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Some new bounds for cover-free families through biclique covers

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ABSTRACT

An $(r, w; d)$ cover-free family (*CFF*) is a family of subsets of a finite set X such that the intersection of any r members of the family contains at least d elements that are not in the union of any other w members. The minimum size of a set X for which there exists an $(r, w; d) - CFF$ with t blocks is denoted by $N((r, w; d), t)$.

In this paper, we show that the value of $N((r, w; d), t)$ is equal to the d-biclique covering number of the bipartite graph $I_t(r, w)$ whose vertices are all w- and r-subsets of a t-element set, where a w-subset is adjacent to an r-subset if their intersection is empty. Next, we provide some new bounds for $N((r, w; d), t)$. In particular, we show that for $r \geq w$ and $r \geq 2$

$$N((r, w; 1), t) \geq c\frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r}\log(t - w + 1),$$

where c is approximately $\frac{1}{2}$. Also, we determine the exact value of $N((r, w; d), t)$ for $t \leq r + w + \frac{r}{w}$ and also for some values of d. Finally, we show that $N((1, 1; d), 4d - 1) = 4d - 1$ if and only if there exists a Hadamard matrix of order $4d$.

1. Introduction

A family of sets is called an (r, w)-cover-free family if no intersection of r sets of the family are covered by a union of any other w sets of the family. Cover-free families have been studied extensively throughout the literature due to both its interesting structure and applications in several respects; see [11,13,16,23,31,33]. As an interesting application of cover-free families, one can consider key predistribution scheme (KPS). In many applications we need to have a KPS in which there is a key for every group of r users, and each such key is secure against any disjoint coalition of at most w users. We can see that if we have an (r, w)-cover-free family then we can construct such a KPS; see [23].

The remainder of the paper is organized as follows. In Section 1, we set up notation and terminology. Section 2 is devoted to study the connection between cover-free families and biclique cover. In Section 3, we present several new lower bounds for $N((r, w; d), t)$. Section 4 concerns the fractional version of biclique cover and we determine the exact value of $N((r, w; d), t)$ for $t \leq r + w + \frac{r}{w}$ and for some values of d. Finally, we show that if there exists a Hadamard matrix of order $4d$, then $N((1, 1; d), 4d - 1) = 4d - 1$.

Throughout this paper, we only consider finite simple graphs. For a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. By a *biclique* we mean a bipartite graph with vertex set (X, Y) such that every vertex in X is adjacent to

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every vertex in Y. Note that every empty graph is a biclique. A *biclique cover* of a graph G is a collection of bicliques of G such that each edge of G is in at least one of the bicliques. The number of bicliques in a minimum biclique cover of G is called the *biclique covering number* of G and denoted by $bc(G)$. This parameter of graphs was studied in the literature [1,2,14].

In this paper, we also need a generalization of the biclique cover as follows.

Definition 1. A *d-biclique cover* of a graph G is a collection of bicliques of G such that each edge of G is in at least d of the bicliques. The number of bicliques in a minimum d-biclique cover of G is called the *d-biclique covering number* of G and denoted by $bc_d(G)$. □

As usual, we denote by $[t]$ the set $\{1, 2, \ldots, t\}$. In this paper, by A^c we mean the complement of the set A. For $0 < w \le r \le t$, the *subset graph* $S_t(w, r)$ is a bipartite graph whose vertices are all w- and r-subsets of a t-element set, where a w-subset is adjacent to an r-subset if and only if one subset is contained in the other. Some properties of this family of graphs have been studied by several researchers; see [27]. In this paper, we consider an isomorphic version of this graph and name it *bi-intersection graph*.

Definition 2. For $0 < w \le r \le t$, the *bi-intersection graph* $I_t(r, w)$ is a bipartite graph whose vertices are all w- and r-subsets of a t-element set, where a w-subset is adjacent to an r-subset if and only if their intersection is empty. □

A *set system* is an ordered pair $(X, \mathcal{B})$, where X is a set of elements and $\mathcal{B}$ is a family of subsets (called blocks) of X. A set system can be described by an incidence matrix. Let $(X, \mathcal{B})$ be a set system, where $X = \{x_1, x_2, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$. The incidence matrix of $(X, \mathcal{B})$ is the $b \times v$ matrix $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in B_i \\ 0 & \text{if } x_j \notin B_i. \end{cases}$$

Definition 3. Let n, t, r, and w be positive integers. A set system $(X, \mathcal{B})$, where $|X| = n$ and $\mathcal{B} = \{B_1, \ldots, B_t\}$ is called an $(r, w) - CFF(n, t)$ if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \varnothing$, $|L| = r$, and $|M| = w$, we have

$$\bigcap_{l \in L} B_l \not\subseteq \bigcup_{m \in M} B_m. \quad \square$$

Stinson and Wei [30] generalized the definition of cover-free families as follows.

