Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 49 (2005) 1487–1498
www.elsevier.com/locate/camwa

A Simple Fractional-Calculus Approach to the Solutions of the Bessel Differential Equation of General Order and Some of Its Applications

SHY-DER LIN AND WEI-CHICH LING
Department of Applied Mathematics
Chung Yuan Christian University
Chung-Li 32023, Taiwan, Republic of China
shyder@cycu.edu.tw wei.chich@msa.hinet.net

KATSUYUKI NISHIMOTO
Institute of Applied Mathematics
Descartes Press Company
2-13-10 Kaguike
Koriyama 963-8833, Fukushima-Ken, Japan

H. M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
harimsri@math.uvic.ca

(Received and accepted September 2004)

Abstract—In many recent works, several authors demonstrated the usefulness of fractional calculus operators in the derivation of (explicit) particular solutions of a significantly large number of linear ordinary and partial differential equations of the second and higher orders. The main object of the present paper is to show how this simple fractional-calculus approach to the solutions of the classical Bessel differential equation of general order would lead naturally to several interesting consequences which include (for example) an alternative investigation of the power-series solutions obtainable usually by the Frobenius method. The methodology presented here is based largely upon some of the general theorems on (explicit) particular solutions of a certain family of linear ordinary fractional differintegral equations. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Fractional calculus, Bessel differential equation, Fuchsian (and non-Fuchsian) equations, Differintegral equations, Linear (ordinary and partial) differential equations, Index law, Linearity property, Generalized Leibniz rule, Frobenius method, Power-series solutions, Bessel functions, Trigonometric functions.

The present investigation was initiated during the fourth-named author's visit to Aletheia University in Tamsui in August 2004.
This work was supported, in part, by the National Science Council of the Republic of China under Grant NSC 93-2115-M-033-008, the Faculty Research Program of Chung Yuan Christian University, and the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

 Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2004.09.009

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The widely-investigated subject of *fractional calculus* (that is, calculus of derivatives and integrals of any *arbitrary* real or complex order) has gained importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details, [1–4]). Recently, by applying the following definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of order $\nu \in \mathbb{R}$, many authors have explicitly obtained particular solutions of a number of families of homogeneous (as well as nonhomogeneous) linear ordinary and partial fractional differintegral equations (see, for details, [5–18], and the references cited in *each* of these earlier works).

DEFINITION. *(See [19–21].) If the function $f(z)$ is analytic (regular) inside and on $\mathcal{C}$, where*

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}, \tag{1.1}$$

$\mathcal{C}^-$ is a contour along the cut joining the points z and $-\infty + i\Im(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathcal{C}^+$ is a contour along the cut joining the points z and $\infty + i\Im(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$\begin{gathered} f_\nu(z) = ((f(z))_\nu := \frac{\Gamma(\nu+1)}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta \\ \left(\nu \in \mathbb{R} \setminus \mathbb{Z}^-;\ \mathbb{Z}^- := \{-1, -2, -3, \dots\}\right) \end{gathered} \tag{1.2}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\} \qquad (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.3}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi, \qquad \text{for } \mathcal{C}^-, \tag{1.4}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi, \qquad \text{for } \mathcal{C}^+, \tag{1.5}$$

then $f_\nu(z) (\nu > 0)$ is said to be the fractional derivative of $f(z)$ of order ν and $f_\nu(z) (\nu < 0)$ is said to be the fractional integral of $f(z)$ of order $-\nu$, provided that

$$|f_\nu(z)| < \infty \qquad (\nu \in \mathbb{R}). \tag{1.6}$$

REMARK 1. Just as in the aforecited earlier works, we shall simply write f_ν for $f_\nu(z)$ whenever the argument of the differintegrated function f is clearly understood by the surrounding context. Moreover, in case f is a many-valued function, we shall tacitly consider the *principal value* of f in our investigation. For the sake of convenience in dealing with their various (known or new) special cases, we choose also to state one of the fundamental results (Theorem 1 below) for homogeneous (as well as nonhomogeneous) linear ordinary fractional differintegral equations of a *general* order $\mu \in \mathbb{R}$.

First of all, we find it to be worthwhile to recall here the following potentially useful lemmas and properties associated with the fractional differintegration which is defined above (cf., e.g., [19,20]).

LEMMA 1. LINEARITY PROPERTY. *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(k_1 f(z) + k_2 g(z))_\nu = k_1 f_\nu(z) + k_2 g_\nu(z) \qquad (\nu \in \mathbb{R};\ z \in \Omega) \tag{1.7}$$

for any constants k_1 and k_2.

