions; Colombeau generalized functions; Group invariance; Symmetry group analysis of generalized solutions

1. Introduction

Lie group analysis of differential equations is an indispensable tool for studying invariance properties of solutions of PDE as well as for finding explicit solutions, with a wealth of applications (cf. [4,25]). In [19,29–31], a study of invariance properties of distributions and distributional solutions of linear partial differential equations was initiated. Later on, symmetry group analysis of PDEs in generalized functions and systematic methods of deriving group invariant fundamental solutions using infinitesimal techniques of group analysis were developed [1–3,11]. Clearly, in the distributional setting a restriction to linear equations and linear projectable transformation groups is unavoidable. On the other hand, many applied problems (e.g., systems of conservation laws) underline the need for an extension of the above techniques in order to handle nonlinear

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problems involving singularities. Algebras of generalized functions provide a setting for addressing such questions in a coherent way. This line of research was initiated in [24,27,28] in the framework of the 'nowhere dense' algebras of E.E. Rosinger.

An alternative approach, based on Colombeau's theory of algebras of generalized functions [5,6,20], was developed in [7,14,21,22] and will form the basis for the present paper. In particular, in [7,14] criteria for classical symmetry groups to transform weak (distributional, Colombeau or associated) solutions of a given (smooth) system of differential equations into other solutions of the same type were given. In [7,14,21,22], additionally both the differential operators and the group actions are allowed to be given by generalized functions. The setting of generalized functions employed in these works is that of $\mathcal{G}_\tau$, the space of tempered Colombeau functions. As elements of $\mathcal{G}_\tau$ are characterized by global bounds, this setting appears unsuitable for an extension of the theory to the manifold setting. To lift this limitation, in the present work we employ the recently developed theory of Colombeau generalized functions taking values in differentiable manifolds [13,18] as well as the theory of generalized flows of singular vector fields [15] to extend symmetry group analysis in Colombeau generalized functions to a global setting.

The paper is divided into 6 sections. Section 2 provides basic notations and definitions from Colombeau's theory of algebras of generalized functions (in particular in the manifold setting) and symmetry group analysis. In Section 3 we consider generalized group actions and provide a notion of rank of a generalized function, which will be crucial for the infinitesimal criteria to be developed in Section 5. The question of localizing Colombeau generalized functions and an analysis of solution sets of generalized equations is the focus of Section 4. By borrowing a notion from nonstandard analysis we introduce the concept of near-standard points and show that these suffice to characterize equality of Colombeau functions. In Section 5 we prove an infinitesimal criterion for symmetry groups of generalized algebraic equations and apply the obtained results in Section 6 to symmetry group analysis of differential equations in the Colombeau framework. Finally, in Section 7 we turn to the topic of group invariant generalized functions in this setting. Based on a recent result of Pilipović et al. [26] we provide an affirmative answer to an open question posed by M. Oberguggenberger in [22] whether standard rotations suffice to characterize rotational invariance of Colombeau generalized functions.

2. Notations

In what follows, M and N will denote smooth, connected, paracompact Hausdorff manifolds of dimensions m and n, respectively.

Set $I = (0, 1]$ and denote by $\mathcal{P}(M)$ the space of linear differential operators on M. The spaces of moderate respectively negligible nets in M are defined as

$$\mathcal{E}_M(M) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)^I\colon \forall K \Subset M,\ \forall P \in \mathcal{P}(M)\ \exists p \in \mathbb{N}\colon \sup_{x\in K}\big|Pu_\varepsilon(x)\big| = O(\varepsilon^{-p})\Big\},$$

$$\mathcal{N}(M) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M)\colon \forall K \Subset M,\ \forall m \in \mathbb{N}_0\colon \sup_{x\in K}\big|u_\varepsilon(x)\big| = O(\varepsilon^{m})\Big\}$$

(due to [10, Chapter 1, Theorem 1.2.3], for the characterization of $\mathcal{N}(M)$ as a subspace of $\mathcal{E}_M(M)$ it is sufficient to estimate only the 0th order derivative). Clearly, $\mathcal{N}(M)$ is an ideal of the differential algebra $\mathcal{E}_M(M)$. The special Colombeau algebra $\mathcal{G}(M)$ on M is defined as the quotient $\mathcal{E}_M(M)/\mathcal{N}(M)$; it is an associative, commutative differential algebra whose elements are equivalence classes denoted by $u = [(u_\varepsilon)_\varepsilon]$. $\mathcal{G}(\_)$ is a fine sheaf of differential algebras with respect to the Lie derivative along smooth vector fields. $\mathcal{C}^\infty(M)$ is a subalgebra of $\mathcal{G}(M)$ and there exist injective sheaf morphisms embedding $\mathcal{D}'(\_)$ linearly into $\mathcal{G}(\_)$.

A point value characterization of Colombeau generalized functions is based on the concept of compactly supported generalized points [9,23]. The space of compactly supported generalized points M_c is the set of all nets $(x_\varepsilon)_\varepsilon \in M^I$ for which x_ε stays in a fixed compact set for ε small. In M_c one introduces an equivalence relation $\sim$ in the following way: for $(x_\varepsilon)_\varepsilon, (y_\varepsilon)_\varepsilon \in M_c$, $(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow d_h(x_\varepsilon, y_\varepsilon) = O(\varepsilon^m)$, for each $m > 0$, where d_h denotes the distance function induced on M by one (hence any) Riemannian metric h. The quotient space $\tilde{M}_c := M_c/\sim$ is called the space of compactly supported generalized points on M, and we denote its elements by $\tilde{x} = [(x_\varepsilon)_\varepsilon]$. In the case $M = \mathbb{R}$ one also defines the ring of generalized numbers $\tilde{\mathbb{R}}$ as the quotient of the set of moderate nets of numbers $(r_\varepsilon)_\varepsilon \in \mathbb{R}^I$ with $|r_\varepsilon| = O(\varepsilon^{-p})$ for some $p \in \mathbb{N}$ modulo the set of negligible nets $(r_\varepsilon)_\varepsilon$ with $|r_\varepsilon| = O(\varepsilon^m)$ for each m. It is the ring of constants in the Colombeau algebra. Insertion of a compactly supported generalized point into any representative of a Colombeau generalized function produces a well-defined element of $\tilde{\mathbb{R}}$. Moreover, elements of $\mathcal{G}(M)$ are uniquely determined by their values on $\tilde{M}_c$.

In order to describe generalized functions on the manifold M taking values in the manifold N one introduces the space $\mathcal{G}[M,N]$ of compactly supported (or c-bounded for short) generalized functions. A net $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M,N)^I$ is called c-bounded if

$$\forall K \Subset M\ \exists \varepsilon_0 > 0\ \exists K' \Subset N\ \forall \varepsilon < \varepsilon_0\colon\ u_\varepsilon(K) \subseteq K'.$$

A c-bounded net is moderate if it satisfies:

$\forall k \in \mathbb{N}$, for each chart (V,φ) in M, each chart (W,ψ) in N, each $L \Subset V$ and each $L' \Subset W$ there exists $p \in \mathbb{N}$ with

$$\sup_{x \in L \cap u_\varepsilon^{-1}(L')} \bigl\| D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1})\bigl(\varphi(x)\bigr)\bigr\| = O(\varepsilon^{-p}).$$

Denote by $\mathcal{E}_M[M,N]$ the set of all moderate c-bounded nets. Introduce an equivalence relation $\sim$ in $\mathcal{E}_M[M,N]$ in the following way: $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_M[M,N]$, $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$ if

(i) $\forall K \Subset M$, $\sup_{x\in K} d_h(u_\varepsilon(x), v_\varepsilon(x)) \to 0$ $(\varepsilon \to 0)$ for some (hence every) Riemannian metric h on N.
(ii) $\forall k \in \mathbb{N}_0\ \forall m \in \mathbb{N}$, for each chart (V,φ) in M, each chart (W,ψ) in N, each $L \Subset V$ and each $L' \Subset W$:

$$\sup_{x \in L \cap u_\varepsilon^{-1}(L') \cap v_\varepsilon^{-1}(L')} \bigl\| D^{(k)}(\psi \circ u_\varepsilon \circ \varphi^{-1} - \psi \circ v_\varepsilon \circ \varphi^{-1})\bigl(\varphi(x)\bigr)\bigr\| = O(\varepsilon^m).$$

The space of c-bounded Colombeau generalized functions from M to N is defined as the quotient $\mathcal{G}[M,N] := \mathcal{E}_M[M,N]/\sim$.