Definition 4. Let d, n, t, r, and w be positive integers. A set system $(X, \mathcal{B})$, where $|X| = n$ and $\mathcal{B} = \{B_1, \ldots, B_t\}$ is called an $(r, w; d) - CFF(n, t)$ if for any two sets of indices $L, M \subseteq [t]$ such that $L \cap M = \varnothing$, $|L| = r$, and $|M| = w$, we have

$$\left| \left(\bigcap_{l \in L} B_l \right) \setminus \left(\bigcup_{m \in M} B_m \right) \right| \ge d. \quad \square$$

Let $N((r, w; d), t)$ denote the minimum number of elements in any $(r, w; d) - CFF$ having t blocks. For convenience, we use the notation $N((r, w), t)$ instead of $N((r, w; 1), t)$. Obviously, we have $N((r, w; d), t) = N((w, r; d), t)$. Hence, unless otherwise stated we assume that $w \le r$.

2. Biclique cover

In this section, we show that the existence of a cover-free family can result from the existence of biclique cover of bi-intersection graph and vice versa. Our viewpoint sheds some new light on cover-free family. Using this observation, we introduce several new bounds.

Theorem 1. *Let r, w, and t be positive integers, where $t \ge r + w$, then*

$$N((r, w), t) = bc(I_t(r, w)).$$

Proof. First, consider an optimal $(r, w) - CFF(n, t)$, i.e., $n = N((r, w), t)$, with incidence matrix $A = (a_{ij})$. Assign to the jth column of A, the set A_j as follows

$$A_j \stackrel{\text{def}}{=} \{i | 1 \le i \le t, a_{ij} = 1\}.$$

Now, for any $1 \le j \le n$, construct a bipartite graph G_j with vertex set (X_j, Y_j), where the vertices of X_j are all r-subsets of A_j and the vertices of Y_j are all w-subsets of A_j^c, i.e.,

$$X_j = \{U | U \subseteq A_j, |U| = r\} \quad \text{and} \quad Y_j = \{V | V \subseteq A_j^c, |V| = w\}.$$

Also, an r-subset is adjacent to a w-subset if their intersection is empty. One can see that G_j, for $1 \le j \le n$, is a biclique. Let UV be an arbitrary edge of $I_t(r, w)$, where $U \cap V = \varnothing$, $|U| = r$ and $|V| = w$. In view of definition of *CFF* and since A is the incidence matrix of the *CFF*, there is a column of A, say j, where $a_{ij} = 1$ if $i \in U$ and $a_{ij} = 0$ if $i \in V$. Clearly, $U \in X_j$, $V \in Y_j$, and $UV \in G_j$. Hence, $\{G_1, G_2, \ldots, G_n\}$ is a biclique cover of $I_t(r, w)$. So $bc(I_t(r, w)) \le N((r, w), t)$.

Conversely, assume that $G_1, \ldots, G_l$ constitute a biclique cover of $I_t(r, w)$, where $l = bc(I_t(r, w))$ and G_i has as its vertex set (X_i, Y_i). Let A_i be the union of sets that lie in X_i. Consider the indicator vector of the set A_i, for $i = 1, \ldots, l$, and construct the matrix A whose ith column is the indicator vector of the set A_i. We claim that A is the incidence matrix of an $(r, w)-CFF(l, t)$. To see this, let U and V be two arbitrary disjoint sets of $[t]$, where $|U| = r$ and $|V| = w$. Thus, UV is an edge of the graph $I_t(r, w)$. Hence, there exists a biclique G_j, where $U \in X_j$ and $V \in Y_j$. Now, in view of definition of A_j, all entries corresponding to the elements of U and V in the jth column are 1 and 0, respectively. So $N((r, w), t) \le bc(I_t(r, w))$. This completes the proof. □

By the same argument we obtain the following theorem.

Theorem 2. *Let r, w, d, and t be positive integers, where $t \ge r + w$, then*

$$N((r, w; d), t) = bc_d(I_t(r, w)).$$

A *weakly separating system* on $[t]$ is a collection $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ of disjoint pairs of subsets of $[t]$ such that for every $i, j \in [t]$ with $i \neq j$ there is a k with either $i \in X_k$ and $j \in Y_k$ or $i \in Y_k$ and $j \in X_k$. Similarly, a *strongly separating system* on $[t]$ is a collection $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ of disjoint pairs of subsets of $[t]$ such that for every ordered pair (i, j) with $1 \le i, j \le t$ and $i \neq j$, there is a $k \in [n]$ with $i \in X_k$ and $j \in Y_k$. The study of separating systems was started by Rényi [25] in 1961. Other researchers have studied the properties of separating systems in the literature; see [5,6,24,28]. One can construct a $(1, 1) - CFF(n, t)$ from a strongly separating system on $[t]$ of size n and vice versa (see the proof of Theorem 1). So if we denote by $\mathcal{R}(t)$, the minimum size of a strongly separating system, then $N((1, 1), t) = \mathcal{R}(t)$. Let $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ be a weakly separating system. The complete bipartite graphs with vertex classes X_i and Y_i cover the edges of the complete graph K_t with vertex set $[t]$. Also, if the family $\{G_1, \ldots, G_n\}$ is a biclique cover of K_t, where G_i has as its vertex set (X_i, Y_i), then $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ is a weakly separating system. So if we denote by $s(t)$, the size of the minimum weakly separating system, then $s(t) = bc(K_t)$. Also, in [2], it was proved that $\mathcal{R}(t) = bc(K^-_{t,t})$, where $K^-_{t,t}$ is the complete bipartite graph $K_{t,t}$ with a perfect matching removed. The exact value of $\mathcal{R}(t)$ was determined by Sperner.

