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RELATIONS BETWEEN THE MILNOR AND QUILLEN K-THEORY OF FIELDS

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In this paper we will investigate the connection between two versions of the algebraic K-theory of fields. The definition of $K_n^M(F)$, $n>2$, given by Milnor in [4] involves a straightforward generalization of $K_2(F)$ for a field F. The groups $K_0^M(F)$, $K_1^M(F)$ and $K_2^M(F)$ are the standard K-theory groups introduced in [5]. Putting these groups together we get

$$K_*^M(F) \stackrel{\text{def}}{\equiv} \coprod_{n\geq 0} K_n^M(F).$$

There is a product structure defined on $K_*^M(F)$ in such a way that if $x\in K_i^M(F)$, $y\in K_j^M(F)$ then $xy\in K_{i+j}^M(F)$. With this added structure we get a ring which is called the Milnor ring of F. All the definitions are strictly algebraic and the theory evolved bears a strong relation with both the Witt ring and Galois cohomology [4].

For any ring A we can define the infinite general linear group $\mathrm{GL}(A)$ and its classifying space $\mathrm{BGL}(A)$. For $n\geq 1$ Quillen defines the group $K_n^Q(A)$ as the nth homotopy group of the space $\mathrm{BGL}(A)^+$, derived from $\mathrm{BGL}(A)$ by the plus construction. For $n=0$ take $K_0^Q(A)$ as the standard group $K_0(A)$ defined in [5]. For $n=1, 2$ it is known that $K_n^M(F)\cong K_n^Q(F)$ for any field F [3]. On the other hand if n is an odd integer greater than one and if F is a finite field, then $K_n^M(F)$ is trivial while $K_n^Q(F)$ is cyclic.

We can define the Quillen K-ring analagously [3].

$$K_*^Q(F) \stackrel{\text{def}}{\equiv} \coprod_{n\geq 0} K_n^Q(F).$$

We also get a natural ring homomorphism $\lambda_F: K_*^M(F)\to K_*^Q(F)$ which is uniquely determined by the isomorphism $K_1^M(F) \xrightarrow{\cong} K_1^Q(F)$ (uniqueness follows since $K_*^M(F)$ is generated as a ring by $K_1^M(F)$). We will refer to the image of λ_F as the set of *decomposable elements.*

The calculations for finite fields shows that the decomposable elements do not in general cover $K_*^Q(F)$. One might then ask if λ_F gives us a faithful copy of $K_*^M(F)$ in the Quillen K-ring, i.e. is the map λ_F injective. As a corollary to the work of Quillen on the J-homomorphism we will show that for a global field, F, λ_F is injective on $K_3^M(F)$. This will give us a subgroup of $K_3^Q(F)$ which is a direct sum of copies of $Z/2Z$, one for each real completion of F (see Section 1). On the other hand, if $\mathbb{Q}$ is the field of rational numbers then λ_F is the zero map on $K_4^M(Q) \cong Z/2Z$.

Another natural question to ask is what happens to decomposable elements under field extensions. Suppose E is a finite extension of F. The inclusion $i: F \hookrightarrow E$ induces a ring homomorphism $i^*: K_*^Q(F) \to K_*^Q(E)$. Since it is a ring homomorphism it automatically takes decomposable elements to decomposable elements. In the other direction we have a group homomorphism, i_*, called the Quillen transfer. It is not at all clear that i_* preserves decomposable elements.

One way to attack this problem is by introducing a transfer map for the Milnor theory which matches up, via λ_F, with i_*. If E is separable over F then for each choice of a primitive element, α, Bass and Tate have introduced a transfer map $N_\alpha: K_*^M(E) \to K_*^M(F)$. In general it is not known whether N_α is independent of the choice of α (for some partial results see [1] and [8]). In Section 2 we deduce two results matching up N_α and i_*. The first case is where E has prime degree over F. In that case N_α and i_* correspond via λ_F and λ_E. The second case is where E is Galois over F. Then we can show that N_α and i_* correspond on p-torsion where p is a prime which does not divide the degree of E over F. It is easy to see that in any case where N_α and i_* agree i_* automatically takes decomposable elements in $K_*^Q(E)$ to decomposable elements in $K_*^Q(F)$.

