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Hitting Spheres with Brownian Motion and Sommerfeld's Radiation Condition

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By taking the Laplace transform of the scalar wave equation an inhomogeneous problem of the Helmholtz type is obtained. For this problem a probabilistic representation of the solution exists. Using the hitting distribution of a sphere by Brownian motion, we show that this representation satisfies a Sommerfeld radiation condition.

1. INTRODUCTION

Consider the following mixed problem for the scalar wave equation in an unbounded domain $D \subset \mathbb{R}^3$, whose complement is a compact set

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t, x) = c^2(x)\, \Delta u(t, x) + f(t, x), \qquad x \in D,\ t > 0 \\ &u(0, x) = \varphi(x), \qquad \frac{\partial u}{\partial t}(0, x) = \psi(x), \qquad x \in D \\ &u(t, x) = h(t, x) \qquad \text{for} \quad x \in \partial D. \end{aligned} \tag{1.1}$$

We assume that $0 < c_1 < c(x) < c_2 < \infty$ for some constants c_1, c_2, and that $c(x) = c_0$ for $|x| > R_0$. A probabilistic approach to study this and more general problems was developed in [1]. By considering the Laplace transform $\tilde{u}(\alpha, x)$ of $u(t, x)$, problem (1.1) can be transformed into

$$\begin{aligned} \Delta \tilde{u}(\alpha, x) - \alpha^2 q(x)\, \tilde{u}(\alpha, x) &= -g(\alpha, x) \qquad x \in D \\ \tilde{u}(\alpha, x) &= \tilde{h}(\alpha, x) \end{aligned} \tag{1.2}$$

with $g(\alpha, x) = (\tilde{f}(\alpha, x) + \alpha\varphi(x) + \psi(x))\, q(x)$ and $q(x) = 1/c^2(x)$.

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The solution to (1.2) can be represented as a path integral with respect to an auxiliary standard Brownian motion (B_t) on $\mathbb{R}^3$ as follows: If $\alpha > 0$,

$$\tilde{u}(\alpha, x) = E^x[M(T)\,\tilde{h}(\alpha, B(T))] + \tfrac{1}{2} E^x \int_0^T M(t)\, g(\alpha, B(t))\, dt, \quad (1.3)$$

where

$$M(t) = \exp -\frac{\alpha^2}{2} \int_0^t q(B_s)\, ds,$$

$$T = \inf\{t > 0 : B_t \in \partial D\}$$

and (1.3) provides the unique solution to (1.2) when $\alpha > 0$.

If $\alpha = 0$,

$$\tilde{u}(0, x) = E^x[\tilde{h}(0, B(T)); T < \infty] + \tfrac{1}{2} E^x \int_0^T g(0, B_t)\, dt + A \cdot P^x(T = \infty), \quad (1.4)$$

where to ensure uniqueness we have set $\lim_{x \to \infty} \tilde{u}(0, x) = A$. The essential fact here is that $P^x(T < \infty)$ tends to zero as $|x|$ goes to infinity (of course, adequate conditions must be imposed on $\tilde{h}, g$ to get finite integrals). For details about all these facts, the reader is directed to [2].

Besides providing us with a starting point for theoretical analysis, (1.3) or (1.4) can be used for numerical evaluation of $\tilde{u}(\alpha, x)$ by Monte Carlo simulation. But none of this is the issue here. We are interested in enquiring whether (1.3) and (1.4) satisfy a Sommerfeld radiation condition like the one imposed on the Helmholtz equation satisfied by the Fourier transform $v(w, x)$ of (1.1), namely:

$$\left(\frac{\partial v}{\partial r} + i\frac{w}{c_0} v\right) = o\left(\frac{1}{r}\right) \quad \text{as } r \text{ tends to infinity.} \quad (1.5)$$

See [3, 4, 5] for a nonprobabilistic approach to these matters.

We show in Section 3 that the strong Markov property of (B_t), combined with the hitting distribution of a sphere from an exterior point, allows us to obtain

$$\frac{\partial \tilde{u}}{\partial r} + \frac{\alpha}{c_0} \tilde{u} = o\left(\frac{1}{r}\right) \quad (1.6)$$

for $\alpha > 0$ and a corresponding condition for $\alpha = 0$.

In Section 2 we reobtain some results by Wendel [6] for the joint distribution of the hitting time and place of a sphere by Brownian motion, starting from an exterior point.