Theorem A ([29]). *If $C = \min\{c \mid \binom{c}{\lfloor \frac{c}{2} \rfloor} \ge t\}$, then $C = \mathcal{R}(t)$.*

Theorem A implies

$$\mathcal{R}(t) = \log_2 t + \frac{1}{2} \log_2 \log_2 t + O(1).$$

It is simple to see that $bc(G) \ge m(G)$, where $m(G)$ is the maximum size of induced matchings of G. Let $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ be a family of pairs of subsets of an arbitrary set. The family $\mathcal{F}$ is called an (r, w)-system if for all $1 \le i \le h$, $|A_i| = r$, $|B_i| = w$, $A_i \cap B_i = \varnothing$, and for all distinct i, j with $1 \le i, j \le h$, $A_i \cap B_j \neq \varnothing$. Bollobás [4] proved that the maximum size of an (r, w)-system is equal to $\binom{r+w}{r}$. Obviously, $m(I_t(r, w))$ is the maximum size of an (r, w)-system, so $N((r, w), t) \ge \binom{r+w}{r}$.

3. Bounds

In this section, we introduce several bounds for $N((r, w; d), t)$. Engel [11], using the fractional matching and fractional cover of ordered interval hypergraph, obtained the following bounds.

Theorem B ([11]). *For any positive integers r, w, and t, where $r \ge w$ and $t \ge r + w$, we have*

$$N((r, w), t) \ge \binom{r + w - 1}{r} \mathcal{R}(t - r - w + 2).$$

Theorem C ([11]). *For any $\epsilon > 0$, it holds that*

$$N((r, w), t_\epsilon) \ge (1 - \epsilon) \frac{(w + r - 2)^{w+r-2}}{(w - 1)^{w-1}(r - 1)^{r-1}} \mathcal{R}(t_\epsilon - r - w + 2),$$

for all sufficiently large t_ϵ.

Here is the best known lower bound for $N((r, 1), t)$.

Theorem D ([8,15,26]). *Let $r \ge 2$ and $t \ge r + 1$ be positive integers. Then*

$$N((r, 1), t) \ge C_{r,t} \frac{r^2}{\log r} \log t,$$

where $\lim_{r+t\to\infty} C_{r,t} = c$ for some constant c.

Several proofs have been presented for the preceding theorem. In [8,15,26], it was shown that c is approximately $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$, respectively.

Lemma A ([31]). *For any positive integers r, w, and t, where $t \geq r + w$, we have*

$$N((r, w), t) \geq N((r, w-1), t-1) + N((r-1, w), t-1).$$

Stinson et al. [31], using Lemma A and Theorem D, improved the bounds of Engel in some cases and obtained the following bounds.

Theorem E ([31]). *For any positive integers r, w, and t, where $t \geq r + w$, we have*

$$N((r, w), t) \geq 2c \frac{\binom{w+r}{r}}{\log(w+r)} \log t,$$

where c is a constant satisfies Theorem D.

Theorem F ([31]). *For any positive integers r, $w \geq 1$, there exists an integer $t_{r,w}$ such that for all $t > t_{r,w}$*

$$N((r, w), t) \geq 0.7c(r+w) \frac{\binom{w+r}{r}}{\log\left(\binom{w+r}{r}\right)} \log t,$$

where c is a constant satisfies Theorem D.

In [22], it was shown $t_{r,w} \leq \max\{\lfloor \frac{r+w+1}{2} \rfloor^2, 5\}$. Consider the case $d = 1$. The rate of $(r, w) - CFF$ is defined by

$$R(r, w) = \limsup_{t \to \infty} \frac{\log t}{N((r, w), t)}.$$

In [10], it was shown that

$$R(r, w) \leq \min_{0<x<r} \min_{0<y<w} \frac{x^x y^y R(r-x, w-y)}{(x+y)^{x+y}}.$$

For a fixed $w \geq 2$ and $r \to \infty$, the previous bound gives the following lower bound.

Theorem G ([10]). *For any fixed positive integer $w \geq 2$ and every sufficiently large positive integers r, we have*

$$N((r, w), t) \geq \frac{2e^{w-1} r^{w+1} \log t}{(w+1)^{w+1} \log r}.$$

The bound of Theorem G is better than the bound

$$N((r, w), t) \geq \frac{r^{w+1} \log t}{(w+1)! \log r},$$

obtained in [9], and also the bounds of Theorems E and F provided that r is sufficiently large. In [19,20], the following recursive upper bound was proved.

