Analytic Version of Test Functionals, Fourier Transform, and a Characterization of Measures in White Noise Calculus

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It is shown that the space \( (\mathcal{F}) \) of test white noise functionals has an analytic version \( \mathcal{A}_p \), which is an algebra as well as a topological linear space topologized by the projective limit of a sequence \( \{ \mathcal{A}_p : p \in \mathbb{N} \} \) of Banach spaces with respective norm given by \( \| f \|_{\mathcal{A}_p} = \sup_{z \in \mathbb{C}} \{ |f(z)| \exp[-2^{-1} \|z\|^2] \} \), where \( \mathcal{A}_p \) denotes the complexification of \( \mathcal{A}_p \). Furthermore, it is shown that the \( \mathcal{A}_p \)-topology and the \( (\mathcal{F}) \)-topology are equivalent. In the course of the proof, it is also shown that the space \( (\mathcal{F}) \) is isometrically isomorphic to the Bargmann-Segal analytic functions on \( \mathcal{S}\mathcal{P}_p \) under \( S \)-transform. Employing this new version, we are able to define the Fourier transform of a generalized white noise functional as the adjoint of Fourier-Wiener transform \( \mathcal{F}_L \), with parameter \((1, -i)\), and, moreover, we show that every measure in \( (\mathcal{F})^* \) always satisfies a certain "growth condition."

1. INTRODUCTION

The generalized function theory on infinite dimensional spaces in terms of white noise calculus has been initiated by T. Hida in his celebrated paper \([6]\). There the generalized functions, regarded as functions of white noise \( \{ \hat{B}(t), t \in \mathbb{R} \} \), are defined and studied through their \( \mathcal{F} \)-transforms. Later, Kubo and Takenaka \([12]\) reformulated Hida's theory by using \( S \)-transform as machinery in place of \( \mathcal{F} \)-transform in their investigation. If the underlying infinite dimensional space is taken to be the tempered distribution \( \mathcal{F}^* \), the dual of the Schwartz space \( \mathcal{S} \) on \( \mathbb{R}^1 \), the space \( (\mathcal{F}) \), to be defined later, has become an increasingly important one among those existent spaces of test white noise functionals in the recent development of white noise calculus and its application (see \([6, 7, 9, 12-14, 18, 20, 21, 23]\).

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We describe the construction of $(\mathcal{S})$ as follows:

Let $A$ denote the operator $A = 1 + t^2 - (d/dt)^2$ with domain $\mathcal{D}(A) \subset L^2 = L^2(\mathbb{R}^1)$ and $\{e_n, n \in \mathbb{N}_0\}$ the CONS of $L^2$ consisting of eigenfunctions of $A$ with corresponding eigenvalues $\{2n + 2, n \in \mathbb{N}_0\}$. $e_n$'s are known as Hermite functions.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}^*(\mathbb{R}^n)$ be respectively the Schwartz space and tempered distributions on $\mathbb{R}^n$. For $p \in \mathbb{Z}$, $n \in \mathbb{N}$, let $\mathcal{S}_p(\mathbb{R}^n)$ denote the space of functions $f$ in $\mathcal{S}^*(\mathbb{R}^n)$ satisfying the condition

$$\|f\|_p^2 := \sum_{i_1, \ldots, i_n = 0}^{n} \left( \prod_{k=1}^{n} (2i_k + 2)^{2p} \right) \left( \left( f, \bigotimes_{k=1}^{n} e_{i_k} \right) \right)^2 < \infty,$$

where $(\cdot, \cdot)$ denotes the $\mathcal{S}^*(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ pairing. Then $\mathcal{S}_p(\mathbb{R}^n)$ forms a real separable Hilbert space with inner product given by

$$\langle f, g \rangle_p = \sum_{i_1, \ldots, i_n = 0}^{n} \left( \prod_{k=1}^{n} (2i_k + 2)^{2p} \right) \left( f, \bigotimes_{k=1}^{n} e_{i_k} \right) \left( g, \bigotimes_{k=1}^{n} e_{i_k} \right).$$

The dual of $\mathcal{S}_p(\mathbb{R}^n)$ is given by $\mathcal{S}_{-p}(\mathbb{R}^n)$. Also we have the relation

$$\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}_p(\mathbb{R}^n) \subset \mathcal{S}_p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}_{-p}(\mathbb{R}^n) \subset \mathcal{S}^*(\mathbb{R}^n),$$

where $q > p \geq 0$ and $\mathcal{S}(\mathbb{R}^n) = \bigcap_{p \geq 0} \mathcal{S}_p(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ is provided with the projective limit topology. For simplicity, we denote $\mathcal{S}_p = \mathcal{S}_p(\mathbb{R}^1)$.

Now we proceed to define $(\mathcal{S})$. Let $\mu$ be the centered white noise Gaussian measure (with variance parameter 1) on $\mathcal{S}^*$. Identifying $(L^2)$ with $L^2(\mathcal{S}^*, \mu)$, the square integrable Borel functions on $\mathcal{S}^*$ with respect to $\mu$, and applying the Wiener–Itô decomposition theorem, one can write a function $f$ in $(L^2)$ in a unique manner as the orthogonal direct sum of multiple Wiener integrals $I_n(f_n)$ of order $n$ with kernel functions $f_n \in L^2(\mathbb{R})$ ($n \geq 0$), where $L^2(\mathbb{R}^0) = \mathbb{R}^1$ and $L^2(\mathbb{R}^n)$ denotes the symmetric $L^2$-functions of $n$ variables. Moreover, we have

$$\|f\|_{L^2}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\mathbb{R}^n)}^2$$

and $f(x) = \sum_{n=0}^{\infty} I_n(f_n)(x)$ for almost all $x$ in $\mathcal{S}^*$ with respect to $\mu$.