1. Injectivity of λ_F

For any ring A we have the algebraic K-groups $K_i(A)$, $i=0, 1, 2$ introduced in [5]. For a field F Milnor has defined the groups $K_n(F)$ for $n>2$ and has given $K_*(F) \overset{\text{def}}{\equiv} \coprod_{n\geq 0} K_n(F)$ the structure of a graded ring [4]. Following Milnor we introduce the canonical isomorphism $l: F^* \xrightarrow{\cong} K_1(F)$, where F^* is the multiplicative group of units and $l(ab) = l(a) + l(B)$ for $a, b \in F^*$. In terms of generators and relations $K_*^M(F)$ is the associative ring with identity which is generated by symbols $l(a)$, $a \in F^*$, subject only to the relations $l(ab) = l(a) + l(b)$ and $l(a)l(1-a) = 0$, a, $1-a \neq 0$.

For any ring A Quillen has introduced the space $\mathrm{BGL}(A)^+$ constructed from the classifying space of $\mathrm{GL}(A)$ by the addition of certain 2-cells and 3-cells [3]. Recall that

$$\mathrm{GL}(A) \overset{\text{def}}{\equiv} \varinjlim_n \mathrm{GL}_n(A)$$

where the inclusion of $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$ is defined by putting 1 in the lower right hand corner and by putting zeros in the rest of the last column and last row. The group $K_n^Q(A)$ is defined as the nth homotopy group $\pi_n(\mathrm{BGL}(A)^+)$ for $n \geq 1$, and we will take $K_0^Q(A)$ to be $K_0(A)$. The total group $K_*^Q(A)$ is endowed with a multiplicative structure in such a way that it becomes a graded ring. Let us also recall that for $n=0, 1, 2$ $K_i^Q(A) \cong K_i^M(A)$ [3].

It follows from the description of $K_*^M(F)$ in terms of generators and relations that the isomorphism $K_1^M(F) \xrightarrow{\cong} K_1^Q(F)$ induces a ring homomorphism $\lambda_F : K_*^M(F) \to K_*^Q(F)$. This follows because both relations (being essentially in K_2) go to zero in $K_*^Q(F)$. In particular λ_F takes an additive generator

$$l(a_1) \cdots l(a_n) \quad \text{to} \quad l(a_1) * \cdots * l(a_n)$$

in $K_n^Q(F)$, where the $*$ denotes the product in $K_*^Q(F)$. As noted in the introduction the calculations of the K theory of finite fields implies that λ_F is not surjective. As far as injectivity is concerned we have the following two propositions. The first gives us a class of examples where λ_F is injective and the second an example where injectivity fails.

I would like to thank Stuart Priddy for explaining to me the pertinent facts from homotopy theory which led to the proofs of these propositions.

Proposition 1. *If F is a global field, then $\lambda_F : K_3^M(F) \to K_3^Q(F)$ is injective.*

The following corollary follows from the second part of the proof of the proposition.

Corollary. *If F is a global field, then $K_3^Q(F)$ contains as a subgroup a direct sum of $Z/2Z$, one for each real completion of F.*

Proof (Proposition 1). (i) Suppose F is the rational numbers, $\mathbb{Q}$. $K_3^M(\mathbb{Q}) \cong Z/2Z$ with the nontrivial element represented by $l(-1)^3$ [4, Example 1.8]. We must show, therefore, that $x = l(-1) * l(-1) * l(-1)$ is nontrivial in $K_3^Q(\mathbb{Q})$. It follows from [2, Theorem 4.8], that $K_3^Q(Z) \to K_3^Q(\mathbb{Q}) \to K_3^Q(\mathbb{R})$ is injective, where Z is the rational integers and $\mathbb{R}$ the real numbers. Therefore it suffices to show that $x \neq 0$ as an element of $K_3^Q(Z)$.

If we view the elements of the symmetric group, Σ_n, as permutation matrices we get an embedding of Σ_n in $\mathrm{GL}_n(Z)$. By passing to $\Sigma_\infty = \varinjlim_n \Sigma_n$ we derive a map from Σ_∞ to $\mathrm{GL}(Z)$. This in turn induces a homomorphism $\pi_n(B\Sigma_\infty^+) \to K_n^Q(Z)$ for all n. Using the same technique as that used to induce a product structure on $K_*^Q(A)$ (i.e. tensor product, see [3]) we can define a ring structure on $\pi_*(B\Sigma_\infty^+)$. It follows also that the map from $\pi_*(B\Sigma_\infty^+)$ to $K_*^Q(Z)$ is a ring homomorphism (the definitions of the respective product structures are totally compatible).