LEMMA 2. INDEX LAW. *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$\begin{gathered}(f_\mu(z))_\nu = f_{\mu+\nu}(z) = (f_\nu(z))_\mu \\ (f_\mu(z) \neq 0;\ f_\nu(z) \neq 0;\ \mu, \nu \in \mathbb{R};\ z \in \Omega).\end{gathered} \tag{1.8}$$

LEMMA 3. GENERALIZED LEIBNIZ RULE. *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f(z) \cdot g(z))_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} f_{\nu-n}(z) \cdot g_n(z) \qquad (\nu \in \mathbb{R};\ z \in \Omega), \tag{1.9}$$

where $g_n(z)$ is the ordinary derivative of $g(z)$ of order n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), it being tacitly assumed (for simplicity) that $g(z)$ is the polynomial part (if any) of the product $f(z) \cdot g(z)$.

PROPERTY 1. For a constant λ,

$$\left(e^{\lambda z}\right)_\nu = \lambda^\nu e^{\lambda z} \qquad (\lambda \neq 0;\ \nu \in \mathbb{R};\ z \in \mathbb{C}). \tag{1.10}$$

PROPERTY 2. For a constant λ,

$$\left(e^{-\lambda z}\right)_\nu = e^{-i\pi\nu} \lambda^\nu e^{-\lambda z} \qquad (\lambda \neq 0;\ \nu \in \mathbb{R};\ z \in \mathbb{C}). \tag{1.11}$$

PROPERTY 3. For a constant λ,

$$\begin{gathered}\left(z^\lambda\right)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu - \lambda)}{\Gamma(-\lambda)} z^{\lambda-\nu} \\ \left(\nu \in \mathbb{R};\ z \in \mathbb{C};\ \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right).\end{gathered} \tag{1.12}$$

Some of the most recent contributions on the subject of explicit particular solutions of linear ordinary and partial fractional differintegral equations are those given by Tu *et al.* [5] who presented unification and generalization of a significantly large number of widely scattered results on this subject (see also the many relevant earlier works cited by Tu *et al.* [5]). For the sake of ready reference, we choose to recall here one of the *main* results of Tu *et al.* [5], involving a family of linear ordinary fractional differintegral equations, as Theorem 1 below.

THEOREM 1. *(See [5, p. 295, Theorem 1; p. 296, Theorem 2].) Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$\begin{aligned}P(z;p) &:= \sum_{k=0}^{p} a_k z^{p-k} \\ &= a_0 \prod_{j=1}^{p} (z - z_j) \qquad (a_0 \neq 0;\ p \in \mathbb{N})\end{aligned} \tag{1.13}$$

and

$$Q(z;q) := \sum_{k=0}^{q} b_k z^{q-k} \qquad (b_0 \neq 0;\ q \in \mathbb{N}). \tag{1.14}$$

Suppose also that $f_{-\nu} (\neq 0)$ exists for a given function f.

Then the following nonhomogeneous linear ordinary fractional differintegral equation:

$$\begin{gathered}P(z;p)\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q)\right] \phi_{\mu-k}(z) \\ + \binom{\nu}{q} q! b_0 \phi_{\mu-q-1}(z) = f(z) \\ (\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N})\end{gathered} \tag{1.15}$$

has a particular solution of the form:

$$\phi(z) = \left(\left(\frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)} \right)_{-1} \cdot e^{-H(z;p,q)} \right)_{\nu-\mu+1} \tag{1.16}$$
$$(z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}),$$

where, for convenience,

$$H(z;p,q) := \int^z \frac{Q(\zeta;q)}{P(\zeta;p)} d\zeta \qquad (z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}), \tag{1.17}$$

provided that the second member of (1.16) exists.

Furthermore, the following homogeneous linear ordinary fractional differintegral equation:

$$\begin{gathered} P(z;p)\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q) \right] \phi_{\mu-k}(z) \\ + \binom{\nu}{q} q! b_0 \phi_{\mu-q-1}(z) = 0, \\ (\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N}) \end{gathered} \tag{1.18}$$

has solutions of the form

$$\phi(z) = K \left(e^{-H(z;p,q)} \right)_{\nu-\mu+1}, \tag{1.19}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (1.17), it being provided that the second member of (1.19) exists.

REMARK 2. As already remarked in conclusion by Tu *et al.* [5, p. 301], it is fairly straightforward to observe that either or both of the polynomials $P(z;p)$ and $Q(z;q)$, involved in Theorem 1, can be of degree 0 as well. Thus, in the definitions (1.13) and (1.14), and in analogous situations appearing elsewhere in this paper, $\mathbb{N}$ may easily be replaced (if and where needed) by $\mathbb{N}_0$. The definitions (1.13) and (1.14) do serve the *main* purpose of this paper.

