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A GENERALIZATION OF THE REEB STABILITY THEOREM

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IN 1947 G. Reeb [2] proved a theorem fundamental to the theory of codimension one foliations.

REEB'S STABILITY THEOREM. *Let $\mathscr{F}$ be a codimension one foliation of a compact manifold M^n. Suppose L is a compact leaf of $\mathscr{F}$ such that $\pi_1(L)$ is finite. Then all leaves of $\mathscr{F}$ are diffeomorphic with L (up to two-fold covers if there are leaves with one-sided tubular neighborhoods). We assume here that if M^n has boundary, ∂M^n is a union of leaves of $\mathscr{F}$.*

It is clear from Reeb's proof that the hypothesis on $\pi_1(L)$ can be weakened somewhat. The crucial property is that the holonomy around such a leaf must be trivial; thus, for instance, if $\pi_1(L)$ is generated by elements of finite order, it is easy to see that suffices to obtain Reeb's conclusion. In fact, we give the necessary and sufficient condition, in the C^1 case, to obtain this conclusion:

THEOREM 1. *Let $\mathscr{F}$ be a codimension one, C^1, transversely oriented foliation of a compact manifold M^{n-1} with a compact leaf L such that $H^1(L; \mathbb{R}) = 0$. Then all leaves of $\mathscr{F}$ are diffeomorphic with L, and the leaves of $\mathscr{F}$ are the fibers of a fibration of M^{n-1} over S^1 or I. We assume here that if M^{n-1} has boundary, then ∂M^{n-1} is a union of leaves of $\mathscr{F}$.*

Example 1. We observe here that the hypothesis about L is necessary: suppose L^{n-1} is a manifold such that $H^1(L; \mathbb{R}) \neq 0$. Let α be a cohomologically non-trivial one-form. Then on $L \times S^1$, the one-form $d\theta + f(\theta)\alpha$, for an arbitrary function f, is non-singular and integrable, and defines a foliation whose closed leaves correspond exactly to the zeros of f, and are diffeomorphic to L. Examples on other manifolds can be generated from these.

Here is a counter-example in the C^0 case:

Example 2. There is a homology three-sphere, M^3, with fundamental group $(a, b, c: a^2 = b^3 = c^7 = abc)$ which occurs as a leaf of a foliation $\mathscr{F}$ of $M^3 \times S^1$, where $\mathscr{F}$ has also non-compact leaves.

Construction. The fundamental group of M^3 is a discrete subgroup Γ of $\widetilde{Sl}(2, \mathbb{R})$ with compact quotient M^3. Γ is the group of transformations covering transformations of the non-Euclidean plane which preserve the tiling generated by the triangle with angles $\pi/2$, $\pi/3$, and $\pi/7$. $\widetilde{Sl}(2, \mathbb{R})$ acts on $\mathbb{R} = \tilde{S}^1$, so $\widetilde{Sl}(2, \mathbb{R})$ acts topologically on $S^1 = \mathbb{R} \cup \{\infty\}$. This gives a C^0 action of Γ on S^1, which determines a foliation of $M^3 \times S^1$ as required. $M^3 \times \{\infty\}$ is a leaf, and all other leaves are non-compact.

It would be interesting to have a characterization of compact leaves L for which Reeb's conclusion holds in the C^0 case. It seems reasonable to conjecture that if $\pi_1(L)$ is amenable (e.g. if $\pi_1(L)$ has polynomial growth, cf. Hirsch and Thurston [1] for applications of amenability to foliations), and if L has a non-trivial C^0 foliated neighborhood, then $H^1(L; \mathbb{R}) \neq 0$.

Reeb also obtained a local stability theorem which applies in all codimensions: if a leaf L has finite fundamental group, then all nearby leaves are (finite) covering spaces of L, and L has a saturated tubular neighborhood. Our method also gives a generalization of this local stability theorem. Given a leaf L of a codimension k, let H be the holonomy along L, so H is a representation of $\pi_1(L)$ in the group of germs of diffeomorphisms of a small k-disk normal to L. $\mathrm{d}H$ is then the linear holonomy. H_α, where $\alpha \in \pi_1(L)$, denotes the holonomy around α.

THEOREM 2. *Let $\mathscr{F}$ be a codimension k foliation, and let L be a compact leaf of $\mathscr{F}$. Then either*

(i) $\mathrm{d}H\colon \pi_1(L) \to Gl(k, \mathbb{R})$ *is non-trivial*;

or (ii) $H^1(L; \mathbb{R})$ *is non-trivial*;

or (iii) *the holonomy H is trivial*;

and L has a saturated tubular neighborhood with a product foliation.