2. Hitting Spheres from the Outside

Let (B_t) denote the standard three-dimensional Brownian motion and let

$$T=\inf\{t>0 : |B_t|=R\}$$

be the hitting time of the sphere of radius R. We denote by $B(T)$ the position where the Brownian motion hits the sphere.

The joint distribution of $(T, B(T))$, when the starting point of the process is either inside or outside the sphere, was obtained for all dimensions in [6]. At the end of that paper is the description of how the joint distribution may be obtained by using an exponential martingale when the process starts inside the sphere, and it is asserted that no similar approach is known, or hoped for, for the exterior problem. In this section we prove that, by considering the appropriate martingale, the joint distribution of $(T, B(T))$ can indeed be obtained for initial points outside the sphere. We treat the three-dimensional case here and the general case elsewhere. We prove the following result:

Theorem 2.1. *Let $\Theta_t = \angle B_0OB_t$ be the angle between the positions at times 0 and t of the Brownian motion process in $\mathbb{R}^3$. Let $P_n(\xi)$ be the nth-degree Legendre polynomial. Then, for $|x|>R$ and $\lambda>0$,*

$$E^x[e^{-(\lambda^2/2)T}P_n(\cos\Theta_T)]=\left(\frac{R}{|x|}\right)^{1/2}\frac{K_{n+1/2}(\lambda\,|x|)}{K_{n+1/2}(\lambda R)}.$$

($K_n(\xi)$ are the modified spherical Bessel functions, see [7]. The limits $\lambda\to 0$ and $|x|\to R$ follow readily from (2.2).)

Proof. Let x_0 be a point inside the ball $B(0, R)$ and x be an exterior point, i.e., $|x|>R>|x_0|$. Consider the function

$$F(t, x)=e^{-\lambda^2 t/2}\frac{e^{-\lambda|x-x_0|}}{|x-x_0|} \tag{2.3}$$

which satisfies

$$\frac{\partial F}{\partial t}+\frac{1}{2}\Delta F=0 \qquad \text{at} \quad (t, x)\in(0,\infty)\times(\mathbb{R}^3-\{x_0\}) \tag{2.4}$$

and consider the process $F(t, B_t)$. For $|x| > R > |x_0|$ and $T = \inf\{t > 0 : |B_t| = R\}$, it follows from Ito's formula and (2.4) that $F(t, B_t)$ is a martingale on $[0, T]$ with respect to P^x. Actually, since x_0 is polar for B_t, $F(t, B_t)$ is a martingale on $[0, \infty)$ with respect to P^x. Also, since (B_t) is transient, $F(t, B_t) \to 0$ as $t \to \infty$. From the martingale property of $F(t, B_t)$, we obtain

$$E^x\left[e^{-\lambda^2 T/2}\frac{e^{-\lambda|B(T)-x_0|}}{|B(T)-x_0|}\right] = \frac{e^{-\lambda|x-x_0|}}{|x-x_0|}. \tag{2.5}$$

We now use the following expansion, which can be found in [7] (see 10.2.35): for arbitrary complex numbers r, ρ, Θ, λ, and $D = \sqrt{r^2+\rho^2-2r\rho\cos\Theta}$ one has that

$$\begin{aligned}\frac{e^{-\lambda D}}{\lambda D} = \frac{2}{\pi}\sum_0^\infty (2n+1)\left[\sqrt{\frac{\pi}{2}}\,\lambda r\, I_{n+1/2}(\lambda r)\right]\\ \times\left[\sqrt{\frac{\pi}{2}}\,\lambda\rho\, K_{n+1/2}(\lambda\rho)\right]P_n(\cos\Theta).\end{aligned} \tag{2.6}$$

Expanding both sides of equality (2.5) and equating coefficients of $I_{n+1/2}(\lambda\,|x_0|)$, we obtain

$$\begin{aligned}E^x\left[e^{-\lambda^2 T/2}P_n(\cos(\Theta_T))\sqrt{\frac{\pi}{2}}\,\lambda R\, K_{n+1/2}(\lambda R)\right]\\ = \sqrt{\frac{\pi}{2}}\,\lambda\,|x|\,K_{n+1/2}(\lambda\,|x|)\,P_n(\cos\alpha),\end{aligned}$$

where $\alpha = \angle x_0 O x$ and $\Theta_t = \angle B_t O x$.