For various interesting applications of Theorem 1, one may refer to the earlier works [5–18], in each of which numerous further references on this subject can be found. The main object of the present paper is to investigate solutions of some general families of second-order linear ordinary differential equations, which are associated with the familiar Bessel differential equation of general order ν (cf. [22, Chapter 7; 23; 24, Chapter 17]):

$$z^2 \frac{d^2w}{dz^2} + z \frac{dw}{dz} + \left(z^2 - \nu^2\right) w = 0, \tag{1.20}$$

which is named after Friedrich Wilheim Bessel (1784–1846). Specially, we aim at demonstrating how the underlying simple fractional-calculus approach to the solutions of the classical differential equation (1.20) would lead naturally to several interesting consequences including (for example) an alternative investigation of the power-series solutions of (1.20) in terms of the familiar Bessel function $J_\nu(z)$ defined by

$$\begin{aligned} J_\nu(z) &:= \sum_{k=0}^{\infty} \frac{(-1)^k ((1/2)z)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \\ &= \frac{((1/2)z)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\text{——};\ \nu+1;\ -\frac{1}{4}z^2 \right) \\ &= \frac{((1/2)z)^\nu}{\Gamma(\nu+1)} \exp(\pm iz)\, {}_1F_1\left(\nu+\frac{1}{2};\ 2\nu+1;\ \mp 2iz \right), \end{aligned} \tag{1.21}$$

which are derived usually by appealing to the familiar method attributed to Ferdinand Georg Frobenius (1849–1917) (cf., e.g., [25, Chapter 16]).

REMARK 3. The last hypergeometric ${}_1F_1$ representation in (1.21) follows readily from the usual hypergeometric ${}_0F_1$ representation by means of the following hypergeometric transformation (known as *Kummer's second theorem*):

$$ {}_0F_1\left(\text{—};\ \lambda+\frac{1}{2};\ \frac{1}{4}z^2\right) = e^{-z}\,{}_1F_1\left(\lambda;\ 2\lambda;\ 2z\right) $$
$$ (2\lambda \neq -1,-3,-5,\dots). $$

REMARK 4. It is fairly obvious that the Bessel differential equation (1.20) remains *unaltered* when z is replaced by $-z$ (and also when ν is replaced by $-\nu$), so the functions $J_{\pm\nu}(-z)$ are solutions of the equation (1.20) satisfied by $J_{\pm\nu}(z)$.

2. APPLICATIONS OF THEOREM 1 TO A FAMILY OF GENERALIZED BESSEL DIFFERENTIAL EQUATIONS

Motivated essentially by the *celebrated* Bessel differential equation (1.20), Lin *et al.* [7] presented a systematic investigation of the following general family of second-order nonhomogeneous linear ordinary differential equations:

$$ Az^2\frac{d^2\varphi}{dz^2} + (Bz+C)\frac{d\varphi}{dz} + \left(Dz^2+Ez+F\right)\varphi(z) = f(z), \tag{2.1} $$

which obviously corresponds to (1.20) when the parameters $A \neq 0$, B, C, $D \neq 0$, E, and F are specialized as follows:

$$ A = B = D = 1, \qquad C = E = 0, \qquad \text{and} \qquad F = -\nu^2. \tag{2.2} $$

With a view to applying Theorem 1 in order to find (*explicit*) particular solutions of the nonhomogeneous non-Fuchsian differential equation (2.1), Lin *et al.* [7] made use of the following transformation:

$$ \varphi(z) = z^{\rho}e^{\lambda z}\phi(z), \tag{2.3} $$

so that

$$ \frac{d\varphi}{dz} = z^{\rho-1}e^{\lambda z}\left[z\frac{d\phi}{dz} + (\rho+\lambda z)\,\phi(z)\right] \tag{2.4} $$

and

$$ \frac{d^2\varphi}{dz^2} = z^{\rho-2}e^{\lambda z}\left[z^2\frac{d^2\phi}{dz^2} + 2(\rho+\lambda z)\,z\frac{d\phi}{dz} + \left\{\lambda^2z^2+2\rho\lambda z+\rho(\rho-1)\right\}\phi(z)\right]. \tag{2.5} $$

Upon substituting from (2.3), (2.4), and (2.5) into the nonhomogeneous non-Fuchsian differential equation (2.1), Lin *et al.* [7] *finally* arrived at the following application of Theorem 1.

THEOREM 2. *(See [7, p. 39, Theorem 3].) If the given function f satisfies the constraint (1.6) and $f_{-\nu} \neq 0$, then the following nonhomogeneous linear ordinary differential equation:*

$$ Az^2\frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2+Ez+F\right)\varphi(z) = f(z) \qquad (A \neq 0;\ D \neq 0) \tag{2.6} $$

has a particular solution in the form

$$ \varphi(z) = z^{\rho}e^{\lambda z}\left(\left(A^{-1}z^{-\nu-1+(2A\rho+B)/A}\cdot e^{2\lambda z}\left(z^{-\rho-1}\cdot e^{-\lambda z}\cdot f(z)\right)_{-\nu}\right)_{-1} \cdot z^{\nu-(2A\rho+B)/A}\cdot e^{-2\lambda z}\right)_{\nu-1} \qquad \left(A \neq 0;\ D \neq 0;\ z \in \mathbb{C}\setminus\{0\}\right), \tag{2.7} $$

where ρ and λ are given by

$$\rho = \frac{A - B \pm \sqrt{(A-B)^2 - 4AF}}{2A} \qquad \text{and} \qquad \lambda = \pm i\sqrt{\frac{D}{A}}, \tag{2.8}$$

and

$$\nu = \frac{(2A\rho + B)\lambda + E}{2A\lambda}, \tag{2.9}$$

it being provided that the second member of (2.7) exists.