Alternative characterizations of the notions of moderateness and equivalence for the elements of $\mathcal{C}^\infty(M,N)^I$ are: $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M[M,N] \Leftrightarrow (f \circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M)$, $\forall f \in \mathcal{C}^\infty(N)$ [18, Proposition 3.2] and $(u_\varepsilon)_\varepsilon \sim (v_\varepsilon)_\varepsilon$ $((u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon) \in \mathcal{E}_M[M,N]) \Leftrightarrow (f \circ u_\varepsilon - f \circ v_\varepsilon)_\varepsilon \in \mathcal{N}(M)$, $\forall f \in \mathcal{C}^\infty(N)$ [18, Theorem 3.3]. Similarly as for the elements of $\mathcal{G}(M)$, if $u \in \mathcal{G}[M,N]$ and $\tilde{x} \in \tilde{M}_c$ then $u(\tilde{x}) = [(u_\varepsilon(x_\varepsilon))_\varepsilon]$ is a well-defined element of $\tilde{N}_c$, and elements of $\mathcal{G}[M,N]$ are uniquely determined by their values on all compactly supported generalized points on M, i.e. $u = v \Leftrightarrow u(\tilde{x}) = v(\tilde{x})$, $\forall \tilde{x} \in \tilde{M}_c$ [18, Theorem 3.5].

If $E \to M$ is any vector bundle over M, denote by $\Gamma(M, E)$ the space of smooth sections of E, and by $\mathcal{P}(M, E)$ the space of linear differential operators $\Gamma(M, E) \to \Gamma(M, E)$. The module of generalized sections of E, $\Gamma_{\mathcal{G}}(M, E)$, is defined as the quotient $\Gamma_{\mathcal{E}_M}(M, E)/\Gamma_{\mathcal{N}}(M, E)$ where

$$\Gamma_{\mathcal{E}_M}(M, E) := \Big\{(s_\varepsilon)_\varepsilon \in \Gamma(M, E)^I \colon \forall P \in \mathcal{P}(M, E)\ \forall K \Subset M\ \exists p \in \mathbb{N}\colon \sup_{x\in K} \big\| P u_\varepsilon(x) \big\|_h = O(\varepsilon^{-p})\Big\},$$

$$\Gamma_{\mathcal{N}}(M, E) := \Big\{(s_\varepsilon)_\varepsilon \in \Gamma_{\mathcal{E}_M}(M, E) \colon \forall K \Subset M\ \forall m \in \mathbb{N}\colon \sup_{x\in K} \big\| u_\varepsilon(x) \big\|_h = O(\varepsilon^{m})\Big\},$$

where $\| \, \|_h$ is the norm on the fibers of E induced by any Riemannian metric on M. $\Gamma_{\mathcal{G}}(\_, E)$ is a fine sheaf of projective and finitely generated $\mathcal{G}(M)$-modules, and

$$\Gamma_{\mathcal{G}}(M, E) = \mathcal{G}(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, E).$$

If E is some tensor bundle $T^r_s M$ we write $\mathcal{G}^r_s(M)$ instead of $\Gamma_{\mathcal{G}}(M, T^r_s M)$; in particular, if E is the tangent bundle $TM (= T^1_0 M)$ then $\mathcal{G}^1_0(M)$ is the space of generalized vector fields on M.

We say that a generalized vector field $\xi \in \mathcal{G}^1_0(M)$ is locally bounded, respectively locally of L^∞-log type, if for all $K \Subset M$ and one (hence every) Riemannian metric h on M we have for any representative $(\xi_\varepsilon)_\varepsilon$ and ε sufficiently small

$$\sup_{x\in K} \big\| \xi_\varepsilon|_x \big\|_h \leqslant C, \quad \text{respectively} \quad \sup_{x\in K} \big\| \xi_\varepsilon|_x \big\|_h \leqslant C|\log\varepsilon|.$$

ξ is called globally bounded with respect to h if for some (hence every) representative $(\xi_\varepsilon)_\varepsilon$ of ξ there exists $C > 0$ with

$$\sup_{x\in M} \big\| \xi_\varepsilon|_x \big\|_h \leqslant C,$$

for ε small (cf. [15, Definition 3.4]).

To conclude this section we fix some notations from symmetry group analysis of differential equations, following [25]. Let X and U be spaces of independent and dependent variables and suppose that G is a local Lie group of transformations acting regularly on some open subset $M \subseteq X \times U$; for the group action we write $g \cdot (x, u) = (\Xi_g(x, u), \Psi_g(x, u))$, with appropriate smooth functions Ξ_g and Ψ_g. If Ξ_g does not depend on the dependent variables the group action is called projectable. The n-jet space of M will be denoted by $M^{(n)}$ and the nth prolongation of a group action g, respectively vector field $\mathbf{v}$, by $\mathrm{pr}^{(n)} g$, respectively $\mathrm{pr}^{(n)}\mathbf{v}$. If $\Delta_\nu(x, u^{(n)}) = 0$ $(1 \leqslant \nu \leqslant l)$ is a system of nth order differential equations on M, where $\Delta : X \times U^{(n)} \to \mathbb{R}^l$ is a smooth function, then the solution set of Δ is the subvariety $S_\Delta := \{(x, u^{(n)})\colon \Delta(x, u^{(n)}) = 0\}$ of $X \times U^{(n)}$. We say that a function f is a solution of the system if the n-jet of the graph $\Gamma_f = \{(x, f(x))\colon x \in \Omega\} \subset X \times U$ of f, i.e. $\Gamma^{(n)}_f$ is contained in S_Δ. A symmetry group of Δ is a local transformation group G acting on M with the property that whenever $u = f(x)$ is a solution of the system and $g \cdot f$ $(g \in G)$ is defined, then $g \cdot f$ is again a solution of Δ.

3. Generalized group actions

To begin with we recall the following definitions from [15].

Definition 3.1. A generalized group action on a manifold M is an element $\Phi \in \mathcal{G}[\mathbb{R} \times M, M]$ with the following properties:

(i) $\Phi(0,\cdot) = id$ in $\mathcal{G}[M, M]$,
(ii) $\Phi(\eta_1 + \eta_2, x) = \Phi(\eta_1, \Phi(\eta_2, x))$ in $\mathcal{G}[\mathbb{R}^2 \times M, M]$.

In the following definition we make use of $\mathcal{G}^h$, the space of hybrid Colombeau functions defined on a manifold and taking values in a vector bundle which was introduced in [17] (see also [10]).

Definition 3.2. Let $\xi \in \mathcal{G}^1_0(M)$ be a generalized vector field such that there exists a unique generalized group action $\Phi \in \mathcal{G}[\mathbb{R} \times M, M]$ satisfying

$$\frac{d}{d\eta}\Phi(\eta, x) = \xi\bigl(\Phi(\eta, x)\bigr) \quad \text{in } \mathcal{G}^h[\mathbb{R} \times M, TM]. \tag{1}$$

Then ξ is called the infinitesimal generator of Φ and both ξ and its generalized flow Φ are called $\mathcal{G}$-complete. We call ξ and Φ strictly $\mathcal{G}$-complete if, in addition, there exist representatives $(\xi_\varepsilon)_\varepsilon$, $(\Phi_\varepsilon)_\varepsilon$ such that Φ_ε is the flow of ξ_ε for each $\varepsilon \in I$.

Even for not necessarily $\mathcal{G}$-complete group actions Φ we shall call a generalized vector field ξ an infinitesimal generator of Φ if (1) holds. In practice, since in order to show $\mathcal{G}$-completeness one usually works componentwise, the condition of strict $\mathcal{G}$-completeness is normally no additional restriction, cf. the following remark.