Since x_0 is arbitrary, we can choose it lying along x, so that $\alpha = 0$, to obtain (2.2), namely

$$E^x[e^{-\lambda^2 T/2}P_n(\cos\Theta_T)] = \left(\frac{R}{|x|}\right)^{1/2}\frac{K_{n+1/2}(\lambda\,|x|)}{K_{n+1/2}(\lambda R)}.$$

3. Sommerfeld's Conditions

As mentioned in the Introduction, when the scalar wave equation (1.1) in the complement D of a compact set in $\mathbb{R}^3$ is Laplace transformed in time, problem (1.2) is obtained. If the boundary ∂D of D is smooth enough (D^c is regular for every $x \in \partial D$) and the boundary data are continuous, then (1.3) provides us with an explicit representation of the unique solution to (1.2), for $\alpha > 0$, continuous on $\bar{D}$. In case $\alpha = 0$, we have the representation (1.4) for the solution. In this section we consider the homogeneous case, $f \equiv 0$, $\varphi = \psi \equiv 0$.

A natural question for wave theorists is whether (1.3) satisfies Sommerfeld's radiation condition. We now see that if, as we assumed, $c(x) = c_0$ for $|x| \geq R_0$, then the answer is yes. In fact, we have

PROPOSITION 3.1. *Let $\alpha > 0$. For the function $\tilde{u}(\alpha, x)$ given by* (1.3), *the following identities hold*

$$\tilde{u}(\alpha, x) = O\left(\frac{1}{r}\right)$$

$$\frac{\partial \tilde{u}}{\partial r} + \left(\frac{\alpha}{c_0}\right) \tilde{u} = o\left(\frac{1}{r}\right)$$

as r tends to infinity.

Proof. Let $R > R_0$ be such that $D^c \subset B(0, R)$ and let $S = \inf\{t > 0 : |B_t| = R\}$. Then, for $|x| > R$, $\{T < \infty\} \subset \{S < \infty\}$ and $T = S + T \circ \Theta_S$ a.s. P^x. From the strong Markov property it follows that (1.3) can be rewritten as

$$\begin{aligned} \tilde{u}(\alpha, x) &= E^x[e^{-(1/2)(\alpha/c_0)^2 S} \tilde{u}(\alpha, B(S)); S < \infty] \\ &= E^x[e^{-(1/2)(\alpha/c_0)^2 S} \tilde{u}(\alpha, B(S))]. \end{aligned} \tag{3.2}$$

The last equality is due to the fact that the exponential vanishes on $\{S = \infty\}$.

Set $v(x) = \tilde{u}(\alpha, x)$ restricted to $\partial B(0, R)$, then $v(x)$ is two times continuously differentiable and can be expanded in terms of spherical harmonics as follows (see [4, Vol. I])

$$v(x) = \sum_0^\infty a_{n,0} P_n(\cos \Theta) + \sum_{h=1}^n a_{nh} e^{i\varphi h} P_{n,h}(\cos \Theta), \tag{3.3}$$

where P_n is the nth-degree Legendre polynomial and $P_{n,h}$ denote the Legendre functions of hth order. The series converges absolutely and uniformly, which allows us to perform the exchanges of sums needed later.

Using (3.2), (3.3), and the axial symmetry of the Brownian motion with respect to rotations around the $\overline{0x}$ axis, we can write $\tilde{u}(\alpha, x)$ as

$$\begin{aligned} \tilde{u}(\alpha, x) &= E^x[e^{-(1/2)(\alpha/c_0)^2 S} v(B(S))] \\ &= \sum_{n=0}^\infty a_{n,0} E^x[e^{-(1/2)(\alpha/c_0)^2 S} P_n(\cos \Theta_S)]. \end{aligned}$$

We now use Theorem 2.1 to obtain

$$\tilde{u}(\alpha, x) = \left(\frac{R}{|x|}\right)^{1/2} \sum_{n=0}^{\infty} a_{n,0} \frac{K_{n+1/2}((\alpha/c_0)\,|x|)}{K_{n+1/2}((\alpha/c_0) R)}. \tag{3.4}$$

The Bessel functions in (3.4) can be expanded as (see [7])

$$K_{n+1/2}(\xi) = \sqrt{\frac{\pi}{2}} \frac{e^{-\xi}}{\xi^{1/2}} \sum_{k=0}^{n} (n+1/2, k) \left(\frac{1}{2\xi}\right)^k, \tag{3.5}$$

where $(n+1/2, k) = (n+k)!/k!\,(n-k)!$.