Theorem H ([19,20]). *Let r and w be positive integers. We have*

$$R(r, w) \leq \min_{0<x<r} \min_{0<y<w} \frac{R(r-x, w-y)}{R(r-x, w-y) + \frac{(x+y)^{x+y}}{x^x y^y}}.$$

Theorem H improves the bound from [10] for all w and r (for a fixed $w \geq 2$ and $r \to \infty$ it also gives Theorem G). In fact, Theorem H is currently the best known lower bound for $N((r, w), t)$.

Here we introduce some new lower bounds for $N((r, w; d), t)$ which improve Theorem B and also we present a lower bound (Theorem 4) which can be considered as an improvement of Theorems E, F and G in some cases. We first prove the following preliminary lemma which will be needed in the proof of Theorem 3.

Lemma 1. *Let G be a graph and $G_1, G_2, \ldots, G_k$ be some pairwise vertex disjoint subgraphs of G. Also, assume that for every four cycle C_4 of G and $1 \leq i \neq j \leq k$, we have $E(C_4) \cap E(G_i) = \varnothing$ or $E(C_4) \cap E(G_j) = \varnothing$. Then*

$$bc_d(G) \geq \sum_{i=1}^{k} bc_d(G_i).$$

Proof. Let $\{H_1, H_2, \ldots, H_l\}$ be an optimal d-biclique cover of G, i.e., $l = bc_d(G)$. Also, assume that H_i' is a subgraph of $G_1 \cup G_2 \cup \cdots \cup G_k$ induced by H_i, i.e., $E(H_i') = E(G_1 \cup G_2 \cup \cdots \cup G_k) \cap E(H_i)$. If H_i' is a non-empty graph, by the assumption, it is clear that H_i' has exactly one non-empty connected component and this component is a biclique of exactly one of G_i's. Now, H_j''s cover all edges of G_i's at least d times. So $bc_d(G) \geq \sum_{i=1}^{k} bc_d(G_i)$, as desired. □

Before embarking on the proof of the next theorem, we need the following definition. The family $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ is called a *weakly cross-intersecting set-pairs* (resp. cross-intersecting set-pairs) of size g on a ground set of size h whenever all A_i's and B_i's are subsets of an h-set and for every i, where $1 \le i \le g$, A_i and B_i are disjoint subsets, and furthermore, for every $i \ne j$, $(A_i \cap B_j) \cup (A_j \cap B_i) \ne \varnothing$ (resp. $(A_i \cap B_j) \ne \varnothing$ and $(A_j \cap B_i) \ne \varnothing$). This concept is a variant of the generalization of (r, w)-weakly cross-intersecting set-pairs which was introduced first by Tuza [32]. The weakly cross-intersecting set-pairs $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ is called an (r, w)-*weakly cross-intersecting set-pairs* whenever for every $1 \le i \le g$, $|A_i| = r$ and $|B_i| = w$. Hereafter, we adopt the convention that $N((r, 0; d), t) = N((0, w; d), t) = 1$.

Theorem 3. *Suppose that g, h, r, w, and t are positive integers. Also, assume that $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ is a weakly cross-intersecting set-pairs on a ground set of size h such that for any $1 \le i \le g$, $|A_i| \le r$ and $|B_i| \le w$. If $t \ge \max\{h, r + w\}$, then*

$$N((r, w; d), t) \ge \sum_{i=1}^{g} N((r - |A_i|, w - |B_i|; d), t - |A_i| - |B_i|).$$

Proof. Assume that $\mathcal{F} = \{(A_1, B_1), \ldots, (A_g, B_g)\}$ is a weakly cross-intersecting set-pairs. For every $1 \le k \le g$, construct a bipartite graph G_k with vertex set (X_k, Y_k), where the vertices of X_k are all r-subsets of $[t]$ which contain A_k and their intersections with B_k are empty. Also, the vertices of Y_k are all w-subsets of the set $[t]$ which contain B_k and their intersections with A_k are empty, i.e.,

$$X_k = \{U \mid U \subseteq [t], |U| = r,\ A_k \subseteq U,\ U \cap B_k = \varnothing\}$$
$$Y_k = \{V \mid V \subseteq [t], |V| = w,\ B_k \subseteq V,\ V \cap A_k = \varnothing\},$$

where a vertex $U \in X_k$ is adjacent to a vertex $V \in Y_k$ if $U \cap V = \varnothing$. Obviously, if $|A_k| = r$ or $|B_k| = w$, then G_k is isomorphic to a star graph. Otherwise, one can check that every G_k is isomorphic to $I_{t-|A_k|-|B_k|}(r - |A_k|, w - |B_k|)$. Since if we delete the elements of A_k from the vertices of X_k, every vertex is mapped to an $(r - |A_k|)$-subset of the set $[t] \setminus (A_k \cup B_k)$ and also if we remove the elements of B_k from the vertices of Y_k, every vertex is mapped to a $(w - |B_k|)$-subset of the set $[t] \setminus (A_k \cup B_k)$. Clearly, this mapping is an isomorphism between G_k and $I_{t-|A_k|-|B_k|}(r - |A_k|, w - |B_k|)$. Also, since $\mathcal{F}$ is a weakly cross-intersecting set-pairs, G_k's are pairwise vertex disjoint. On the other hand, for any $1 \le i \ne j \le k$, there is no four cycle C_4 of $I_t(r, w)$ such that $E(C_4) \cap E(G_i) \ne \varnothing$ and $E(C_4) \cap E(G_j) \ne \varnothing$. So, in view of Lemma 1,