We are now ready to show that $x \neq 0$. The generator $\eta \in \pi_1(B\Sigma_\infty^+) \cong Z/2Z$ maps to $l(-1)$ in $K_1^Q(Z) \cong Z/2Z$ [7]. Therefore η^3 maps to x. $\eta^3 \neq 0$ [2, 4.4] and $\pi_3(B\Sigma_\infty^+) \to K_3(Z)$ is injective [7, proposition]. Therefore $x \neq 0$.

(ii) Now let F be a general global field. We have an isomorphism, induced by the inclusion maps, $K_3^M(F) \to \coprod_v k_3^M(F_v)$, where $\{F_v\}$ is the family of real completions of F and

$$k_3^M(F) \overset{\text{def}}{\equiv} K_3^M(F)/2K_3^M(F).$$

By [4, Example 1.6], $k_3^{\mathrm{M}}(\mathbb{R}) \cong Z/2Z$ with the non-zero element represented by $l(-1)^3$. The inclusion maps $F \hookrightarrow F_v$ also induce a homomorphism from $K_3^{\mathrm{Q}}(F)$ to $\coprod_v k_3^{\mathrm{Q}}(F_v)$, where the lower case again stands for the group mod 2. We therefore derive the following commutative square

$$\begin{array}{ccc} K_3^{\mathrm{M}}(F) & \xrightarrow{\cong} & \coprod_v k_3^{\mathrm{M}}(F_v) \\ \downarrow{\scriptstyle \lambda_F} & & \downarrow{\scriptstyle (\lambda_v)} \\ K_3^{\mathrm{Q}}(F) & \longrightarrow & \coprod_v k_3^{\mathrm{Q}}(F_v). \end{array}$$

Part (i) of the proof implies that λ_v is injective for each v. Therefore λ_F is injective. □

We now use the fact that $\eta^4 \in \pi_4(B\Sigma_\infty^+)$ is zero and that $l(-1)^4 \neq 0$ in $K_4^{\mathrm{M}}(\mathbb{Q})$ to derive the second proposition.

Proposition 2. $\lambda_Q : K_4^{\mathrm{M}}(\mathbb{Q}) \to K_4^{\mathrm{Q}}(\mathbb{Q})$ *is the zero map. In particular* λ_Q *is not injective.*

2. Decomposable elements and transfer

For a field F let us call the image of $K_*^{\mathrm{M}}(F)$ in $K_*^{\mathrm{Q}}(F)$ the *decomposable elements* and denote that subring by $K_*^{\mathrm{Q}}(F)^{\mathrm{dec}}$. In Section 1 we have noted that in general $K_*^{\mathrm{Q}}(F)^{\mathrm{dec}}$ does not provide us with a faithful copy of the Milnor K-ring. On the other hand it does provide a portion of an algebraic object for which calculation has proved difficult. It is with this in mind that we investigate this subring in greater detail. In particular we would like to study the effect of field extensions on the decomposable elements.

Suppose E is a field extension of F with inclusion map $i: F \hookrightarrow E$. This map induces a ring homomorphism $i^*: K_*^{\mathrm{Q}}(F) \to K_*^{\mathrm{Q}}(E)$. It follows trivially that $i^*(K_*^{\mathrm{Q}}(F)^{\mathrm{dec}}) \subseteq K_*^{\mathrm{Q}}(E)^{\mathrm{dec}}$. Let us also note that i^* makes $K_*^{\mathrm{Q}}(E)$ into a $K_*^{\mathrm{Q}}(F)$-module. If E is finite over F we also have the *Quillen transfer* map $i_* : K_*^{\mathrm{Q}}(E) \to K_*^{\mathrm{Q}}(F)$ which is a $K_*^{\mathrm{Q}}(F)$-module homomorphism [6, Section 4]. It is not at all obvious that the map i_* sends $K_*^{\mathrm{Q}}(E)^{\mathrm{dec}}$ to $K_*^{\mathrm{Q}}(F)^{\mathrm{dec}}$. Let us note that if we can find a map $N: K_*^{\mathrm{M}}(E) \to K_*^{\mathrm{M}}(F)$ such that the diagram

$$\begin{array}{ccc} K_*^{\mathrm{M}}(E) & \xrightarrow{\lambda_E} & K_*^{\mathrm{Q}}(E) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle i_*} \\ K_*^{\mathrm{M}}(F) & \xrightarrow{\lambda_F} & K_*^{\mathrm{Q}}(F) \end{array} \qquad (1)$$

is a commutative square of groups and group homomorphisms then we automatically get the desired result.