Setting $b_n = a_{n,0} R^{1/2}/K_{n+1/2}((\alpha/c_0) R)$ and using (3.5), we can write (3.4) as

$$\begin{aligned}
\tilde{u}(\alpha, x) &= \sum_n b_n \frac{K_{n+1/2}((\alpha/c_0)\,|x|)}{|x|^{1/2}} \\
&= \sqrt{\frac{\pi}{2}} \sqrt{\frac{c_0}{2}} \frac{e^{-(\alpha/c_0)\,|x|}}{|x|} \sum_{k=0}^{\infty} \left(\frac{c_0}{2\alpha\,|x|}\right)^k \sum_{n \geqslant k} (n+1/2, k) b_n \\
&= \sqrt{\frac{\pi}{2}} \sqrt{\frac{c_0}{2}} \frac{e^{-(\alpha/c_0)|x|}}{|x|} \sum_{n=0}^{\infty} b_n + \sum_{k=1}^{\infty} d_k \left(\frac{c_0}{2\alpha\,|x|}\right)^k \\
&= \sqrt{\frac{\pi}{2}} \sqrt{\frac{c_0}{2}} \frac{e^{-(\alpha/c_0)|x|}}{|x|} \sum_{n=0}^{\infty} b_n + O\left(\frac{1}{|x|}\right).
\end{aligned}$$

Setting $r = |x|$, we see that $\tilde{u}(\alpha, x) = O(1/r)$. Note that the identities follow from the absolute and uniform convergence of the original series (3.3).

Since for any $p \geqslant 1$, $r^p e^{-(\alpha/c_0)r}$ tends to zero as r tends to infinity, it is easy to see from the last identity that

$$\frac{\partial \tilde{u}}{\partial r}(\alpha, r) + \left(\frac{\alpha}{c_0}\right) \tilde{u} = o\left(\frac{1}{r}\right).$$

To conclude, we examine the case $\alpha = 0$, for which

$$\tilde{u}(0, x) = E^x[\bar{h}(0, B(T));\, T < \infty] + AP^x(T = \infty). \tag{3.6}$$

PROPOSITION 3.7. *The function $\tilde{u}(0, x)$ given by (3.6) satisfies*

$$\tilde{u}(0, x) = O\left(\frac{1}{r}\right)$$

$$\frac{\partial \tilde{u}}{\partial r} = o\left(\frac{1}{r}\right) \qquad \textit{as } r \textit{ tends to infinity.}$$

Proof. Let T, S be as in Proposition 3.1, $|x| > R$. Then, since $\{T < \infty\} \subset \{S < \infty\}$ a.s. P^x, using the strong Markov property we obtain

$$u(0, x) = E^x[v(B_S); S < \infty] + AP^x(S = \infty),$$

where $v(x) = u(x)$ is restricted to $\partial B(0, R)$. But

$$P^x(S = \infty) = 1 - \frac{R}{|x|}$$

and, using (3.3) and Theorem 2.1, we can write

$$\begin{aligned} E^x[v(B_S); S < \infty] &= \sum_n a_n E^x[P_n(\cos \Theta_S); S < \infty] \\ &= \sum_n a_n \lim_{\lambda \to 0} E^x[P_n(\cos \Theta_S) e^{-\lambda S}] \\ &= \sum_n a_n \left(\frac{R}{|x|}\right)^{1/2} \lim_{\lambda \to 0} \frac{K_{n+1/2}(\lambda |x|)}{K_{n+1/2}(\lambda R)}. \end{aligned}$$

Using (3.5) one gets

$$\lim_{\lambda \to 0} \frac{K_{n+1/2}(\lambda |x|)}{K_{n+1/2}(\lambda R)} = \left(\frac{R}{|x|}\right)^{n+1/2}$$

and therefore

$$\tilde{u}(0, x) = \sum_n a_n \left(\frac{R}{|x|}\right)^{n+1} = \frac{R}{|x|} \sum_n a_n \left(\frac{R}{|x|}\right)^n,$$

from which the desired conclusion follows.

Final comment. It is rather easy to verify when any of φ, ψ, or f are not identically zero, that is, when $g(\alpha, x)$ is not identically zero; start from (1.3) and apply the strong Markov property, and arriving at (3.2) is as easy as above. From this point on, a similar argument to that above gives us the same conclusion.

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