$$bc_d(I_t(r, w)) \ge \sum_{k=1}^{h} bc_d(G_k).$$

Hence, the result easily follows. □

Here, we mention some consequences of the above theorem. Let M be an s-subset of $[t]$. For any non-negative integers i and j, where $s - w \le i \le r$ and $s - r \le j \le w$, set

$$\mathcal{F}_i = \{(A^i, B^i) : A^i \subseteq M, |A^i| = i,\ B^i = M \setminus A^i\},$$
$$\mathcal{E}_j = \{(A^j, B^j) : A^j \subseteq M, |A^j| = j,\ B^j = M \setminus A^j\}.$$

It is easy to see that $|\mathcal{F}_i| = \binom{s}{i}$ and $|\mathcal{E}_j| = \binom{s}{j}$. Also, $\mathcal{F} = \cup_{s-w \le i \le r} \mathcal{F}_i$ (resp. $\mathcal{E} = \cup_{s-r \le j \le w} \mathcal{E}_j$) is a weakly cross-intersecting set-pairs. Therefore, in view of Theorem 3, the next corollary which is a generalization of Lemma A follows.

Corollary 1. *For any positive integers $0 < s \le r + w$ and $t \ge r + w$, it holds that*

1. $N((r, w; d), t) \ge \sum_{s-w \le i \le r} \binom{s}{i} N((r - i, w - s + i; d), t - s)$,
2. $N((r, w; d), t) \ge \sum_{s-r \le j \le w} \binom{s}{j} N((r - s + j, w - j; d), t - s)$.

Let $T((r, w); n)$ denote the maximum number of blocks in an $(r, w) - CFF$ with n points. Erdős et al. [12] discussed $(1, 2)$-CFFs in detail, and showed that

$$1.134^n \le T((1, 2); n) \le 1.25^n.$$

The upper bound is asymptotic and for sufficiently large n is useful. Hence, for large n, $N((1, 2); t) \ge \frac{1}{\log(1.25)} \log t$. If we set $s = r + w - 3$ in the above corollary, then the following bound can be concluded which can be considered as an improvement of Theorem B.

Corollary 2. *For any positive integers r and w, where $r \ge 2$, it holds that*

$$N((r, w), t) \ge \binom{r + w - 2}{r - 1} N((2, 1); t - r - w + 3) + \binom{r + w - 3}{r} + \binom{r + w - 3}{r - 3}.$$

In view of Theorem 3, if there exists an (i, j)-weakly cross-intersecting set-pairs, then the following corollary can be concluded. We should mention that Engel [11] obtained a result that is similar to the following corollary.

Corollary 3. *Let i, j, r, and w be positive integers, where $1 \leq i \leq r-1$ and $1 \leq j \leq w-1$. If there exists an (i, j)-weakly cross-intersecting set-pairs of size $g(i, j)$ on a ground set of cardinality h, then for any t, where $t \geq \max\{h, r+w\}$, we have*

$$N((r, w; d), t) \geq g(i, j)N((r-i, w-j; d), t-i-j).$$

By a lattice path we mean a path on an $i \times j$ grid from $(0, 0)$ to (i, j), where each move is to the right or up. Assume that $\mathcal{L}(i, j)$ is the set of lattice paths such that the path is strictly below the line $y = \frac{j}{i}x$ except at the two endpoints. Tuza [32] showed that if $f(i, j)$ is the maximum size of a weakly cross-intersecting set-pairs, then $f(i, j) < \frac{(i+j)^{i+j}}{i^i j^j}$. Recently, Király et al. [18], by a charming idea and using lattice paths, presented an (i, j)-weakly cross-intersecting set-pairs of size $(2i+2j-1)|\mathcal{L}(i, j)|$ on a ground set of size $2i+2j-1$. Unfortunately, for general (i, j), there is no explicit formula for $|\mathcal{L}(i, j)|$. However, Bizley [3] showed that for relatively prime numbers i and j, $|\mathcal{L}(i, j)| = \frac{\binom{i+j}{i}}{i+j}$. In [18], it is shown $g(i, j) \geq (2-o(1))\binom{i+j}{i}$, where $f \in o(1)$ means that $\lim_{i+j\to\infty} f = 0$.