If E is a finite separable extension of F, then for each choice of a primitive element, α, Bass and Tate have defined a transfer map $N_\alpha: K_*^{\mathrm{M}}(E) \to K_*^{\mathrm{M}}(F)$ [1]. If the degree of E over F, $[E:F]$, is a prime number then this map is independent of α. The proof of this assertion follows because in this case $K_*^{\mathrm{M}}(E)$ is generated by $K_0(E)$ and $K_1(E)$ as a $K_*^{\mathrm{M}}(F)$-module [1, 5.9]. Since $K_i^{\mathrm{M}}(F) \cong K_i^{\mathrm{Q}}(F)$ for $i=0, 1$ and since in these cases $N_\alpha = i_*$ (via the above isomorphisms) the commutativity of diagram (1) is immediate. More generally, we have the following proposition.

Proposition 3. *Suppose E is a finite separable extension of F with the property that there exists a tower of intermediate extensions $E = E_0 \geq E_1 \geq \cdots \geq E_n = F$ such that for each i, $0 \leq i \leq n-1$, $[E_i : E_{i+1}]$ is a prime. Then i_* takes $K_*^{\mathrm{Q}}(E)^{\mathrm{dec}}$ into $K_*^{\mathrm{Q}}(F)^{\mathrm{dec}}$.*

Example. E is Galois over F with nilpotent Galois group.

Proof. In this case we define $N_i: K_*^{\mathrm{M}}(E_i) \to K_*^{\mathrm{M}}(E_{i+1})$ for each i, $0 \leq i \leq n-1$. By iterating the above argument n-times we arrive at the fact that diagram (1) commutes, where N is the n-fold composite of the N_i. The result on the decomposable elements then follows immediately. □

Now suppose E is a finite Galois extension of F. In this case we will be able to show that the p-torsion part of diagram (1) commutes, where p is any prime not dividing the degree of E over F. This in turn will imply that the Quillen transfer takes

$$K_*^{\mathrm{Q}}(E)_p^{\mathrm{dec}} \quad \text{to} \quad K_*^{\mathrm{Q}}(F)_p^{\mathrm{dec}},$$

where for an abelian group B, B_p denotes the elements having p-power order.

Theorem. *Suppose E is a finite Galois extension of F and p is a prime not dividing $[E:F]$. Let N_α denote the Bass–Tate transfer induced by any primitive element, α. Then we get a commutative square as follows.*

$$\begin{array}{ccc} K_*^{\mathrm{M}}(E)_p & \xrightarrow{\lambda_E} & K_*^{\mathrm{Q}}(E)_p \\ \downarrow{\scriptstyle N_\alpha} & & \downarrow{\scriptstyle i_*} \\ K_*^{\mathrm{M}}(F)_p & \xrightarrow{\lambda_F} & K_*^{\mathrm{Q}}(F)_p. \end{array} \tag{1_p}$$

Corollary. *With the hypothesis as in the above theorem it follows that the Quillen transfer takes*

$$K_*^{\mathrm{Q}}(E)_p^{\mathrm{dec}} \quad \textit{to} \quad K_*^{\mathrm{Q}}(F)_p^{\mathrm{dec}}.$$

For the rest of this paper we will fix a prime p which is prime to $[E:F]$. The method of proof, as in [8], will be to introduce a special extension F' which contains F and is contained in a separable closure of F, F^{sep}.

Let F^{sep} be chosen so that it also contains E. Following Tate [9] *F' is a maximal extension of F, contained in F^{sep}, which is a union (set theoretic) of finite prime-to-p extensions of F.* As noted in [8] this field F' has a number of important properties which we now recall.

Property 1. F' is a direct limit of fields $F' = \varinjlim_{\alpha} F_{\alpha}$ where $\{F_{\alpha}\}$ is the family of prime-to-p extensions of F contained in F'.

Property 2. If K is a finite extension of F' contained in F^{sep}, then $[K:F'] = p^a$ where "a" is a non-negative integer. Furthermore there exists a tower of fields $F' = K_0 \leq K_1 \leq \cdots \leq K_a = K$ such that $[K_{i+1}:K_i] = p$, $1 \leq i \leq a$.