Remark 3.3. Sufficient conditions for $\mathcal{G}$-completeness of a generalized vector field ξ have been derived in [15, Theorem 3.5], for the case of (M, h) a complete Riemannian manifold, to wit:

(i) ξ is globally bounded with respect to h, and
(ii) for each first-order differential operator $P \in \mathcal{P}(M, TM)$, $P\xi$ is locally of L^∞-log-type.

In fact, these conditions even ensure strict $\mathcal{G}$-completeness of ξ.

One of our main interest in generalized group actions in this work will be symmetry properties in the following sense.

Definition 3.4. Let $F \in \mathcal{G}(M)$ and let Φ be a $\mathcal{G}$-complete generalized group action on M. Φ is called a symmetry group of the equation

$$F(x) = 0$$

in $\mathcal{G}(M)$ if for any $\tilde{x} \in \tilde{M}_c$ with $F(\tilde{x}) = 0 \in \tilde{\mathbb{R}}$ we have $F(\Phi(\tilde{\eta}, \tilde{x})) = 0$ in $\tilde{\mathbb{R}}$, for every $\tilde{\eta} \in \tilde{\mathbb{R}}_c$ (i.e., $\eta \mapsto F(\Phi(\eta, \tilde{x})) = 0$ in $\mathcal{G}(\mathbb{R})$). If $F = (F_\nu)_{\nu=1}^l \in \mathcal{G}(M)^l$, then Φ is called a symmetry group of the equation $F = 0$ if it is a symmetry group of each equation $F_\nu = 0$ $(1 \leqslant \nu \leqslant l)$.

We note that, since $\mathcal{G}[M, \mathbb{R}]$ is naturally contained in $\mathcal{G}(M)$, the above definitions and results directly apply to c-bounded generalized functions as well.

As in the classical case (cf. [25, Chapter 2]) our first aim is to derive infinitesimal criteria characterizing symmetries of "algebraic" equations as in Definition 3.4. In the smooth setting, one supposes a maximal rank condition on F and then uses distinguished local charts to obtain the desired result. In our present context, however, a direct transfer of classical methods is impossible due to the lack of structure of the space $\tilde{M}_c$ of compactly supported generalized points on M. In particular, elements of $\tilde{M}_c$ are only very weakly localized in the sense that every $\tilde{x} \in \tilde{M}_c$ possesses a representative contained in a suitable compact set in M. We therefore call an open set $U \subseteq M$ a neighborhood of $\tilde{x} = [(x_\varepsilon)_\varepsilon]$ if

$$\exists \varepsilon_0 \; \exists K \Subset U \text{ s.t. } x_\varepsilon \in K \; \forall \varepsilon < \varepsilon_0.$$

Moreover, in the absence of an inverse function theorem in the generalized function setting, it is a priori not clear how to define the rank of a generalized function. Since, on the positive side, inversion of generalized functions is possible in $\mathcal{G}[M, N]$ we suggest the following notion of rank of a generalized map.

Definition 3.5. Let $F \in \mathcal{G}[M, N]$, $k \in \{0, \ldots, \min(m, n)\}$ and $\tilde{x} \in \tilde{M}_c$. F is called of rank k in $\tilde{x}$ if there exist open neighborhoods $U \subseteq M$ of $\tilde{x}$, $V \subseteq N$ of $F(\tilde{x})$, open sets $U' \subseteq \mathbb{R}^m$, $V' \subseteq \mathbb{R}^n$, $\varepsilon_0 > 0$ and diffeomorphisms $\varphi_\varepsilon : U \to U'$, $\psi_\varepsilon : V \to V'$ for each $\varepsilon \in (0, \varepsilon_0]$ with $\varphi = [(\varphi_\varepsilon)_\varepsilon] \in \mathcal{G}[U, U']$, $\varphi^{-1} := [(\varphi_\varepsilon^{-1})_\varepsilon] \in \mathcal{G}[U', U]$, $\psi = [(\psi_\varepsilon)_\varepsilon] \in \mathcal{G}[V, V']$, $\psi^{-1} := [(\psi_\varepsilon^{-1})_\varepsilon] \in \mathcal{G}[V', V]$ such that $F|_U \in \mathcal{G}[U, V]$ and

$$\psi \circ F \circ \varphi^{-1} = (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$$

in $\mathcal{G}[U', V']$. If $A \subseteq U$ then F is called of rank k globally on A.

It is straightforward to adapt this definition also to the case where $F \in \mathcal{G}(M)^l$ (set $N = \mathbb{R}^n$ and $\psi_\varepsilon = id$ for all ε).

According to the above discussion it is natural to ask whether a more strict localization than the one used in Definition 3.5 is attainable in general. Before we proceed with the theory of symmetry groups of generalized algebraic equations we should therefore investigate the possibility of localizing Colombeau generalized functions, respectively solution sets of generalized equations. The following section is devoted to this purpose.

4. Localization

By the point value characterization of Colombeau generalized functions (cf. [16,18,23]), elements of $\mathcal{G}(M)$ as well as of $\mathcal{G}[M, N]$ are uniquely determined by their values on compactly supported generalized points on M.

As was mentioned in the previous section, elements of $\tilde{M}_c$ are only weakly localized, so in particular the existence of suitable open neighborhoods of $\tilde{x} \in \tilde{M}_c$ as in Definition 3.5 is not necessarily guaranteed. Therefore the question arises whether we need all elements of $\tilde{M}_c$ to characterize elements of $\mathcal{G}(M)$ (or $\mathcal{G}[M, \mathbb{R}]$) or if more strongly localized generalized points suffice. The following definition borrows a concept from nonstandard analysis to specify what is meant by this notion.

Definition 4.1. A point $\tilde{x} \in \tilde{M}_c$ is called near-standard if there exists $x \in M$ such that $\tilde{x} \approx x$ (i.e., $x_\varepsilon \to x$ $(\varepsilon \to 0)$ for every representative of $\tilde{x}$).

In particular, any neighborhood of x is a neighborhood of $\tilde{x} \approx x$ in the sense of Section 3. Near-standard points indeed suffice to characterize Colombeau generalized functions.

Proposition 4.2.

(i) *Let $u \in \mathcal{G}(M)$. Then $u = 0$ if and only if $u(\tilde{x}) = 0$, for all near-standard points $\tilde{x} \in \tilde{M}_c$.*
(ii) *Let $u, v \in \mathcal{G}[M, N]$. Then $u = v$ if and only if $u(\tilde{x}) = v(\tilde{x})$, for all near-standard points $\tilde{x} \in \tilde{M}_c$.*

Proof. (i) One direction is clear. So, suppose that $u(\tilde{x}) = 0$, for all near-standard points $\tilde{x} \in \tilde{M}_c$ and suppose that $u \neq 0$. Then

$$\exists K \Subset M\ \exists m\ \forall k \in \mathbb{N}\ \exists x_k \in K\ \exists \varepsilon_k < \min\left(\frac{1}{k}, \varepsilon_{k-1}\right):\ \left|u_{\varepsilon_k}(x_k)\right| > k\varepsilon_k^m. \tag{2}$$

Since K is a compact set there exists a subsequence x_{k_l} which converges to $x \in K$. Set $x_\varepsilon := x_{k_l}$ for $\varepsilon \in (\varepsilon_{k_{l+1}}, \varepsilon_{k_l}]$ and $\tilde{x} := [(x_\varepsilon)_\varepsilon] \in \tilde{M}_c$. $\tilde{x}$ is a near-standard point and from (2) it follows that $u(\tilde{x}) \neq 0$, which gives a contradiction.