COROLLARY 1. *Let $\mathscr{F}$, L be as above. If $H^1(L, \mathbb{R}) = 0$ and $H^1(L; Gl(k, \mathbb{R})) = 0$ then the holonomy around L is trivial, and L has a saturated tubular neighborhood with a product foliation.*

COROLLARY 2. *Let $\mathscr{F}$, L be as in Th^{m}2. If $\mathrm{d}H : \pi_1(L) \to Gl(k, \mathbb{R})$ has a finite image and if $H^1(\tilde{L}; \mathbb{R}) = 0$, where $\tilde{L}$ is the covering space of L corresponding to* Ker(dH), *then L has a saturated tubular neighborhood in which the foliation is that of a flat disk bundle over L with structure group $\pi_1(L)/\mathrm{Ker}(\mathrm{d}H)$.*

Derivation of corollaries. Corollary 1 is immediate from Theorem 2 [$H^1((L); Gl(k, \mathbb{R})$ is the set of representations of $\pi_1(L)$ in $Gl(k, \mathbb{R})$]. Corollary 2 is proved by considering the finite covering space of a tubular neighborhood of L corresponding to Ker(dH). This tubular neighborhood has a compact leaf $\tilde{L}$ covering L, and $H^1(\tilde{L}; \mathbb{R}) = 0$ by hypothesis. There is a smaller, saturated tubular neighborhood of $\tilde{L}$ with a trivial foliation. This means the holonomy around L is finite, from which Corollary 2 follows (e.g. there is a Riemannian metric invariant by holonomy transformations; begin with a convex disk normal to L, and intersect all its images under holonomy transformations to obtain an invariant normal disk etc.).

THEOREM 3. *Let G be a topological group generated by a compact neighborhood of 1. Suppose G acts continuously in the C^1 topology as a group of C^1 transformations on a connected manifold V^k with a fixed point p. Then either the linear action at p is non-trivial, or there is a non-trivial continuous representation of G in $\mathbb{R}$ (i.e. $H_c^1(G; \mathbb{R}) \neq 0$), or the action of G is trivial.*

Remark. In the case G is compact, this just says, if the linear action at p is trivial, the action is trivial; this is well-known. An interesting special case is when G is finitely generated:

then it is essentially equivalent to Theorem 2. Although Theorem 3 is stated in global terms, it is derived from a local form which says that under the hypotheses, points sufficiently near p must be fixed. The action near p is like the holonomy of a foliation.

Proof of the Theorems

Reduction of Theorem 1 *to Theorem* 2. Under the assumptions of Theorem 1, the linear holonomy around L must be trivial, since otherwise $\log|\mathrm{d}H_\alpha|$ $[\alpha \in \pi_1(L)]$ is a non-trivial representation of $\pi_1(L)$ in $\mathbb{R}$; i.e. a non-zero element of $H^1(L; \mathbb{R})$. Assuming Theorem 2, we conclude the holonomy around L is trivial, and L has a saturated tubular neighborhood with all leaves diffeomorphic to L. Therefore, the set of points on leaves diffeomorphic with L is open. This set is also closed (see, for instance, Thurston [3], or Reeb [2]). Therefore, since we should have mentioned that M^n is connected, all leaves of $\mathscr{F}$ are diffeomorphic to L.

Proof of Theorem 2. We assume L is a compact leaf of $\mathscr{F}$ such that the linear holonomy $\mathrm{d}H$ is trivial, but the holonomy H is not trivial; i.e. there are points x_i arbitrarily near L on the k-dimensional normal disk to L, and elements $\alpha_i \in \pi_1(L)$, such that $H_{\alpha_i}(x_i) \neq x_i$. We need to produce, then, a non-trivial element of $H^1(L; \mathbb{R})$. In fact, we will construct a non-trivial representation of $\pi_1(L)$ in the group $\mathbb{R}^k$. This will suffice. The idea is that as x_k approaches L, $\mathrm{d}H_\alpha(x_i)$ approaches 1, so H_α becomes more and more nearly a translation, in a neighborhood of x_i; if H_α were a translation for each α, this would be our desired representation.