Corollary 4. *Let r, w, and t be positive integers, where $t \geq \max\{2r+2w-5, r+w\}$. Then*

$$N((r, w), t) \geq (2-o(1))\binom{r+w-2}{r-1}\mathcal{R}(t-r-w+2).$$

Also, in [18], it is shown that there exists an $(r-1, r-1)$-weakly cross-intersecting set-pairs of size $(2-\frac{1}{2r-2})\binom{2r-2}{r-1}$ on a ground set of size $4r-6$.

Corollary 5. *Assume that r and t are positive integers, where $t \geq \max\{4r-6, 2r\}$. Then*

$$N((r, r), t) \geq \left(2 - \frac{1}{2r-2}\right)\binom{2r-2}{r-1}\mathcal{R}(t-2r+2).$$

The following theorem improves Theorems F and G in some cases, e.g., whenever $r^{0.50} \ll w \ll r^{0.65}$. Moreover, this bound holds for any $t \geq r+w$.

Theorem 4. *For any positive integers r, w, and t, where $t \geq r+w$, $r \geq w$, and $r \geq 2$, we have*

$$N((r, w), t) \geq c\,\frac{\binom{r+w}{w+1} + \binom{r+w-1}{w+1} + 3\binom{r+w-4}{w-2}}{\log r}\log(t-w+1),$$

where c is a constant satisfies Theorem D.

Proof. We prove the assertion by induction on w. By Theorem D, the assertion holds for $w = 1$. Assume that the assertion is true for every w' where $w' < w$. Obviously,

$$\mathcal{F} = \{(\emptyset, \{1\}), (\{1\}, \{2\}), (\{1, 2\}, \{3\}), \ldots, (\{1, 2, \ldots, r-w\}, \{r-w+1\}), (\{1, 2, \ldots, r-w+1\}, \{\emptyset\})\}$$

is a weakly cross-intersecting set-pairs. Hence, in view of Theorem 3,

$$\begin{aligned} N((r, w), t) &\geq \left(\sum_{i=0}^{r-w} N((r-i, w-1), t-i-1)\right) + N((w-1, w), t-r+w-1) \\ &= \left(\sum_{i=0}^{r-w} N((r-i, w-1), t-i-1)\right) + N((w, w-1), t-r+w-1). \end{aligned}$$

Now by induction

$$\begin{aligned} N((r, w), t) \geq{}& \sum_{i=0}^{r-w} c\,\frac{\binom{r+w-i-1}{w} + \binom{r+w-i-2}{w} + 3\binom{r+w-i-5}{w-3}}{\log(r-i)}\log(t-w+1-i) \\ &+ c\,\frac{\binom{2w-1}{w} + \binom{2w-2}{w} + 3\binom{2w-5}{w-3}}{\log(w)}\log(t-r+1). \end{aligned}$$

Since $\frac{\log x}{\log(x-1)}$ is a decreasing function, it holds that

$$\begin{aligned} N((r,w),t) &\geq c\frac{\log(t-w+1)}{\log r}\left(\sum_{i=0}^{r-w}\binom{r+w-i-1}{w}+\binom{r+w-i-2}{w}+3\binom{r+w-i-5}{w-3}\right) \\ &\quad + c\frac{\log(t-w+1)}{\log r}\left(\binom{2w-1}{w}+\binom{2w-2}{w}+3\binom{2w-5}{w-3}\right) \\ &\geq c\frac{\log(t-w+1)}{\log r}\left(\sum_{i=0}^{r-w}\binom{r+w-i-1}{w}+\binom{r+w-i-2}{w}+3\binom{r+w-i-5}{w-3}\right) \\ &\quad + c\frac{\log(t-w+1)}{\log r}\left(\binom{2w-1}{w+1}+\binom{2w-2}{w+1}+3\binom{2w-5}{w-2}\right) \\ &= c\frac{\binom{r+w}{w+1}+\binom{r+w-1}{w+1}+3\binom{r+w-4}{w-2}}{\log r}\log(t-w+1). \quad \square \end{aligned}$$

4. Fractional biclique cover

The next result concerns the fractional version of biclique cover. If $\mathcal{R}$ is the set of all bicliques of a graph G, then each biclique cover of G can be described by a function $\phi : \mathcal{R} \to \{0, 1\}$ such that $\phi(G_i) = 1$ if and only if G_i belongs to the cover. Hence, $bc(G)$ is the minimum of $\sum_{G_i\in\mathcal{R}} \phi(G_i)$ over all function $\phi : \mathcal{R} \to \{0, 1\}$ such that for any edge e of G,

$$\sum_{G_i\in\mathcal{R}:e\in E(G_i)} \phi(G_i) \geq 1. \tag{1}$$

The fractional biclique covering number $bc^*(G)$ is the minimum of $\sum_{G_i\in\mathcal{R}} \phi(G_i)$ over all functions $\phi : \mathcal{R} \to [0, 1]$ satisfying (1).
Fractional graph theory is the modification of integer-valued graph parameters to take its value on non-integer values. For more on fractional graph theory and other fractional graph parameters; see [27]. In the fractional cover, using linear programming, it is proved that

$$bc^*(G) = \inf_d \frac{bc_d(G)}{d} = \lim_{d\to\infty} \frac{bc_d(G)}{d}.$$

Also, we have the following theorem.