(ii) Necessity is again obvious. For the converse direction we use the characterization of c-bounded generalized functions given in [18]. Let $f \in \mathcal{C}^\infty(N)$. Then $f \circ u$ and $f \circ v$ are well-defined elements of $\mathcal{G}(M)$. For any near-standard point $\tilde{x} \in \tilde{M}_c$ we have by (i) that

$$(f \circ u)(\tilde{x}) = (f \circ v)(\tilde{x}),$$

so $f \circ u = f \circ v$ in $\mathcal{G}(M)$. Hence, $(f \circ u_\varepsilon - f \circ v_\varepsilon)_\varepsilon \in \mathcal{N}(M)$ and by [18, Theorem 3.3] it follows that $u = v$. □

In the smooth setting, a maximal rank condition on the set of solutions of an equation $F(x) = 0$ allows to derive an infinitesimal criterion for symmetry groups of the equation (cf. [25, Chapter 2]). In the generalized case, however, the assumption of maximal rank in each near-standard point $\tilde{x} \in \tilde{M}_c$ which is a solution of $F(x) = 0$, $F \in \mathcal{G}(M)$, may be insufficient. We illustrate this by the following example.

Example 4.3. Set $M = \mathbb{R}$, $I = (0, 1]$ and $J = \bigcup_{n=1}^\infty (\frac{1}{2n+1}, \frac{1}{2n}]$. Let

$$x_\varepsilon = \begin{cases} 0, & \varepsilon \in J, \\ 1, & \varepsilon \in I \setminus J, \end{cases} \tag{3}$$

$$F_\varepsilon(x) = \begin{cases} x, & \varepsilon \in J, \\ x - 1, & \varepsilon \in I \setminus J. \end{cases} \tag{4}$$

Then $F_\varepsilon(x_\varepsilon) = 0$ for all ε and $\tilde{x} := [(x_\varepsilon)_\varepsilon]$ is not a near-standard point. We claim that the solution set $S_F = \{\tilde{y} \in \tilde{\mathbb{R}}_c \mid F(\tilde{y}) = 0\}$ does not contain any near-standard point. To see this, suppose that $\tilde{y} = [(y_\varepsilon)_\varepsilon]$ satisfies $F(\tilde{y}) = 0$. Then

$$F_\varepsilon(y_\varepsilon) = y_\varepsilon - 1 \quad \text{on } I \setminus J$$

and

$$F_\varepsilon(y_\varepsilon) = y_\varepsilon \quad \text{on } J.$$

Suppose that $\tilde{y}$ is a near-standard point and choose $y \in \mathbb{R}$ such that $y_\varepsilon \to y$ when $\varepsilon \to 0$. Then since $F(\tilde{y}) = 0$ we obtain $y - 1 = 0 = y$, a contradiction. Moreover, the above reasoning implies that $\tilde{x}$ is in fact the only zero of the equation $F(x) = 0$ in $\tilde{\mathbb{R}}_c$.

This example shows that there exist functions whose solution set is nonempty although it does not contain any near-standard points. In order to obtain infinitesimal criteria for an equation $F(x)=0$ we will therefore have to require a maximal rank condition in a neighborhood of all of S_F, no matter which types of generalized points belong to it.

An alternative localization strategy consists in considering an open covering $\mathcal{U}$ of M. Since $\mathcal{G}$ is a sheaf, $F=0$ on M if and only if $F=0$ on each open set $U\subseteq M$. Also, if Φ is a symmetry group of the equation

$$F(x)=0 \quad \text{in } \mathcal{G}(M)$$

then for every open covering $\mathcal{U}$ of M, Φ is a symmetry group of

$$F|_U(x)=0$$

for each $U\in\mathcal{U}$. However, a localization to near-standard points fails in general: consider again Example 4.3. Let $U_1=(-\infty,\frac{1}{2})$ and $U_2=(\frac{1}{4},\infty)$. Then the intersection of S_F with both $(\tilde{U}_1)_{\mathrm{c}}$ and $(\tilde{U}_2)_{\mathrm{c}}$ is empty, although S_F itself is nonempty, consisting precisely of the generalized point $\tilde{x}$ from (3).

5. Infinitesimal criteria

Our aim in this section is to derive infinitesimal criteria for symmetry groups of algebraic equations in the Colombeau setting. To this end we will need the following auxiliary result.

Lemma 5.1. *Let $\psi_\varepsilon: M\to N$ $(\varepsilon\in I)$ be a net of diffeomorphisms such that $\psi=[(\psi_\varepsilon)_\varepsilon]\in\mathcal{G}[M,N]$ and $\psi^{-1}=[(\psi_\varepsilon^{-1})_\varepsilon]\in\mathcal{G}[N,M]$. If Φ is a strictly $\mathcal{G}$-complete group action on N with generator $\xi\in\mathcal{G}_0^1(N)$ then $\psi^*\Phi=[(\psi^*\Phi_\varepsilon)_\varepsilon]$ is a strictly $\mathcal{G}$-complete group action on M with infinitesimal generator $\psi^*\xi=[(\psi^*\xi_\varepsilon)_\varepsilon]$.*

Proof. Choose representatives $(\xi_\varepsilon)_\varepsilon$, $(\Phi_\varepsilon)_\varepsilon$ as in the definition of strict $\mathcal{G}$-completeness. Then for each fixed $\varepsilon\in I$, $\psi^*\Phi_\varepsilon(\eta,x)=\psi^{-1}\circ\Phi_\varepsilon(\eta,\psi(x))$ is a group action on M with generator $\psi^*\xi_\varepsilon=T\psi^{-1}\circ\xi_\varepsilon\circ\psi$. Since Eq. (1) transfers componentwise from N to M, strict $\mathcal{G}$-completeness of the pullback follows. □

In the formulation of Theorem 5.2 we will make use of the following definition: a subset of $\mathbb{R}^n$ is called an n-dimensional box if it is a product $I_1\times\cdots\times I_n$ of n finite or infinite open intervals in $\mathbb{R}$.

Theorem 5.2. *Let Φ be a strictly $\mathcal{G}$-complete group action on M with generator ξ. Let $F\in\mathcal{G}(M)^l$ be of maximal rank on some U with $\tilde{U}_{\mathrm{c}}\supseteq\Phi((-\eta_0,\eta_0)_{\mathrm{c}}^{\sim}\times S_F)$ $(\eta_0>0)$ via a generalized chart $\psi\in\mathcal{G}[U,V]$, where $S_F:=\{\tilde{x}\in\tilde{M}_{\mathrm{c}}\mid F(\tilde{x})=0\}$. Set $\bar{\xi}:=(\psi^{-1})^*\xi$ and suppose that one of the following conditions holds*:

(i) *$V=\mathbb{R}^m$ and $\bar{\xi}$ possesses a representative $(\bar{\xi}_\varepsilon)_\varepsilon$ satisfying*:

$$\exists C,\ \varepsilon_0>0 \text{ such that } \left|\bar{\xi}_\varepsilon(x)\right|\leqslant C\bigl(1+|x|\bigr)\quad (x\in\mathbb{R}^n,\ \varepsilon<\varepsilon_0).$$

(ii) *V is a box and $D\bar{\xi}$ is locally of L^∞-*log*-type.*

Then Φ is a symmetry group of

$$F_\nu(x)=0 \quad (1\leqslant \nu\leqslant l) \tag{5}$$

if and only if

$$\xi(F_\nu)|_{\tilde x}=0 \quad 1\leqslant \nu\leqslant l,\ \forall \tilde x\in S_F. \tag{6}$$