Furthermore, the following homogeneous linear ordinary differential equation:

$$Az^2\frac{d^2\varphi}{dz^2} + Bz\frac{d\varphi}{dz} + \left(Dz^2 + Ez + F\right)\varphi(z) = 0 \tag{2.10}$$

has solutions of the form

$$\varphi(z) = Kz^\rho e^{\lambda z}\left(z^{\nu-(2A\rho+B)/A}\cdot e^{-2\lambda z}\right)_{\nu-1} \qquad (A \neq 0;\ D \neq 0;\ z \in \mathbb{C}\setminus\{0\}), \tag{2.11}$$

where K is an arbitrary constant, ρ and λ are given by (2.8), and ν is given by (2.9), it being provided that the second member of (2.11) exists.

REMARK 5. By *first* setting $\nu \longmapsto \nu + 1/2$ and *then* specializing the involved parameters A, B, D, E, and F as in (2.2), Theorem 2 would immediately yield the following special case involving the Bessel differential equation (1.20).

THEOREM 3. *(See [7, p. 40, Corollary 1; 26, p. 27, Theorem 1, p. 29, Theorem 2].) Under the hypotheses of Theorem 2, the following nonhomogeneous linear ordinary differential equation:*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \nu^2\right)\varphi(z) = f(z) \tag{2.12}$$

has a particular solution in the form

$$\varphi(z) = z^\nu e^{\lambda z}\left(\left(z^{\nu-\frac{1}{2}}\cdot e^{2\lambda z}\left(z^{-\nu-1}\cdot e^{-\lambda z}\cdot f(z)\right)_{-\nu-1/2}\right)_{-1}\cdot z^{-\nu-\frac{1}{2}}\cdot e^{-2\lambda z}\right)_{\nu-1/2} \qquad (\nu \in \mathbb{R};\ \lambda = \pm i;\ z \in \mathbb{C}\setminus\{0\}), \tag{2.13}$$

provided that the second member of (2.13) exists.

Furthermore, the following homogeneous linear ordinary differential equation:

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \nu^2\right)\varphi(z) = 0 \tag{2.14}$$

has solutions of the form

$$\varphi(z) = Kz^\nu e^{\lambda z}\left(z^{-\nu-1/2}\cdot e^{-2\lambda z}\right)_{\nu-1/2} \qquad (\nu \in \mathbb{R};\ \lambda = \pm i;\ z \in \mathbb{C}\setminus\{0\}), \tag{2.15}$$

where K is an arbitrary constant, it being provided that the second member of (2.15) exists.

Since $\lambda = \pm i$ in the assertions (2.13) and (2.15) of Theorem 3, by *further* setting

$$\nu = \frac{1}{2} \qquad \text{and} \qquad f(z) = z\sqrt{z}, \tag{2.16}$$

and making use of the *principle of superposition* of solutions of linear differential equations, Lin *et al.* [7] deduced the following interesting consequence of Theorem 3.

COROLLARY 1. *(See [7, p. 40, Corollary 2].) The general solution of the following nonhomogeneous Bessel equation of order $1/2$:*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \frac{1}{4}\right)\varphi(z) = z\sqrt{z} \tag{2.17}$$

is given by

$$\varphi(z) = K_1\frac{\cos z}{\sqrt{z}} + K_2\frac{\sin z}{\sqrt{z}} + \frac{1}{\sqrt{z}} \qquad (z \in \mathbb{C}\setminus(-\infty, 0]), \tag{2.18}$$

where K_1 and K_2 are arbitrary constants.

3. SOLUTIONS OF THE BESSEL DIFFERENTIAL EQUATION (1.20) WHEN $\nu = n + \frac{1}{2}$ $(n \in \mathbb{N}_0)$

By setting

$$\nu = n + \frac{1}{2}, \quad (n \in \mathbb{N}_0) \qquad \text{and} \qquad \lambda = -i \tag{3.1}$$

in assertion (2.15) of Theorem 3, we obtain

$$\varphi^{(1)}(z) = K z^{n+1/2} e^{-iz} \frac{d^n}{dz^n} \left\{ z^{-n-1} \cdot e^{2iz} \right\} \qquad (n \in \mathbb{N}_0), \tag{3.2}$$

where K is an arbitrary constant. If we now apply the familiar case of the Leibniz rule (1.9) for *ordinary* derivatives, we find from (3.2) that

$$\varphi^{(1)}(z) = K z^{n+1/2} e^{-iz} \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dz^k} \left\{ z^{-n-1} \right\} \cdot \frac{d^{n-k}}{dz^{n-k}} \left\{ e^{2iz} \right\}, \tag{3.3}$$

which can readily be simplified to the following form:

$$\varphi^{(1)}(z) = K \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!\,(n-k)!} 2^{n-k} z^{-k-1/2} \cdot \exp\left(i \left[z + \frac{1}{2}(n-k)\pi \right] \right). \tag{3.4}$$