Definitions. Let K be a subset of a group G. Let $\varepsilon \geq 0$. Then a (K, ε) cocycle γ with values in $\mathbb{R}^k$ is an $\mathbb{R}^k$-valued function on K satisfying $\|\delta\gamma(\alpha, \beta)\| \leq \varepsilon$, when α, β and $\alpha\beta$ are elements of K, and $\delta\gamma : G \times G \to \mathbb{R}^k$ is defined by:

$$\delta\gamma(\alpha, \beta) = \gamma(\alpha) + \gamma(\beta) - \gamma(\alpha\beta).$$

Suppose that B is a finite set of generators for G. Then, for $K \supset B$, we define a *normal* (K, ε)-cocycle γ to be a (K, ε)-cocycle such that

$$\max_{\beta \in B} \|\gamma(\beta)\| = 1.$$

In general, a (K, ε)-cocycle is not a really-truly cocycle. If $K' \supset K$ and $\varepsilon' \leq \varepsilon$ then a (K', ε')-cocycle is a (K, ε)-cocycle. A $(G, 0)$-cocycle is a 1-cocycle. A 1-cocycle is a representation of G in $\mathbb{R}^k$.

We assume now that B is a generating set containing 1 and closed under inversion. Let B^l denote the set of products of at most l elements of B.

LEMMA 1. *Non-trivial cocycles exist iff normal (B^l, ε)-cocycles exist, for every $\varepsilon > 0$ and every positive l.*

Proof of Lemma 1. The set of normal (B^l, ε)-cocycles is compact, since they form a closed, bounded subset of the set of functions from B^l to $\mathbb{R}^k$. The set of normal (B^l, ε)-cocycles which are restrictions of normal $(B^{l'}, \varepsilon')$-cocycles, for $l' \geq l$, $0 < \varepsilon' \leq \varepsilon$, is non-empty, and the family of such sets has the finite intersection property. Taking the intersection over ε, we obtain normal $(B^l, 0)$-cocycles for every l. Forming the intersection over l, we obtain a normal $(B^l, 0)$-cocycle γ which extends to a normal $(B^{l'}, 0)$-cocycle for every $l' > l$. Choose an extension of γ to a normal $(B^{l+1}, 0)$-cocycle which extends to a normal $(B^{l'}, 0)$-cocycle for every $l' > l + 1$. Continue by induction to obtain a normal $(G, 0)$-cocycle, with values in $\mathbb{R}^k$. (Alternatively, we could have used the Tychanoff product theorem.) This concludes the proof of Lemma 1.

Now, suppose x is a point in the normal disk D through L which is not fixed by H. (From now on, we take a fixed parametrization of D by $\mathbb{R}^k$.) Let B be a finite set of generators for $\pi_1(L)$, closed under inversion and containing 1. Define a function γ_x on the subset of $\pi_1(L)$ for which $H_\alpha(x)$ is defined, by:

$$\gamma_x(\alpha) = \frac{1}{m}(H_\alpha(x) - x) \tag{1}$$

where

$$m = \max_{\alpha \in B} \|H_\alpha(x) - x\|.$$

The rest of the proof is devoted to estimates needed to prove the following lemma:

LEMMA 2. *For every natural number l, and every $\varepsilon > 0$, there is a δ such that if $\|x\| \le \delta$, then γ_x is a (B^l, ε)-cocycle.*

Proof of Lemma 2. We define recursively a function $f(r, \varepsilon')$, r a natural number, $\varepsilon' \varepsilon \mathbb{R}$, by:

$$\begin{aligned} f(0, \varepsilon') &= 0 \\ f(r + 1, \varepsilon') &= 1 + f(r, \varepsilon')(1 + \varepsilon'). \end{aligned} \tag{2}$$

$f(r, \varepsilon')$ is a polynomial in ε' with constant term r.

Let $\varepsilon' > 0$ be sufficiently small that

$$\varepsilon' \cdot f(l, \varepsilon') \le \varepsilon \tag{3}$$

and let $\delta' > 0$ be sufficiently small that if $\|x\| \le \delta'$, and $\alpha \in B^l$, then

$$\|dH_\alpha(x) - I\| \le \varepsilon'. \tag{4}$$

Finally, let $0 < \delta < \delta'$ be sufficiently small that if $\|x\| \le \delta$, then $\|H_\alpha(x)\| \le \delta'$ for $\alpha \in B^l$. This, we claim, is a δ suitable for Lemma 2. Let x be such that $\|x\| \le \delta$, and $H_\alpha(x) \ne x$, for some α.