Proof. Let Φ be a symmetry group of (5). Then for each $\tilde x\in S_F$, the generalized function

$$\eta\mapsto F\big(\Phi(\eta,\tilde x)\big)$$

equals 0 in $\mathcal{G}(\mathbb{R})$. Therefore,

$$0=\frac{d}{d\eta}\bigg|_0 F\big(\Phi(\eta,\tilde x)\big)=\xi(F)|_{\tilde x}.$$

Conversely, by assumption we have $F\circ\psi^{-1}=\mathrm{pr}:V\subseteq\mathbb{R}^m\to\mathbb{R}^l$. By Lemma 5.1, $(\psi^{-1})^*\xi$ is a strictly $\mathcal{G}$-complete vector field on V with flow $\bar\Phi(\eta,x):=(\eta,x)\mapsto\psi\circ\Phi(\eta,\psi^{-1}(x))$. Write

$$\bar\xi=(\psi^{-1})^*\xi=\sum_{i=1}^m\bar\xi_i\,\partial_{x_i}$$

and

$$\bar F:=(\psi^{-1})^*F=(x_1,\dots,x_m)\mapsto(x_1,\dots,x_l).$$

Then $S_{\bar F}=\psi(S_F)=\{\tilde x\in\tilde V_{\mathrm c}\mid \bar F(\tilde x)=0 \text{ in } \tilde{\mathbb{R}}^l_{\mathrm c}\}=\{\tilde x\in\tilde V_{\mathrm c}\mid(\tilde x_1,\dots,\tilde x_l)=0\text{ in }\tilde{\mathbb{R}}^l_{\mathrm c}\}$. Moreover, $\bar\xi(\bar F)=0$ on $S_{\bar F}$ means that $\bar\xi_i|_{V\cap(\{0\}\times\mathbb{R}^{m-l})}=0$ in $\mathcal{G}(V\cap(\{0\}\times\mathbb{R}^{m-l}))$, for $i=1,2,\dots,l$. Hence, $\bar\xi|_{V\cap(\{0\}\times\mathbb{R}^{m-l})}$ has a representative $(\bar\xi_\varepsilon)_\varepsilon$ with $\bar\xi_{1\varepsilon},\dots,\bar\xi_{l\varepsilon}\equiv 0$. Write $\bar\xi=(\bar\xi',\bar\xi'')\in\mathcal{G}(V)^l\times\mathcal{G}(V)^{m-l}$ and let $\tilde x\in S_{\bar F}$. Then $\tilde x$ has a representative $(x_\varepsilon)_\varepsilon$ such that $x_\varepsilon=(0,x''_\varepsilon)\in(\mathbb{R}^l\times\mathbb{R}^{m-l})\cap V$ for all ε.

Suppose now that assumption (i) is satisfied. Then the initial value problem

$$\begin{aligned}&\frac{d}{d\eta}\phi(\eta)=\bar\xi''\big(0,\phi(\eta)\big),\\&\phi(0)=\tilde x''\end{aligned}\tag{7}$$

possesses a solution on $\{0\}\times\mathbb{R}^{m-l}$ (see the existence part of the proof of Theorem 3.2 in [15]). Set $\tilde x'':=[(x''_\varepsilon)_\varepsilon]$ with x''_ε as above. Let ϕ be a solution of (7). Then $\bar\Phi(\eta,\tilde x)=(0,\phi(\eta))$. Indeed, let $\tilde\Phi(\eta,x):=(0,\phi(\eta))$. Then

$$\tilde\Phi(0,\tilde x)=(0,\tilde x'')=\tilde x$$

and

$$\frac{d}{d\eta}\tilde\Phi(\eta,\tilde x)=\big(0,\phi'(\eta)\big)=\big(0,\bar\xi''\big(0,\phi(\eta)\big)\big)=\bar\xi\big(0,\phi(\eta)\big)=\bar\xi\big(\tilde\Phi(\eta,\tilde x)\big).$$

Hence $\bar\Phi$ and $\tilde\Phi$ both solve the initial value problem

$$\begin{aligned}&\frac{d}{d\eta}\Phi(\eta)=\bar\xi\big(\Phi(\eta)\big),\\&\Phi(0)=\tilde x.\end{aligned}\tag{8}$$

Since $\bar{\xi}$ is $\mathcal{G}$-complete it follows that $\bar{\Phi}(\cdot,\tilde{x}) = \tilde{\Phi}(\cdot,\tilde{x})$, for all $\tilde{x} \in (\{0\} \times \mathbb{R}^{m-l})^{\sim}_{c}$. Therefore $\bar{\Phi} = \tilde{\Phi}$ on $(\{0\} \times \mathbb{R}^{m-l})^{\sim}_{c}$ and $\bar{\Phi}(\eta,\tilde{x}) \in S_{\bar{F}}$, for all η and all $\tilde{x} \in S_{\bar{F}}$, i.e. $\bar{\Phi}$ is a symmetry of $\bar{F} = 0$.

Alternatively, let us assume that (ii) obtains. We have to show that $\mathrm{pr}_1 \circ \bar{\Phi}(\tilde{\eta},\tilde{x}) = 0$ for all $\tilde{\eta} \in \tilde{\mathbb{R}}_c$ and $\tilde{x} \in S_F$. Let $1 \leqslant k \leqslant l$. Then for representatives as above, $\bar{\Phi}_{k\varepsilon}(0,x_\varepsilon) = 0$ and since V is a box, $(\sigma\bar{\Phi}'_\varepsilon(\tau,x_\varepsilon), \bar{\Phi}''_\varepsilon(\tau,x_\varepsilon)) \in V$ for $\sigma \in [0,1]$ and $\tau \in (-\eta_0,\eta_0)$. Therefore,

$$\begin{aligned}
\bar{\Phi}_{k\varepsilon}(\eta,x_\varepsilon) &= \int_0^\eta \frac{d}{d\tau}\bar{\Phi}_{k\varepsilon}(\tau,x_\varepsilon)\,d\tau = \int_0^\eta \bar{\xi}_{k\varepsilon}\big(\bar{\Phi}_\varepsilon(\tau,x_\varepsilon)\big)\,d\tau \\
&= \int_0^\eta \Big(\bar{\xi}_{k\varepsilon}(\underbrace{\bar{\Phi}_{1\varepsilon}(\tau,x_\varepsilon),\ldots,\bar{\Phi}_{l\varepsilon}(\tau,x_\varepsilon)}_{=:\bar{\Phi}'_\varepsilon(\tau,x_\varepsilon)},\underbrace{\bar{\Phi}_{l+1\varepsilon}(\tau,x_\varepsilon),\ldots,\bar{\Phi}_{m\varepsilon}(\tau,x_\varepsilon)}_{=:\bar{\Phi}''_\varepsilon(\tau,x_\varepsilon)}) \\
&\qquad - \bar{\xi}_{k\varepsilon}\big(0,\bar{\Phi}''_\varepsilon(\tau,x_\varepsilon)\big)\Big)\,d\tau \\
&= \int_0^\eta \int_0^1 \frac{d}{d\sigma}\bar{\xi}_{k\varepsilon}\big(\sigma\bar{\Phi}'_\varepsilon(\tau,x_\varepsilon),\bar{\Phi}''_\varepsilon(\tau,x_\varepsilon)\big)\,d\sigma\,d\tau \\
&= \int_0^\eta \int_0^1 \sum_{j=1}^{l} D_j\bar{\xi}_{k\varepsilon}\big(\sigma\bar{\Phi}'_\varepsilon(\tau,x_\varepsilon),\bar{\Phi}''_\varepsilon(\tau,x_\varepsilon)\big)\cdot\bar{\Phi}_{j\varepsilon}(\tau,x_\varepsilon)\,d\sigma\,d\tau.
\end{aligned}$$

Since $\bar{\Phi}$ is c-bounded and $D\bar{\xi}$ is locally of L^∞-log type, the claim therefore follows by applying Gronwall's inequality. □

Remark 5.3. We list some sufficient conditions for the respective assumptions of the above theorem:

(i) In case M is a Riemannian manifold with Riemannian metric h (e.g., a submanifold of $\mathbb{R}^n$ with the induced metric) it suffices to assume that ξ and $P\psi$ are globally bounded with respect to h for each differential operator P of first order.
(ii) To secure this condition it suffices to suppose that $P\xi$ is locally bounded for each differential operator P of order $\leqslant 1$ and that $P\psi$ is locally bounded for each differential operator P of order $\leqslant 2$.

Examples 5.4. In certain algebraically special cases a global chart ψ as in Theorem 5.2 can immediately be read off.