Similarly, by setting

$$\nu = n + \frac{1}{2} \quad (n \in \mathbb{N}_0) \qquad \text{and} \qquad \lambda = i \tag{3.5}$$

in assertion (2.15) of Theorem 3, we get

$$\varphi^{(2)}(z) = K \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!\,(n-k)!} 2^{n-k} z^{-k-1/2} \cdot \exp\left(-i \left[z + \frac{1}{2}(n-k)\pi \right] \right), \tag{3.6}$$

where K is an arbitrary constant.

Since the *homogeneous* Bessel differential equation (1.20) is also *linear*, it obviously admits itself of solutions in the following forms:

$$\begin{aligned} w^{(1)}(z) &:= \varphi^{(1)}(z) + \varphi^{(2)}(z) \\ &= K \sum_{k=0}^{n} (-1)^k \frac{(n+k)!}{k!\,(n-k)!} 2^{n-k+1} z^{-k-1/2} \cdot \cos\left(z + \frac{1}{2}(n-k)\pi \right) \end{aligned} \tag{3.7}$$

and (cf. Remark 4 above)

$$\begin{aligned} w^{(2)}(z) &:= \varphi^{(2)}(-z) - \varphi^{(1)}(-z) \\ &= K \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!} 2^{n-k+1} z^{-k-1/2} \cdot \sin\left(z - \frac{1}{2}(n-k)\pi \right), \end{aligned} \tag{3.8}$$

where we have also used the fact that $i := \sqrt{-1}$.

In view of the following elementary series identity:

$$\sum_{k=0}^{\infty} \Omega(k) = \sum_{k=0}^{\infty} \Omega(2k) + \sum_{k=0}^{\infty} \Omega(2k+1), \tag{3.9}$$

it is not difficult to reduce the solutions (3.7) and (3.8) as follows:

$$\begin{aligned} w^{(1)}(z) = K\frac{2^{n+1}}{\sqrt{z}} \Bigg[& \cos\left(z+\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!\,(n-2k)!}(2z)^{-2k} \\ & - \sin\left(z+\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!\,(n-2k-1)!}(2z)^{-2k-1} \Bigg] \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} w^{(2)}(z) = K\frac{2^{n+1}}{\sqrt{z}} \Bigg[& \sin\left(z-\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!\,(n-2k)!}(2z)^{-2k} \\ & + \cos\left(z-\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!\,(n-2k-1)!}(2z)^{-2k-1} \Bigg]. \end{aligned} \tag{3.11}$$

Now, from the vast available literature on the Bessel functions, we recall that [23, p. 55, Equation 3.4 (5)]

$$\begin{aligned} J_{-n-1/2}(z) = \sqrt{\frac{2}{\pi z}} \Bigg[& \cos\left(z+\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!\,(n-2k)!}(2z)^{-2k} \\ & - \sin\left(z+\frac{1}{2}n\pi\right) \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!\,(n-2k-1)!}(2z)^{-2k-1} \Bigg] \end{aligned} \tag{3.12}$$

and that [23, p. 53, Equation 3.4 (2)]

$$\begin{aligned} J_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}} \Bigg[& \sin\left(z-\frac{1}{2}n\pi\right) \cdot \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!\,(n-2k)!}(2z)^{-2k} \\ & + \cos\left(z-\frac{1}{2}n\pi\right) \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!\,(n-2k-1)!}(2z)^{-2k-1} \Bigg]. \end{aligned} \tag{3.13}$$

For

$$K = \frac{2^{-n}}{\sqrt{2\pi}} \qquad (n \in \mathbb{N}_0), \tag{3.14}$$

the expressions for $w^{(1)}(z)$ and $w^{(2)}(z)$, given by (3.10) and (3.11), coincide precisely with the known results (3.12) and (3.13), respectively. Thus, by means of our fractional-calculus approach, we have shown that the *homogeneous* Bessel differential equation (1.20) of order

$$\nu = n + \frac{1}{2} \qquad (n \in \mathbb{N}_0)$$

has its general solution given by

$$w(z) = K_1 J_{-n-1/2}(z) + K_2 J_{n+1/2}(z), \qquad (n \in \mathbb{N}_0), \tag{3.15}$$

where K_1 and K_2 are arbitrary constants.