We also have the following proposition.

Proposition 4. *Let $j: F \hookrightarrow F'$ be the natural inclusion. Then $j^*: K_*^{Q}(F)_p \to K_*^{Q}(F')_p$ is injective.*

Proof. By Property 1 we see that $F' = \varinjlim_{\alpha} F_{\alpha}$ where $\{F_{\alpha}\}$ is the family of prime-to-p extensions of F contained in F'. For the Quillen K-theory we know that $K_*^{Q}(\varinjlim_{\alpha} F_{\alpha}) \cong \varinjlim_{\alpha} K_*^{Q}(F_{\alpha})$ [6, Section 2]. It is therefore sufficient to prove that for a prime-to-p extension E of F, $K_*^{Q}(F)_p \to K_*^{Q}(E)_p$ is injective. If $j: F \hookrightarrow E$ is the inclusion map then $j_* \circ j^*: K_*^{Q}(F) \to K_*^{Q}(F)$ is multiplication by the degree $[E:F]$. Since $[E:F]$ is prime-to-p it follows that j^* is injective. □

Let us recall that we are working with a finite separable extension E over F and we have chosen a fixed primitive element, α, for E over F. This element induces a homomorphism $N_{\alpha}: K_*^{M}(E) \to K_*^{M}(F)$. The further hypothesis that E is in fact Galois over F will not be needed until the last part of the proof of the theorem. Suppose now that L is any extension of F contained in F^{sep} (at the end of the proof we will specialize L to be the above extension, F'). Let $p(x)$ be the irreducible polynomial of α over F. As a polynomial in $L[x]$, $p(x)$ may decompose into a product of r monic irreducibles, $p(x) = p_1(x) \cdots p_r(x)$. We choose an ordering so that α is a root of $p_1(x)$. Let us denote α also as α_1 and then choose $\alpha_2, \ldots, \alpha_r$ so that α_i is a root of $p_i(x)$, $1 \leq i \leq r$. The tensor product $E \otimes_F L$ is then isomorphic to a product of fields $\prod_{i=1}^{r} L_i$, where $L_i \overset{\text{def}}{\cong} L(\alpha_i)$.

For each i, $1 \leq i \leq r$, we have a diagram of field extensions

$$\begin{array}{ccc} E & \xrightarrow{\;j_i\;} & L_i \\ | & & | \\ F & \xrightarrow[\;j\;]{} & L. \end{array}$$

The identification of $E=F(\alpha)$ as a subfield of $L_i=L(\alpha_i)$ is made by the unique isomorphism over F extending the map sending α to α_i.

Globally we have a diagram of rings

$$\begin{array}{ccc} E & \xrightarrow{j} & E\otimes_F L \overset{\text{def}}{\equiv} \tilde{L} \\ \big\downarrow & & \big\downarrow \\ F & \xrightarrow{j} & L. \end{array}$$

In the Quillen K-theory we know that $K_*^{\mathrm{Q}}(\tilde{L})\cong\prod_{i=1}^r K_*^{\mathrm{Q}}(L_i)$ [6, Section 2]. For the Milnor K-theory we merely define $K_*^{\mathrm{M}}(\tilde{L})$ as $\prod_{i=1}^r K_*^{\mathrm{M}}(L_i)$.

In order to prove the theorem we need a number of lemmas which describe how the transfer maps are affected when passing to the extension $\tilde{L}$ over L. We will denote by $N_i: K_*^{\mathrm{M}}(L_i)\to K_*^{\mathrm{M}}(L)$ the map induced by α_i, and the map $N: K_n^{\mathrm{M}}(\tilde{L})\to K_n^{\mathrm{M}}(L)$ will be the map taking $(x_i)\in\prod_{i=1}^r K_*^{\mathrm{M}}(L_i)$ to $\sum_{i=1}^r N_i(x_i)$.

Lemma 1. *The following is a commutative diagram of groups and group homomorphisms.*

$$\begin{array}{ccc} K_*^{\mathrm{M}}(E) & \xrightarrow{\prod_{i=1}^r j_i^*} & K_*^{\mathrm{M}}(\tilde{L}) \\ \big\downarrow N_\alpha & & \big\downarrow N \\ K_*^{\mathrm{M}}(F) & \xrightarrow{j^*} & K_*^{\mathrm{M}}(L). \end{array}$$

Proof. This follows from [1, proposition in Section 5.8], as already noted in [8]. □

Notation. The maps $T: K_*^{\mathrm{Q}}(E)\to K_*^{\mathrm{Q}}(F)$, $\tilde{T}: K_*^{\mathrm{Q}}(\tilde{L})\to K_*^{\mathrm{Q}}(L)$ and $T_i: K_*^{\mathrm{Q}}(L_i)\to K_*^{\mathrm{Q}}(L)$ will denote the respective Quillen transfer maps.