(i) Suppose that (after a possible renumbering of the coordinates) $F \in \mathcal{G}(\mathbb{R}^n)^l$ is given in the form

$$\begin{aligned}
F_1(x_1,\ldots,x_n) &= x_1 - f_1(x_2,\ldots,x_n), \\
F_2(x_1,\ldots,x_n) &= x_2 - f_2(x_3,\ldots,x_n), \\
&\vdots \\
F_l(x_1,\ldots,x_n) &= x_l - f_l(x_{l+1},\ldots,x_n)
\end{aligned}$$

with $f_i \in \mathcal{G}[\mathbb{R}^{n-i}, \mathbb{R}]$ for $1 \leqslant i \leqslant l$. Then

$$\psi(y_1, \ldots, y_n) = \big(F_1(y_1, \ldots, y_n), \ldots, F_l(y_1, \ldots, y_n), y_{l+1}, \ldots, y_n\big)$$

and writing $\psi^{-1}(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, ψ^{-1} is determined recursively by $y_i = x_i$ ($l < i \leqslant n$) and $y_i = x_i + f_i(y_{i+1}, \ldots, y_n)$ for $i \leqslant l$. Since composition of c-bounded generalized functions can be carried out unrestrictedly [18, Theorem 3.6], ψ is a global generalized chart. For $l = 1$ we obtain [14, Theorem 4.7] (formulated there in the $\mathcal{G}_\tau$-setting) as a special case.

(ii) $A \in \tilde{\mathbb{R}}_c^{n\times n}$ is an invertible matrix of generalized numbers if and only if $\det(A)$ is strictly nonzero in $\tilde{\mathbb{R}}$ (cf. [10, Theorem 1.2.38 and Lemma 1.2.41]). If, in addition, $\psi : x \mapsto A^{-1} \cdot x$ is c-bounded then it is a global chart for the map $F \in \mathcal{G}(\mathbb{R}^n)^l$, $F(x) = \mathrm{pr}_{\mathbb{R}^n \to \mathbb{R}^l}(A \cdot x)$. As a concrete example one may take for A a generalized rotation, i.e., an element of the special orthogonal group $\mathsf{SO}(n, \tilde{\mathbb{R}})$ over the ring $\tilde{\mathbb{R}}$ of generalized numbers (cf. [21,22] and Section 7).

6. Differential equations

Based on the previous section, it is possible to derive a theory of symmetry groups of differential equations in the space of c-bounded generalized functions. This development largely parallels the one presented in [14, Section 4.2], though with the additional benefit of being formulated in a global setting. Therefore we only point out the technical differences and omit proofs which are analogous to the $\mathcal{G}_\tau$-setting used there.

Definition 6.1. A generalized group action $\Phi \in \mathcal{G}[\mathbb{R} \times \mathbb{R}^{p+q}, \mathbb{R}^{p+q}]$ is called projectable if

$$\Phi\big(\eta, (x, u)\big) = \big(\Xi_\eta(x), \Psi_\eta(x, u)\big), \tag{9}$$

where $\Xi \in \mathcal{G}[\mathbb{R} \times \mathbb{R}^p, \mathbb{R}^p]$ and $\Psi \in \mathcal{G}[\mathbb{R} \times \mathbb{R}^{p+q}, \mathbb{R}^q]$.

The group properties

$$\Xi_{\eta_1+\eta_2}(x) = \Xi_{\eta_1}\big(\Xi_{\eta_2}(x)\big), \qquad \Psi_{\eta_1+\eta_2}(x, u) = \Psi_{\eta_1}\big(\Xi_{\eta_2}(x), \Psi_{\eta_2}(x, u)\big)$$

are to be understood as equations in $\mathcal{G}[\mathbb{R}^2 \times \mathbb{R}^p, \mathbb{R}^p]$ and $\mathcal{G}[\mathbb{R}^2 \times \mathbb{R}^{p+q}, \mathbb{R}^q]$, respectively. Ref. [18, Theorem 3.5], shows that any element u of $\mathcal{G}[M, N]$ is uniquely determined by its graph Γ_u. We have

Proposition 6.2. *Let $u \in \mathcal{G}[\mathbb{R}^p, \mathbb{R}^q]$ and let Φ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$. Then $\Phi_\eta(\Gamma_u) = \Gamma_{\Phi_\eta(u)}$ in $\tilde{\mathbb{R}}_c^{p+q}$ for each $\eta \in \tilde{\mathbb{R}}_c$, where $\Phi_\eta(u)$ denotes the element*

$$x \mapsto \Psi_\eta\big(\Xi_{-\eta}(x), u \circ \Xi_{-\eta}(x)\big) \in \mathcal{G}[\mathbb{R}^p, \mathbb{R}^q].$$

Proposition 6.3. *Consider a system of PDEs*

$$\Delta_\nu(x, u^{(n)}) = 0 \quad (1 \leqslant \nu \leqslant l) \tag{10}$$

in $\mathcal{G}[\mathbb{R}^p, \mathbb{R}^q]$, where $\Delta \in \mathcal{G}[(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}, \mathbb{R}^l]$. Set

$$S_\Delta = \big\{\tilde{z} \in \big(\tilde{\mathbb{R}}_c^p\big)^{(n)}\colon\ \Delta_\nu(\tilde{z}) = 0,\ 1 \leqslant \nu \leqslant l\big\}.$$

Then $u \in \mathcal{G}[\mathbb{R}^p, \mathbb{R}^q]$ is a solution of the system if and only if $\Gamma_{\mathrm{pr}^{(n)}u} \subseteq S_\Delta$.

Prolongations of generalized group actions are constructed as in the classical theory. Let Φ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$, $z \in (\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$ and choose a function $h \in \mathcal{C}^\infty(\mathbb{R}^p, \mathbb{R}^q)$ such that $(z_1, \ldots, z_p, \mathrm{pr}^{(n)} h(z_1, \ldots, z_p)) = z$. The nth prolongation of Φ is defined as

$$\mathrm{pr}^{(n)}\Phi(\eta, z) := \bigl(\Xi_\eta(z_1, \ldots, z_p), \mathrm{pr}^{(n)}\bigl(\Phi_\eta(h)\bigr)\bigl(\Xi_\eta(z_1, \ldots, z_p)\bigr)\bigr).$$

By [10, 3.2.59], it follows that $\mathrm{pr}^{(n)}\Phi \in \mathcal{G}[\mathbb{R} \times (\mathbb{R}^{p+q})^{(n)}, (\mathbb{R}^{p+q})^{(n)}]$. As in [14, Lemma 4.12 and Proposition 4.13], it is seen that this definition does not depend on the particular choice of h and that $\mathrm{pr}^{(n)}\Phi$ is a generalized group action on $(\mathbb{R}^p \times \mathbb{R}^q)^{(n)}$.

Proposition 6.4. *Let Φ be a projectable generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ such that $\mathrm{pr}^{(n)}\Phi$ is a symmetry group of the algebraic equation $\Delta(z) = 0$. Then Φ is a symmetry group of* (10).

Definition 6.5. Let ξ be a $\mathcal{G}$-complete generalized vector field. The nth prolongation of ξ is the infinitesimal generator of the nth prolongation of the generalized group action Φ corresponding to ξ:

$$\mathrm{pr}^{(n)}\xi|_z = \frac{d}{d\eta}\bigg|_0 \mathrm{pr}^{(n)}\Phi_\eta(z).$$

If $\mathrm{pr}^{(n)}\xi$ is $\mathcal{G}$-complete, then both ξ and Φ are called $\mathcal{G}$-n-complete.

Theorem 6.6. *Let*

$$\Delta_\nu\bigl(x, \mathrm{pr}^{(n)}u\bigr) = 0 \quad (1 \leqslant \nu \leqslant l) \tag{11}$$

be a system of partial differential equations with $\Delta \in \mathcal{G}(\mathbb{R}^p)^l$. Let Φ be a generalized group action on $\mathbb{R}^p \times \mathbb{R}^q$ with infinitesimal generator ξ and suppose that Δ and $\mathrm{pr}^{(n)}\Phi$ satisfy the assumptions of Theorem 5.2. *If*

$$\mathrm{pr}^{(n)}\xi(\Delta)(\tilde{z}) = 0 \quad \forall \tilde{z} \in \bigl(\tilde{\mathbb{R}}^p_c \times \tilde{\mathbb{R}}^q_c\bigr)^{(n)} \text{ with } \Delta(\tilde{z}) = 0,$$

then Φ is a symmetry group of (11).