It is easily seen from (3.12) and (3.13) *with* $n = 0$ that

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \qquad \text{and} \qquad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \tag{3.16}$$

which incidentally account for the *complementary function* in the general solution (2.18) asserted by Corollary 1. Furthermore, since

$$J_{-3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(-\frac{\cos z}{z} - \sin z\right) \tag{3.17}$$

and

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\frac{\sin z}{z} - \cos z\right), \tag{3.18}$$

which follow readily from (3.12) and (3.13) *with* $n = 1$, by setting

$$\nu = \frac{3}{2} \qquad \text{and} \qquad f(z) = z^2\sqrt{z} \tag{3.19}$$

in Theorem 3, we obtain the following result.

COROLLARY 2. *The general solution of the following nonhomogeneous Bessel equation of order 3/2:*

$$z^2\frac{d^2\varphi}{dz^2} + z\frac{d\varphi}{dz} + \left(z^2 - \frac{9}{4}\right)\varphi(z) = z^2\sqrt{z} \tag{3.20}$$

is given by

$$\varphi(z) = \frac{K_1}{\sqrt{z}}\left(\frac{\cos z}{z} + \sin z\right) + \frac{K_2}{\sqrt{z}}\left(\frac{\sin z}{z} - \cos z\right) + \sqrt{z} + \frac{2}{z\sqrt{z}}, \qquad (z \in \mathbb{C} \setminus (-\infty, 0]), \tag{3.21}$$

where K_1 *and* K_2 *are arbitrary constants.*

REMARK 6. Numerous *further* consequences of Theorem 3, analogous to Corollary 1 *and* Corollary 2 above, can indeed be deduced from the results presented in this section.

4. THE BESSEL DIFFERENTIAL EQUATION (1.20) OF GENERAL ORDER $\nu \notin \mathbb{Z}$

We turn our attention once again toward the assertion (2.15) of Theorem 3. As a matter of fact, in view of the generalized Leibniz rule (1.9), we find from (2.15) that

$$\varphi(z) = Kz^\nu e^{\lambda z}\sum_{k=0}^{\infty}\binom{\nu - \frac{1}{2}}{k}\left(z^{-\nu-1/2}\right)_k \cdot \left(e^{-2\lambda z}\right)_{\nu-k-1/2}, \tag{4.1}$$

where K is an arbitrary constant.

Now, by appealing appropriately to the fractional differintegral formulas (1.10) and (1.12), we can rewrite (4.1) (*with* $\lambda = -i$) in the following form:

$$\Phi^{(1)}(z) = 2^\nu K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1/2)}{k!\Gamma(\nu-k+1/2)}(2z)^{-k-1/2} \cdot \exp\left(i\left[z + \frac{1}{2}\left(\nu - k - \frac{1}{2}\right)\pi\right]\right). \tag{4.2}$$

In a similar manner, if we apply the fractional differintegral formulas (1.11) and (1.12) in (4.1) (*with* $\lambda = i$), we get

$$\Phi^{(2)}(z) = 2^\nu K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1/2)}{k!\Gamma(\nu-k+1/2)}(2z)^{-k-1/2} \cdot \exp\left(-i\left[z + \frac{1}{2}\left(\nu - k - \frac{1}{2}\right)\pi\right]\right). \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned} W^{(1)}(z) &:= \Phi^{(1)}(z) + \Phi^{(2)}(z) \\ &= 2^{\nu+1}K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1/2)}{k!\Gamma(\nu-k+1/2)}(2z)^{-k-1/2} \cdot \cos\left(z + \frac{1}{2}\left(\nu - k - \frac{1}{2}\right)\pi\right), \end{aligned} \tag{4.4}$$

which, in light of the series identity (3.9), yields

$$W^{(1)}(z) = K\frac{2^{\nu+1}}{\sqrt{2z}}\left[\cos\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2z)^{-2k}\right.$$
$$\left.-\sin\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2z)^{-2k-1}\right]. \tag{4.5}$$

Similarly, we have

$$W^{(2)}(z) := \Phi^{(2)}(-z) - \Phi^{(1)}(-z)$$
$$= 2^{\nu+1}K\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+k+1/2)}{k!\Gamma(\nu-k+1/2)}(2z)^{-k-1/2}\cdot\sin\left(z-\frac{1}{2}\left(\nu-k-\frac{1}{2}\right)\pi\right), \tag{4.6}$$

so that, by virtue of the series identity (3.9) once again, we obtain

$$W^{(2)}(z) = K\frac{2^{\nu+1}}{\sqrt{2z}}\left[\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2z)^{-2k}\right.$$
$$\left.-\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2z)^{-2k-1}\right]. \tag{4.7}$$

By comparing (4.5) and (4.7) with the following known results [23, p. 199, Equations 7.21 (1) and 7.21 (3)]:

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\left[\cos\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2z)^{-2k}\right.$$
$$\left.-\sin\left(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2z)^{-2k-1}\right] \tag{4.8}$$

and

$$J_{-\nu}(z) \sim \sqrt{\frac{2}{\pi z}}\left[\cos\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+1/2)}{(2k)!\Gamma(\nu-2k+1/2)}(2z)^{-2k}\right.$$
$$\left.-\sin\left(z+\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right)\cdot\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\nu+2k+3/2)}{(2k+1)!\Gamma(\nu-2k-1/2)}(2z)^{-2k-1}\right], \tag{4.9}$$

each of which is valid for large values of $|z|$ provided that

$$|\arg(z)| \leqq \pi - \varepsilon \qquad (0 < \varepsilon < \pi), \tag{4.10}$$

we can immediately identify the solutions $W^{(1)}(z)$ and $W^{(2)}(z)$ of the Bessel differential equation (1.20) of general order $\nu \notin \mathbb{Z}$ as the Bessel functions $J_{-\nu}(z)$ and $J_\nu(z)$, respectively, the arbitrary constant K being given here by

$$K = \frac{2^{-\nu}}{\sqrt{\pi}}, \qquad (\nu \notin \mathbb{Z}). \tag{4.11}$$

Thus, we have arrived at the following general solution of (1.20):

$$w(z) = K_1 J_{-\nu}(z) + K_2 J_\nu(z) \qquad (\nu \notin \mathbb{Z}) \tag{4.12}$$

at least for large values of $|z|$ under the constraint (4.10).

5. FURTHER REMARKS AND OBSERVATIONS

Upon writing the assertion (2.15) of Theorem 3 in the following form:

$$\varphi(z) = Kz^{\nu}e^{\lambda z}\sum_{k=0}^{\infty}\binom{\nu-\frac{1}{2}}{k}\left(z^{-\nu-1/2}\right)_{\nu-k-1/2}\left(e^{-2\lambda z}\right)_k \tag{5.1}$$

in place of (4.1), if we apply the above fractional-calculus method *mutatis mutandis*, we can derive

$$\phi^{(1)}(z) := \varphi(z)\Big|_{\lambda=-i} = Kz^{-\nu}\cdot\exp\left(i\left[z-\left(\nu-\frac{1}{2}\right)\pi\right]\right)\cdot\sum_{k=0}^{\infty}\frac{\Gamma(2\nu-k)}{\Gamma(\nu-k+1/2)}\frac{(-2iz)^k}{k!} \tag{5.2}$$

and

$$\phi^{(2)}(z) := \varphi(z)\Big|_{\lambda=i} = Kz^{-\nu}\cdot\exp\left(-i\left[z+\left(\nu-\frac{1}{2}\right)\pi\right]\right)\cdot\sum_{k=0}^{\infty}\frac{\Gamma(2\nu-k)}{\Gamma(\nu-k+1/2)}\frac{(2iz)^k}{k!}. \tag{5.3}$$

Now, since

$$\frac{\Gamma(2\nu-k)}{\Gamma(\nu-k+1/2)} = \frac{2^{2\nu-1}}{\sqrt{\pi}}\Gamma(\nu)\cdot\frac{\Gamma(1/2-\nu+k)}{\Gamma(1/2-\nu)}\cdot\frac{\Gamma(1-2\nu)}{\Gamma(1-2\nu+k)} \qquad (k\in\mathbb{N}_0), \tag{5.4}$$

we find from (5.2) and (5.3) that

$$\phi^{(1)}(z) = \frac{2^{2\nu-1}}{\sqrt{\pi}}Kz^{-\nu}\Gamma(\nu)\cdot\exp\left(i\left[z-\left(\nu-\frac{1}{2}\right)\pi\right]\right)\cdot{}_1F_1\left(\frac{1}{2}-\nu;1-2\nu;-2iz\right) \tag{5.5}$$

and

$$\phi^{(2)}(z) = \frac{2^{2\nu-1}}{\sqrt{\pi}}Kz^{-\nu}\Gamma(\nu)\cdot\exp\left(-i\left[z+\left(\nu-\frac{1}{2}\right)\pi\right]\right)\cdot{}_1F_1\left(\frac{1}{2}-\nu;1-2\nu;2iz\right) \tag{5.6}$$

in terms of (Kummer's) confluent hypergeometric ${}_1F_1$ function with one numerator parameter $1/2-\nu$ and one denominator parameter $1-2\nu$. Thus, for appropriate choice of the arbitrary constant K in (5.5) and (5.6), *both* $\phi^{(1)}(z)$ and $\phi^{(2)}(z)$ can easily be identified with the Bessel function $J_{-\nu}(z)$ given by the last expression in (1.21). Consequently, in light of Remark 4 above, our fractional-calculus approach has led us to the following result.

COROLLARY 3. *The above-stated general solution (4.12) of the classical Bessel differential equation (1.20) of order $\nu \notin \mathbb{Z}$ holds true for all admissible values of $z\in\mathbb{C}$ under the constraint (4.10).*

For appropriate choices of the *nonhomogeneous* term $f(z)$ occurring on the right-hand side of (2.12), we can similarly apply the assertion (2.13) of Theorem 3 with a view to deriving the *particular integrals* of the corresponding *nonhomogeneous* Bessel differential equations of the class given by (2.12).