First, we will give an upper bound for $\gamma_x(\alpha)$, when $a \in B^r$, $r \le l$:

$$\|\gamma_x(\alpha)\| \le f(r, \varepsilon'). \tag{5}$$

This is true for $r = 0$ or $r = 1$. Assuming inductively that (5) is true for $r = k$, let σ be an arbitrary element of B^{k+1}. Write $\sigma = \beta\alpha$, with $\alpha \in B^k$, $\beta \in B^1$. Then

$$\begin{aligned} \|\gamma(\sigma)\| &= \frac{1}{m}\|H_\sigma(x) - x\| \\ &= \frac{1}{m}\|H_\beta(H_\alpha(x)) - H_\alpha(x) + H_\alpha(x) - x\| \\ &\le \frac{1}{m}\|H_\beta(H_\alpha(x)) - H_\alpha(x)\| + \|\gamma_x(\alpha)\|. \end{aligned}$$

Now

$$\frac{1}{m}(H_\beta(H_\alpha(x) - H_\alpha(x)) - \frac{1}{m}(H_\beta(x) - x)$$
$$= \frac{1}{m}\int_x^{H_\alpha(x)} \mathrm{d}H_\beta - I).$$

Thus,

$$\frac{1}{m}\|H_\beta(H_\alpha(x)) - H_\alpha(x)\| \leq \max\|\mathrm{d}H_\beta - I\| \cdot \left\|\frac{1}{m}(H_\alpha(x) - x\right\| + 1 \leq \varepsilon'(f(k, \varepsilon') + 1$$

by (4), (5), for $r = k$, and (1). So

$$\|\gamma_x(\sigma)\| \leq 1 + (1 + \varepsilon')f(k, \varepsilon') \leq f(k + 1, \varepsilon'))$$

by (2).

Equation (5) is established for $r = k + 1$, so by induction, (5) is established for all $r \leq l$.

Now we can estimate $\delta\gamma_x$. Let β_1, β_2, and $\beta_1\beta_2$ be in B^l. Then,

$$\delta\gamma_x(\beta_1, \beta_2)) = \gamma_x(\beta_1) + \gamma_x(\beta_2) - \gamma_x(\beta_1 \cdot \beta_2)$$
$$= \frac{1}{m}[H_{\beta_1}(x) - x + H_{\beta_2}(x) - x - H_{\beta_1\beta_2}(x) + x]$$
$$= -\frac{1}{m}[H_{\beta_1}(H_{\beta_2}(x)) - H_{\beta_2}(x) - (H_{\beta_1}(x) - (x))]$$
$$= -\frac{1}{m}\int_x^{H_{\beta 2}(x)}(\mathrm{d}H_{\beta_1} - I)$$

so

$$\|\delta\gamma_x(\beta_1, \beta_2)\| \leq \left\|\frac{1}{m}(H_{\beta_2}(x) - x)\right\| \cdot \max_{\|y\| \leq 0}\|\mathrm{d}H_{\beta_1}(y) - I\| \leq \varepsilon$$

by (5), (4) and (3). Thus, γ_x is a normal (B^l, ε)-cocycle with values in $\mathbb{R}^k$, and the proof of Theorem 2 is concluded.

Modifications necessary to prove Theorem 3. In the proof of Theorem 3, we take B to be a compact neighborhood of 1, closed under inversion and generating G. We denote the representation of G by H. If p is a fixed point of H, then for x near p (and in some Euclidean neighborhood, which we identify with Euclidean space by a fixed chart) such that x is not fixed by H, a continuous function γ_x on a subset of G containing B is constructed by the same formula (1).

The proof of Lemma 2 works verbatim: by the continuity of H and the compactness of B^l, there are no problems with finding sufficiently small ε's and δ's. Lemma 1 must be helped along, here and there. In the definition of normal (B^l, ε)-cocycles, for a topological group G, we do not add any requirement that they be continuous. However, a normal (B^l, ε)-cocycle is bounded by $l + (l - 1)\varepsilon$. Then, in the topology of pointwise convergence, the space of normal (B^l, ε)-cocycles is compact. Assuming the hypotheses of Lemma 1, we conclude that there is a normal $(G, 0)$-cocycle with values in $\mathbb{R}^k$. For the purposes of Theorem 3, we need a continuous cocycle. However, a cocycle γ which is bounded (say, by 1) on B is automatically continuous. We need only see it is continuous at 1, because of the cocycle condition $\delta\gamma = 0$. But for every n, there is a neighborhood U of 1 such that $U^n \supset B$. Then for $\alpha \in U$, $n\gamma(\alpha) = \gamma(\alpha^n) \leq 1$, so γ is bounded by $1/n$ on U: therefore, γ is continuous. This shows that if the linear action at p is trivial, and if there are no non-trivial continuous representations of G in $\mathbb{R}$, then a neighborhood of p is fixed by H. The set of fixed points of H for which the linear action is trivial is a closed set; the above argument shows it is open, so it is all of V^k.

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