Proof. Immediate from Theorem 5.2 and Proposition 6.4. □

As in [14, Theorem 4.17], we may now conclude that the classical algorithm for determining symmetries of a given system of differential equations carries over to the generalized setting: make an ansatz for the infinitesimal generators, calculate the prolongations according to the classical formulas (cf. [25, Theorem 2.36]) and then apply Theorem 6.6 to derive a system of determining equations in the space of c-bounded Colombeau functions. Solutions of this system verifying the conditions of Theorem 6.6 yield generalized symmetries of (11).

Examples 6.7. (i) Consider a scalar conservation law of the form

$$u_t + F(u)u_x = 0$$

with the propagation velocity F a strictly decreasing function of u which is allowed to suffer one or more jumps (cf. [21]). We are looking for projectable generalized symmetries of the form $\xi(x, t, u) = X(x, t)\partial_x + T(x, t)\partial_t + U(x, t, u)\partial_u$. The determining equations in this case read:

$$U_t + FU_x = 0,$$
$$-X_x + FT_t + TF_t + UF_u - FX_x + F^2 T_x + XF_x = 0.$$

Note that these contain nonlinear terms in the non-smooth function F as well as derivatives thereof, which means that the problem cannot be treated on the distributional level. By embedding F into $\mathcal{G}[\mathbb{R},\mathbb{R}]$, however, the problem becomes accessible to the symmetry group analysis laid out in this section. As a particular solution of the determining equations (now to be viewed as equations in the Colombeau setting) we obtain $\xi(x,t,u) = xt\partial_x + t^2\partial_t + (x - tF(u))/F'(u)\partial_u$. This infinitesimal generator is $\mathcal{G}$-complete, transforming any solution $u \in \mathcal{G}[\mathbb{R}^2,\mathbb{R}]$ into $F^{-1}(\eta x(1+\eta t)^{-1} + F(u(x(1+\eta t)^{-1}, t(1+\eta t)^{-1}))(1+\eta t)^{-1})$. Depending on the particular form of F this new solution may be associated to (i.e., have a distributional limit) of the form of a piecewise smooth function with jumps or kinks which is a new generalized solution of the original equation.

(ii) More generally, F may be assumed to be a symmetric $n \times n$-matrix of $\mathcal{C}^1$-functions. In this case the generalized symmetries of the resulting quasilinear system have been studied in [7] in the $\mathcal{G}_\tau$-setting. The results achieved there carry over to our present situation since the generalized solutions of the system remain c-bounded. In particular, so-called associated symmetries are analyzed in [7], and infinitesimal criteria for the transformation of solutions in the sense of association into other such solutions are derived. These criteria are applicable to the study of weak solutions and extend work of Berest in the linear case [1,2]. An extended study of associated symmetries of conservation laws can be found in [7,12].

7. Group invariant generalized functions

In this final section we analyze the notion of invariance of Colombeau generalized functions under generalized group actions. As in classical analysis and distribution theory this concept plays an important role with respect to applications (cf. the calculation of group invariant fundamental solutions in $\mathcal{D}'$, respectively, $\mathcal{G}$ in [1,2,21]).

We shall need the fact that composition of Colombeau generalized functions and c-bounded generalized functions is always well defined.

Lemma 7.1. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}[M,N]$, $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}(N)$. Then $v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$ is a well-defined element of $\mathcal{G}(M)$.*

Proof. To show that $(v_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M)$, let $K \Subset V$ for some chart (V,φ) in M. Since v is c-bounded, there exist $K' \Subset N$ and $\varepsilon_0 > 0$ such that $v_\varepsilon(K) \subseteq K'$ for all $\varepsilon < \varepsilon_0$. Without loss of generality we may assume that K' is contained in some chart (W,ψ) of N. Then the moderateness estimates for $v_\varepsilon \circ u_\varepsilon = (v_\varepsilon \circ \psi^{-1}) \circ (\psi \circ u_\varepsilon)$ on K follow from the chain rule and the respective estimates for $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$. Suppose now that $[(u_\varepsilon)_\varepsilon] = [(u'_\varepsilon)_\varepsilon]$ in $\mathcal{G}[M,N]$ and let $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{M}_c$. Then by [10, Proposition 3.2.56], $[(u_\varepsilon(x_\varepsilon))] = [(u'_\varepsilon(x_\varepsilon))]$ in $\tilde{M}_c$ and hence $[(v_\varepsilon \circ u_\varepsilon(x_\varepsilon))] = [(v_\varepsilon \circ u'_\varepsilon(x_\varepsilon))]$ in $\tilde{M}_c$ by [10, Proposition 3.2.7]. By the same result, $[(v_\varepsilon \circ u_\varepsilon)_\varepsilon] = [(v_\varepsilon \circ u'_\varepsilon)_\varepsilon]$ in $\mathcal{G}(M)$. Finally, if $(v_\varepsilon)_\varepsilon \in \mathcal{N}(N)$ it is immediate from the c-boundedness of $(u_\varepsilon)_\varepsilon$ that $(v_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathcal{N}(M)$. Hence $v \circ u$ is well defined, as claimed. □

In particular, for Φ a generalized group action on M and $f \in \mathcal{G}(M)$, it follows that $f \circ \Phi$ is a well-defined element of $\mathcal{G}(\mathbb{R} \times M)$.

Definition 7.2. Let Φ be a generalized group action on M and let $f \in \mathcal{G}(M)$. f is called invariant under Φ if $f \circ \Phi = f \circ \pi_2$ in $\mathcal{G}(\mathbb{R} \times M)$ (with $\pi_2 : \mathbb{R} \times M \to M$, $\pi_2(\eta, x) = x$).

By the point value characterization of generalized functions (cf. [10, Theorem 3.2.8]) the above condition can equivalently be stated as follows:

$$f\big(\Phi(\tilde{\eta}, \tilde{x})\big) = f(\tilde{x}) \quad \forall \tilde{\eta} \in \tilde{\mathbb{R}}_c \ \forall \tilde{x} \in \tilde{M}_c.$$

Proposition 7.3. *Let $f \in \mathcal{G}(M)$ and let Φ be a generalized group action on M with infinitesimal generator ξ. Then the following statements are equivalent*:

(i) *f is Φ-invariant*;
(ii) *$\xi(f) = 0$ in $\mathcal{G}(M)$.*

Proof. (i) $\Rightarrow$ (ii). Since f is Φ-invariant we have

$$0 = \frac{d}{d\eta}\bigg|_0 \big(f\big(\Phi(\eta, x)\big)\big) = \xi(f)|_x \quad \text{in } \mathcal{G}(M).$$

(ii) $\Rightarrow$ (i). Conversely, let $\xi(f) = 0$ in $\mathcal{G}(M)$. Then

$$\frac{d}{d\eta} f\big(\Phi(\eta, \tilde{x})\big) = \xi(f)|_{\Phi(\eta, \tilde{x})} = 0 \quad \text{in } \mathcal{G}(\mathbb{R}) \ \forall \tilde{x} \in \tilde{M}_c.$$

Therefore, for each $\tilde{x}$ the map $\eta \mapsto f(\Phi(\eta, \tilde{x}))$ is constant in $\mathcal{G}(\mathbb{R})$, so $f \circ \Phi = f \circ \pi_2$, again by [10, Theorem 3.2.8]. □

Invariance properties of Colombeau generalized functions under generalized group actions have first been studied in [21,22]. In particular, the following basic result was derived ([22, Theorem 2], formulated there in the $\mathcal{G}_\tau$-setting):

Theorem 7.4. *Let $u \in \mathcal{G}(\mathbb{R}^n)$. The following are equivalent*:

(i) *$u(\tilde{x}_1 + \eta, \tilde{x}_2, \ldots, \tilde{x}_n) = u(\tilde{x})$ for all $\tilde{x} \in \tilde{\mathbb{R}}^n_c$, $\eta \in \tilde{\mathbb{R}}_c$.*
(ii) *$\partial_{x_1} u = 0$ in $\mathcal{G}(\mathbb{R}^n)$.*
(iii) *u has a representative $(u_\varepsilon)_\varepsilon$ such that $\partial_{x_1} u_\varepsilon \equiv 0$ for all ε.*

It remained an open question there whether (i)–(iii) is equivalent to

(i′) $u(\tilde{x}_1 + \eta, \tilde{x}_2, \ldots, \tilde{x}_n) = u(\tilde{x})$ for all $\tilde{x} \in \tilde{\mathbb{R}}^n_c$, $\eta \in \mathbb{R}$,

i.e., whether standard translations suffice to characterize translational invariance of Colombeau generalized functions. Meanwhile, Pilipović, Scarpalezos and Valmorin have provided two alternative proofs (based on a Baire argument respectively on the construction of a parametrix) which show that this question can be answered affirmatively [26]. In what follows we shall make use of this result to resolve a further open question raised in [21] in the context of generalized rotations.