REFERENCES

1. R. Hilfer, Editor, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, (2000).
2. H.M. Ozaktas, Z. Zalevsky and M.A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, Wiley Series in Pure and Applied Optics, John Wiley and Sons, New York, (2000).
3. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Mathematics in Science and Engineering, Volume 198*, Academic Press, New York, (1999).
4. K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons, New York, (1993).

5. S.-T. Tu, D.-K. Chyan and H.M. Srivastava, Some families of ordinary and partial fractional differintegral equations, *Integral Transform. Spec. Funct.* **11**, 291–302, (2001).
6. C.-Y. Lee, H.M. Srivastava and W.-C. Yueh, Explicit solutions of some linear ordinary and partial fractional differintegral equations, *Appl. Math. Comput.* **144**, 11–25, (2003).
7. S.-D. Lin, J.-C. Shyu, K. Nishimoto and H.M. Srivastava, Explicit solutions of some general families of ordinary and partial differential equations associated with the Bessel equation by means of fractional calculus, *J. Fract. Calc.* **25**, 33–45, (2004).
8. S.-D. Lin, S.-T. Tu, I-C. Chen and H.M. Srivastava, Explicit solutions of a certain family of fractional differintegral equations, *Hyperion Sci. J. Ser. A Math. Phys. Electr. Engrg.* **2**, 85–90, (2001).
9. S.-D. Lin, H.M. Srivastava, S.-T. Tu and P.-Y. Wang, Some families of linear ordinary and partial differential equations solvable by means of fractional calculus, *Internat. J. Differential Equations Appl.* **4**, 405–421, (2002).
10. S.-D. Lin, S.-T. Tu and H.M. Srivastava, Explicit solutions of certain ordinary differential equations by means of fractional calculus, *J. Fract. Calc.* **20**, 35–43, (2001).
11. S.-D. Lin, S.-T. Tu and H.M. Srivastava, Certain classes of ordinary and partial differential equations solvable by means of fractional calculus, *Appl. Math. Comput.* **131**, 223–233, (2002).
12. S.-D. Lin, S.-T. Tu and H.M. Srivastava, Explicit solutions of some classes of non-Fuchsian differential equations by means of fractional calculus, *J. Fract. Calc.* **21**, 49–60, (2002).
13. S.-D. Lin, S.-T. Tu and H.M. Srivastava, A unified presentation of certain families of non-Fuchsian differential equations via fractional calculus operators, *Computers Math. Applic.* **45** (12), 1861–1870, (2003).
14. S.-D. Lin, S.-T. Tu, H.M. Srivastava and P.-Y. Wang, Certain operators of fractional calculus and their applications to differential equations, *Computers Math. Applic.* **44** (12), 1557–1565, (2002).
15. S. Salinas de Romero and H.M. Srivastava, An application of the N-fractional calculus operator method to a modified Whittaker equation, *Appl. Math. Comput.* **115**, 11–21, (2000).
16. S.-T. Tu, Y.-T. Huang, I-C. Chen and H.M. Srivastava, A certain family of fractional differintegral equations, *Taiwanese J. Math.* **4**, 417–426, (2000).
17. S.-T. Tu, S.-D. Lin, Y.-T. Huang and H.M. Srivastava, Solutions of a certain class of fractional differintegral equations, *Appl. Math. Lett.* **14** (2), 223–229, (2001).
18. S.-T. Tu, S.-D. Lin and H.M. Srivastava, Solutions of a class of ordinary and partial differential equations via fractional calculus, *J. Fract. Calc.* **18**, 103–110, (2000).
19. K. Nishimoto, *Fractional Calculus, Volumes I, II, III, IV, and V*, Descartes Press, Koriyama, (1984), (1987), (1989), (1991), and (1996).
20. K. Nishimoto, *An Essence of Nishimoto's Fractional Calculus, Calculus of the 21st Century: Integrations and Differentiations of Arbitrary Order*, Descartes Press, Koriyama, (1991).
21. H.M. Srivastava, S. Owa and K. Nishimoto, Some fractional differintegral equations, *J. Math. Anal. Appl.* **106**, 360–366, (1985).
22. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Volume II*, McGraw-Hill Book Company, New York, (1953).
23. G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, Cambridge, (1944).
24. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, Fourth Edition, Cambridge University Press, Cambridge, (1927).
25. E.L. Ince, *Ordinary Differential Equations*, Longmans, Green and Company, London, (1927); Reprinted by Dover Publications, New York, (1956).
26. K. Nishimoto, Applications of N-transformation and N-fractional calculus method to nonhomogeneous Bessel equations (I), *J. Fract. Calc.* **8**, 25–30, (1995).
27. F.G. Tricomi, *Funzioni Ipergeometriche Confluenti*, Edizioni Cremonese, Rome, (1954).