Theorem I ([27])**.** *For every non-empty edge-transitive graph G,*

$$bc^*(G) = \frac{|E(G)|}{B(G)},$$

where $B(G)$ is the maximum number of edges among the bicliques of G.

It is in general a challenging problem to determine the exact value of $N((r, w; d), t)$. Also, it is a hard problem to find tight bound for $N((r, w; d), t)$ and this problem has been studied in the literature; see [9–11,15,26,30,31,33]. It is known [11] that $N((r, w), t) = \binom{t}{w}$ whenever $t \leq r + w + \frac{r}{w}$. We should mention that this result was improved in [17].

Theorem 5. *Let r, w, t, and d be positive integers such that $t \leq w + r + \frac{r}{w}$. Then*

$$N((r, w; d), t) = d\binom{t}{w}.$$

Proof. Easily, one can see that

$$B(I_t(r, w)) = \max_{t'+t''=t} \binom{t'}{r}\binom{t''}{w}.$$

Also, we have $|E(I_t(r, w))| = \binom{t}{r}\binom{t-r}{w}$, and $I_t(r, w)$ is an edge-transitive graph. Therefore, in view of Theorem I, we have

$$bc^*(I_t(r, w)) = \min_{t'+t''=t} \frac{\binom{t}{r}\binom{t-r}{w}}{\binom{t'}{r}\binom{t''}{w}}.$$

By a straightforward calculation

$$bc^*(I_t(r,w)) = \min_{t'+t''=t} \frac{\binom{t}{r}\binom{t-r}{w}}{\binom{t'}{r}\binom{t''}{w}} = \min_{w\le m\le t-r} \frac{\binom{t}{m}}{\binom{t-r-w}{m-w}}.$$

The sequence $\{a_m = \frac{\binom{t}{m}}{\binom{t-r-w}{m-w}}\}_{m=w}^{t-r}$ is an increasing sequence for $t \le w + r + \frac{r}{w}$. So $bc^*(I_t(r,w)) = \binom{t}{w}$. Also, we have

$$bc^*(I_t(r,w)) \le \frac{bc_d(I_t(r,w))}{d}.$$

On the other hand, $bc_d(I_t(r,w)) \le d \times bc(I_t(r,w))$ and for $t \le w + r + \frac{r}{w}$ we have $bc(I_t(r,w)) = \binom{t}{w}$. So $bc_d(I_t(r,w)) = d\binom{t}{w}$. □

For any graph G, the following inequality was proved in [21,27]

$$bc^*(G) \ge \frac{bc(G)}{1+\ln(B(G))}.$$

So we have the following corollary.

Corollary 6. *For any positive integers r, w, and t, where $t \ge r + w$, we have*

$$N((r,w),t) \le \min_{w\le m\le t-r} \frac{\binom{t}{m}}{\binom{t-r-w}{m-w}} \left(1 + \ln\left(\max_{t'+t''=t} \binom{t'}{r}\binom{t''}{w}\right)\right).$$

In [11], Engel proved that

$$N((r,w),t) \ge \min_{w-1\le m\le t-r+1} \frac{\binom{t}{m}}{\binom{t-r-w+2}{m-w+1}} (N((1,1),t-r-w+2)).$$

Hence, we have

$$N((r,w),t) \ge \min_{w-1\le m\le t-r+1} \frac{\binom{t}{m}}{\binom{t-r-w+2}{m-w+1}} \left(\log_2(t-r-w+2) + \frac{1}{2}\log_2\log(t-r-w+2) + c\right),$$

where c is a constant. In the next theorem, we specify the exact value of $N((r,w;d),t)$ for some special value of d. In the proof of the next theorem, by S_t we mean the permutation group of the set $[t]$.

Theorem 6. *For any positive integers r, w, t, d_0, and $d = \frac{B(I_t(r,w))}{|E(I_t(r,w))|}t!$, where $t \ge r + w$, we have*

$$N((r,w;d_0 d),t) = d_0(t!).$$

Proof. For every $\sigma \in S_t$, define the function $f_\sigma : V(I_t(r,w)) \to V(I_t(r,w))$ such that for every set $A = \{i_1, i_2, \ldots, i_l\} \in V(I_t(r,w))$, we have $f_\sigma(A) = \{\sigma(i_1), \ldots, \sigma(i_l)\}$ (note that here $l = r$ or $l = w$). Set $G = \{f_\sigma \mid \sigma \in S_t\}$. Then G is a subgroup of $\mathrm{Aut}(I_t(r,w))$ and also G acts transitively on $E(I_t(r,w))$. Now it is simple to check that

$$\frac{bc_d(I_t(r,w))}{d} = \frac{|E(I_t(r,w))|}{B(I_t(r,w))}.$$

To see this, assume that K is a biclique of $I_t(r,w)$, where $|E(K)| = B(I_t(r,w))$. Construct a biclique cover of $I_t(r,w)$ as follows. Set