Lemma 2. *The following is a commutative diagram of groups and group homomorphisms.*

$$\begin{array}{ccccc} K_*^{\mathrm{Q}}(E) & \xrightarrow{j^*} & K_*^{\mathrm{Q}}(\tilde{L}) & \xrightarrow[\cong]{\gamma} & \prod_{i=1}^r K_*^{\mathrm{Q}}(L_i) \\ \big\downarrow T & \text{(i)} & \big\downarrow \tilde{T} & \text{(ii)} & \swarrow \sum_{i=1}^r T_i \\ K_*^{\mathrm{Q}}(F) & \xrightarrow{j^*} & K_*^{\mathrm{Q}}(L) & & \end{array}$$

Proof. (i) For a ring A let $\mathbf{P}(A)$ denote the category of finitely generated projective modules over A. Following Quillen [6] any exact functor from $\mathbf{P}(A)$ to $\mathbf{P}(B)$, where B is also a ring, induces a well defined homomorphism from $K_*^{\mathrm{Q}}(A)$ to $K_*^{\mathrm{Q}}(B)$.

We will show that the appropriate maps in the diagram are induced by the same exact functor.

The map $\tilde{T}\circ\tilde{j}^*$ is induced as follows. Suppose $V\in\mathbf{P}(E)$ i.e. a finite dimensional vector space. Then $\tilde{L}\otimes_E V$ is a free $\tilde{L}$-module of the same rank. Restriction of scalars allows us to view $\tilde{L}\otimes_E V$ as a finite dimensional vector space over L and hence as an element of $\mathbf{P}(L)$. $\tilde{L}$ is $L\otimes_F E$ so $\tilde{L}\otimes_E V=(L\otimes_F E)\otimes_E V\cong L\otimes_F V$ where the action of F on V is by restriction of scalars. The functor sending $E\to L\otimes_F V$ induces the map $j^*\circ T$. Therefore square (1) commutes.

(ii) As in the previous part we will show that $(\sum_{i=1}^r T_i)\circ\gamma$ and T are induced by the same exact functor. Suppose $V\in\mathbf{P}(\tilde{L})$; then the projection $\tilde{L}\to L_i$ (via the isomorphism $\tilde{L}\cong\prod L_i$) induces a map $V\to V_i$ which in turn induces a homomorphism $\gamma_i: K_*^{\mathrm{Q}}(\tilde{L})\to K_*^{\mathrm{Q}}(L_i)$. In this notation γ is the map $\prod_{i=1}^r\gamma_i$. T_i is induced by restriction of scalars. That is we view V_i as a vector space over L, where $\tau_i: L\hookrightarrow L_i$. Addition in the abelian group $K_*^{\mathrm{Q}}(L)$ is induced by the direct sum of modules [6, Section 2]. Therefore the composite $(\sum_{i=1}^r T_i)\circ\gamma$ is induced by the map $V\to\coprod_{i=1}^r V_i$, where the action of L on a vector $(v_1,\dots,v_r)\in\coprod_{i=1}^r V_i$, $v_i\in V_i$, is given by $a(v_1,\dots,v_r)=(j_1(a)v_1,\dots,j_r(a)v_r)$, $a\in L$. Using the isomorphism $\tilde{L}\cong\prod_{i=1}^r L_i$ together with the inclusion $\tau_i: L\to L_i$ we can see that the restriction of scalars from $\tilde{L}$ to L induces the same functor, sending $V\to\coprod_{i=1}^r V_i$. We use the fact that a projective over $\prod_{i=1}^r L_i$ is isomorphic to $V_1\times\cdots\times V_r$ where each $V_i\in\mathbf{P}(L_i)$. □

Lemma 3. *The following is a commutative diagram of rings and ring homomorphisms.*

$$\begin{array}{ccc}
K_*^{\mathrm{M}}(L) & \xrightarrow{\lambda_L} & K_*^{\mathrm{Q}}(L) \\
\uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle j^*} \\
K_*^{\mathrm{M}}(F) & \xrightarrow{\lambda_F} & K_*^{\mathrm{Q}}(F)
\end{array}$$

Here "res" *is the unique ring homomorphism compatible with the inclusion* $K_1(F)\cong F^*\hookrightarrow L^*\cong K_1(L)$.