Recall from Example 5.4(ii) that we denote by $\mathsf{SO}(n, \tilde{\mathbb{R}})$ the space of generalized rotations. Rotational invariance of Colombeau functions has been characterized in [21,22] and has been employed there to provide a new method of calculating rotationally invariant fundamental solutions, e.g., of the Laplace equation. The main characterization result is as follows [21, Theorem 4.2].

Theorem 7.5. *Let $u \in \mathcal{G}(\mathbb{R}^n)$. The following are equivalent*:

(i) $u \circ A = u$ *in* $\mathcal{G}(\mathbb{R}^n)$ *for all* $A \in \mathsf{SO}(n, \tilde{\mathbb{R}})$.
(ii) $\xi u = 0$ *in* $\mathcal{G}(\mathbb{R}^n)$ *for all infinitesimal generators of* $\mathsf{SO}(n, \mathbb{R})$.
(iii) *u possesses a representative* $(u_\varepsilon)_\varepsilon$ *such that each* u_ε *is rotationally invariant.*

The following result affirmatively answers an open question from [22].

Theorem 7.6. *Items* (i)–(iii) *in Theorem* 7.5 *are equivalent with*

(i′) $u \circ A = u$ *in* $\mathcal{G}(\mathbb{R}^n)$ *for all* $A \in \mathsf{SO}(n, \mathbb{R})$,

i.e., standard rotations suffice to characterize rotational invariance of Colombeau generalized functions.

Proof. Obviously (i) implies (i′). To prove the converse we first treat the case $n = 2$. Let $\tilde{A} \in \mathsf{SO}(2, \tilde{\mathbb{R}})$. Then by [22, Section 3, Lemma 1], there exists some $\tilde{\eta} \in \tilde{\mathbb{R}}_c$ such that

$$\tilde{A} = \left[\begin{pmatrix} \cos(\eta_\varepsilon) & -\sin(\eta_\varepsilon) \\ \sin(\eta_\varepsilon) & \cos(\eta_\varepsilon) \end{pmatrix}_\varepsilon \right].$$

Given $\tilde{x}, \tilde{y} \in \tilde{\mathbb{R}}_c$ we have to show that $u(\tilde{A} \cdot (\tilde{x}, \tilde{y})^t) = u(\tilde{x}, \tilde{y})$ in $\tilde{\mathbb{R}}$. We may write $(\tilde{x}, \tilde{y}) = [(r_\varepsilon \cos(\theta_\varepsilon), r_\varepsilon \sin(\theta_\varepsilon))]$ for suitable $r_\varepsilon \geqslant 0, \theta_\varepsilon$. Now set $v_\varepsilon := \theta \mapsto u_\varepsilon(r_\varepsilon \cos(\theta), r_\varepsilon \sin(\theta))$. Then $v = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R})$ and by assumption $v(\tilde{\theta} + \eta) = v(\tilde{\theta})$ in $\tilde{\mathbb{R}}$ for all $\tilde{\theta} \in \tilde{\mathbb{R}}_c$ and all $\eta \in \mathbb{R}$. But then the equivalence of (i) and (i′) in Theorem 7.4 shows that v is, in fact, a generalized constant. This immediately gives the result in the 2D-case.

In the general case $n \geqslant 2$ we verify (ii) of Theorem 7.5. Let $1 \leqslant i < j \leqslant n$ and let $\xi = x_i \partial_{x_j} - x_j \partial_{x_i}$ be an infinitesimal generator of $\mathsf{SO}(n, \mathbb{R})$. Fix compactly supported generalized numbers $\tilde{x}_1, \ldots, \tilde{x}_{i-1}, \tilde{x}_{i+1}, \ldots, \tilde{x}_{j-1}, \tilde{x}_{j+1}, \ldots, \tilde{x}_n$ and consider the maps

$$w_\varepsilon : (x_i, x_j) \mapsto u_\varepsilon(\tilde{x}_1, \ldots, \tilde{x}_{i-1}, x_i, \ldots, \tilde{x}_{j-1}, x_j, \ldots, \tilde{x}_n).$$

Then $w = [(w_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^2)$ and from our assumption it follows that $w \circ A = w$ in $\mathcal{G}(\mathbb{R}^2)$ for all $A \in \mathsf{SO}(\mathbb{R}^2)$. By what we have already proved in the 2D-case and Theorem 7.5 it follows that $\xi w = 0$ in $\mathcal{G}(\mathbb{R}^2)$. Hence from the point value characterization of Colombeau generalized functions it follows that $\xi u = 0$ in $\mathcal{G}(\mathbb{R}^n)$ for each $\xi \in \mathsf{SO}(\mathbb{R}^n)$, as claimed. $\square$

Remark 7.7. Let $\tilde{a}$ be a strictly nonzero (i.e., invertible, cf. [10, Theorem 1.2.38]) generalized number and consider the generalized vector field $\xi = \tilde{a}(y\partial_x - x\partial_y)$ on $\mathbb{R}^2$. Then $\psi = [(\psi_\varepsilon)_\varepsilon]$ with

$$\psi_\varepsilon : (r, \theta) \mapsto \bigl(r\cos(a_\varepsilon \theta), r\sin(a_\varepsilon \theta)\bigr)$$

is a generalized chart in $\mathcal{G}[\mathbb{R}^+ \times (0, 2\pi), \mathbb{R}^2 \setminus (\mathbb{R}_0^+ \times \{0\})]$. Moreover, the pullback $\psi^*\xi$ of ξ under ψ is the smooth vector field $\frac{\partial}{\partial\theta}$. This provides a simple case of "straightening out" a (strictly) nonzero generalized vector field. In the case of standard polar coordinates ($\tilde{a} = 1$) ψ allows to directly transform standard generators of $\mathsf{SO}(2, \mathbb{R})$ to translations, albeit only on $\mathbb{R}^2 \setminus \{0\}$. However, there exist elements of $\mathcal{G}(\mathbb{R}^2)$ with support $\{0\}$ which are not rotationally invariant: choose some $\varphi \in \mathcal{D}(\mathbb{R}^2)$ whose support is not rotationally invariant and set $u = [(\varphi(\frac{\cdot}{\varepsilon}))_\varepsilon]$. Therefore, the above argument does not yield an alternative proof of Theorem 7.6 (by reducing it to the translation setting of Theorem 7.4), since, contrary to the smooth setting, rotational invariance on $\mathbb{R}^2 \setminus \{0\}$ is not equivalent to rotational invariance on $\mathbb{R}^2$ for Colombeau functions. (The situation for $\mathcal{D}'(\mathbb{R}^2)$ is similar: for example, $\partial_1\delta$ is a distribution supported in $\{0\}$ which is not rotationally invariant.)

Nevertheless, generalized charts induced by matrix transformations as above and the related question of straightening out infinitesimal generators of matrix groups over the ring of generalized numbers are likely to play an important role in a further analysis of group invariant generalized functions. They should also provide valuable test cases for the development of inverse function theorems in the Colombeau setting [8].

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