$$\mathcal{C} = \{f_\sigma(K) \mid \sigma \in S_t\}.$$

It is readily seen that $\mathcal{C}$ is a biclique cover and every edge is covered with exactly $d = \frac{B(I_t(r,w))t!}{|E(I_t(r,w))|}$ bicliques. So

$$\frac{bc_d(I_t(r,w))}{d} \le \frac{|E(I_t(r,w))|}{B(I_t(r,w))}.$$

On the other hand, by the definition of fractional biclique cover, for every graph G and every positive integer d we have $bc^*(G) \le \frac{bc_d(G)}{d}$. Particularly,

$$bc^*(I_t(r,w)) \le \frac{bc_d(I_t(r,w))}{d}.$$

Consequently, in view of Theorem I,

$$\frac{bc_d(I_t(r,w))}{d} = \frac{|E(I_t(r,w))|}{B(I_t(r,w))}.$$

Also, for any positive integer d_0,

$$bc_{d_0 d}(I_t(r,w)) \leq d_0 bc_d(I_t(r,w)).$$

Hence,

$$\frac{|E(I_t(r,w))|}{B(I_t(r,w))} = bc^*(I_t(r,w)) \leq \frac{bc_{d_0 d}(I_t(r,w))}{d_0 d} \leq \frac{bc_d(I_t(r,w))}{d} = \frac{|E(I_t(r,w))|}{B(I_t(r,w))}. \quad (2)$$

Consequently, using (2) we obtain the result. □

An $n \times n$ matrix H with entries $+1$ and -1 is called a *Hadamard matrix* of *order* n if $HH^t = nI$. It is seen that any two distinct columns of H are orthogonal. Also, if we multiply some rows or columns by -1, or if we permute rows or columns, then H is still a Hadamard matrix. Two such Hadamard matrices are called equivalent. Easily, for any Hadamard matrix H, we can find an equivalent one for which the first row and the first column consist entirely of $+1$'s. Such a Hadamard matrix is called *normalized*.

Theorem 7. *Let d be a positive integer, then $N((1,1;d), 4d-1) = 4d-1$ if and only if there exists a Hadamard matrix of order $4d$.*

Proof. Let $H = [h_{ij}]$ be a normalized Hadamard matrix of order $4d$. Delete the first row and the first column. Also, assume that $K^-_{4d-1,4d-1}$ has (X, Y) as its vertex set where $X = \{v_1, \ldots, v_{4d-1}\}$ and $Y = \{v'_1, \ldots, v'_{4d-1}\}$. Assign to the jth column of H, two sets X_j and Y_j as follows

$$X_j = \{v_i | h_{ij} = +1\} \quad \text{and} \quad Y_j = \{v'_i | h_{ij} = -1\}.$$

Construct a complete bipartite graph G_j with vertex set (X_j, Y_j). The edge $v_i v'_j$ is covered by the complete bipartite graph G_k if and only if the corresponding entries of column k in row i is $+1$ and in row j is -1. It is well-known that the number of these columns, in a normalized Hadamard matrix of order $4d$, is equal to d. Hence, every edge is covered exactly d times. So $bc_d(K^-_{4d-1,4d-1}) \leq 4d-1$. On the other hand, $K^-_{4d-1,4d-1}$ is an edge-transitive graph. Therefore, in view of Theorem I, we have

$$\frac{4d-1}{d} = \frac{|E(K^-_{4d-1,4d-1})|}{B(K^-_{4d-1,4d-1})} \leq \frac{bc_d(K^-_{4d-1,4d-1})}{d}.$$

Consequently, $4d-1 \leq bc_d(K^-_{4d-1,4d-1})$ and the result follows. Conversely, assume that $N((1,1;d), 4d-1) = 4d-1$. This means that there exists a d-biclique cover of size $4d-1$ for the graph $K^-_{4d-1,4d-1}$. Under the same notation in the first part of the proof, assume that $\{G_1, G_2, \ldots, G_{4d-1}\}$ is the desired d-biclique cover. By a straightforward calculation, it follows that every edge is covered exactly d times. Also, $|X_i| = |Y_i| + 1 = 2d$ or $|Y_i| = |X_i| + 1 = 2d$. Suppose that K_{4d-1} has $\{u_1, u_2, \ldots, u_{4d-1}\}$ as its vertex set. Consider the biclique G_i and construct a biclique H_i of K_{4d-1} as follows. Assign to any vertex v_k (resp. v'_k) of G_i, the vertex u_k. Then, $\{H_1, \ldots, H_{4d-1}\}$ is a biclique cover of K_{4d-1} such that every edge is covered exactly $2d$ times. Now, add a new vertex u_{4d} to each H_i such that the resulting graph is isomorphic to $K_{2d,2d}$. So there exists a biclique cover for K_{4d} that every edge is covered exactly $2d$ times. In [7], it was shown that existence of such biclique cover is equivalent to the existence of a Hadamard matrix. □

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