Proof. Since $\{l(a)\mid a\in F^*\}$ forms a set of multiplicative generators for $K_*^{\mathrm{M}}(F)$ it is enough to check commutativity on that set. In either direction we get the identical map, that is, $l(a)\in K_1(F)$ gets sent to $l(a)\in K_1(L)$ via the inclusion map. □

In the last lemma we inspect an analogous diagram for the extension $\tilde{L}$ over E.

Lemma 4. *The following is a commutative diagram of rings and ring homomorphisms.*

$$\begin{array}{ccccc}
K_*^{\mathrm{M}}(\bar{L}) \equiv \prod_{i=1}^{r} K_*^{\mathrm{M}}(L_i) & \xrightarrow{\prod_{i=1}^{r}\lambda_i} & \prod_{i=1}^{r} K_*^{\mathrm{Q}}(L_i) & \xleftarrow[\cong]{\gamma} & K_*^{\mathrm{Q}}(\bar{L}) \\
\uparrow (\mathrm{res}_i) & \text{(i)} & \uparrow (j_i^*) & \text{(ii)} & \nearrow j^* \\
K_*^{\mathrm{M}}(E) & \xrightarrow{\lambda_E} & K_*^{\mathrm{Q}}(E) & &
\end{array}$$

Proof. (i) It is enough to check square (i) for each i, $1 \le i \le r$. For each i we may use Lemma 3 for the desired result.

(ii) For triangle (ii) we can also look at each i separately. That is we must check that j_i^* and the jth component map $(\gamma \circ j^*)_i$ are induced by the same exact functor. The following diagram gives a schematic rendering of the situation on the vector space level, where $V \in \mathbf{P}(E)$

$$\begin{array}{ccc}
L_i \otimes_E V & \longleftarrow & \left(\prod_{i=1}^{r} L_i\right) \otimes_E V \cong \prod_{i=1}^{r} (L_i \otimes_E V) \\
\uparrow & \nearrow & \\
V & &
\end{array}$$

□

Putting together the global diagram (1) and the diagrams in the four lemmas we obtain a cubical diagram in the category of groups. The sixth face of the cube is the square

$$\begin{array}{ccc}
\prod_{i=1}^{r} K_*^{\mathrm{M}}(L_i) & \xrightarrow{\prod_{i=1}^{r}\lambda_i} & \prod_{i=1}^{r} K_*^{\mathrm{Q}}(L_i) \\
\downarrow \sum_{i=1}^{r} N_i & & \downarrow \sum_{i=1}^{r} (\tau_i)_* \\
K_*^{\mathrm{M}}(L) & \xrightarrow{\lambda_L} & K_*^{\mathrm{Q}}(L)
\end{array} \tag{2}$$

If we can show that diagram (2) commutes and that $j^*: K_*^{\mathrm{Q}}(F) \to K_*^{\mathrm{Q}}(L)$ is injective, then we will have proved the theorem (in fact a stronger (global) result).

Remark. It is easy to choose L so that diagram (2) commutes. For example we will see from the proof of the theorem that the splitting field of $p(x)$ works. In general we can say that the global diagram (1) commutes modulo the kernel of $j^*: K_*^{\mathrm{Q}}(F) \to K_*^{\mathrm{Q}}(L)$. In fact choosing L correctly (e.g. as in the proof) we can always get the global diagram (1) to *commute modulo torsion* in $K_*^{\mathrm{Q}}(F)$.

Proof (Theorem). For the fixed p of the theorem, $p \times [E:F]$, let us choose L to be the maximal extension F' which is a union of prime-to-p extensions of F. Then by Property 2 each L_i over L is a p-power extension. Since E is Galois over F, $[L_i : L]$ must divide $[E:F]$. Hence $[L_i : L] = 1$ for all i, $1 \le i \le r$. Thus the commutativity of diagram (2) is a triviality. Finally by Proposition 4 j^* is 1–1 on the p-torsion part of

$K_*^Q(F)$, hence it follows that the p-torsion diagram (1_p) of the theorem is commutative. □

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