Denote

$$\|f\|_{\mathcal{S}_p}^2 := \sum_{n=0}^{\infty} n! \|f_n\|_{\mathcal{S}_p(\mathbb{R}^n)}^2.$$
For $p \geq 0$, define $(\mathcal{S}_p)$ as the set of functions $f = \sum_{n=0}^{\infty} I_n(f_n)$ such that the number $\|f\|_{2,\sigma}^2$ is finite. For $p < 0$, we define $(\mathcal{S}_p)$ as the completion of $(L^2)$ with respect to the norm $\|\cdot\|_{2,p}$. Then the dual of $(\mathcal{S}_p)$ is given by $(\mathcal{S}_{-p})$, and we have

$$(\mathcal{S}_q) \subset (\mathcal{S}_p) \subset (L^2) \subset (\mathcal{S}_{-p}) \subset (\mathcal{S}_{-q})$$

for $q > p$. Set $(\mathcal{S}) = \bigcap_{p \geq 0} (\mathcal{S}_p)$ and endow $(\mathcal{S})$ with the projective limit topology. This completes the construction of $(\mathcal{S})$. Moreover, it can be shown that

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$$

is a Gelfand triplet [4].

Yet there is another way to study the generalized functions on infinite dimensional space. Being motivated by finite dimensional generalized functions theory in the sense of L. Schwartz and using a class of analytic functions as test functions, we extended the generalized function theory to infinite dimensions in a natural way that defines and studies the generalized functions by means of their linear functional representations (see [18]). This also provides a suitable framework for white noise calculus. For example, the generalized white noise functional $\hat{B}(t_1) \cdots \hat{B}(t_n)$ can be defined by the linear functional $\varphi \rightarrow D^\varphi \mu(0) \delta_{t_1} \cdots \delta_{t_n}$, where $\varphi$ is an arbitrary test function, $D^\varphi \mu(x) = \int \varphi(x+y) \delta(0) \mu(dy)$, $D$ denotes the Fréchet derivative, and $\delta$ denotes the Dirac measure concentrated at $t$. From this example one can also see that the smoothness of test functions plays an important role in this approach. In order to study the connection between the above two approaches, it is desirable to establish the latter framework on $(\mathcal{S})$. Thus it is natural to ask if $(\mathcal{S})$ has a smooth property.

In Section 2, we answer the above question positively by showing that $(\mathcal{S})$ has an analytic version. It is also shown that, for any integer $p$, the S-transform happens to be an isometry from $(\mathcal{S}_p)$ onto the space $K[\mathcal{S}_{-p}, n_{1/2}]$ of Bargmann–Segal analytic functions [1, 22, 23] (see also Dwyer [2] and Krée [10, 11]), where $n$, denotes the Gauss cylinder set measure on $\mathcal{S}_{-p}$ with variance parameter $t > 0$. In Section 3, the space $\mathcal{A}_\mathcal{S}$ of the analytic version of $(\mathcal{S})$ is studied. It is shown that $\mathcal{A}_\mathcal{S}$ is an algebra as well as a topological linear space which is topologized as the Banach spaces $\mathcal{S}_p$ of analytic functions on $\mathcal{S}_{-p}$ with norm given by

$$\|f\|_{\mathcal{A}_p} = \sup_{z \in \mathcal{S}_{-p}} \{ \|f(z)| e^{-1/2} \|_{2,-p} \},$$

where $\mathcal{S}_{-p}$ denotes the complexification of $\mathcal{S}_{-p}$. Moreover, it is shown that the three family of norms $\{\|\cdot\|_{2,p}\}, \{\|\mathcal{S}(\cdot)\|_{\mathcal{A}_p}\}$, and $\{\|\cdot\|_{\mathcal{A}_p}\}$ are equivalent on $\mathcal{S}_{-p}$. In Section 4, the Fourier transform defined in [18] is shown to be also well-defined on $\mathcal{S}^*$. Finally, in Section 5, it is shown that every measure in $(\mathcal{S}^*)$ always satisfies certain "growth condition" which is stronger than the moment
condition. For the sake of clarity, we include in an appendix at the end of this paper a brief discussion about the analytic function on the dual space of a countably normed linear space.

We list below some notation which shall be used frequently in this paper.

\[ \mathbb{N} \quad \text{positive integers.} \]
\[ \mathbb{N}_0 \quad \text{nonnegative integers.} \]
\[ \mathbb{I} \quad \text{integers.} \]
\[ \mathcal{L}^n[X] \quad \text{the space of all } n\text{-linear operators on a normed linear space } X. \]
\[ \mathcal{L}^n_2[H] \quad \text{the space of all } n\text{-linear Hilbert–Schmidt operators on a Hilbert space } H. \]
\[ \mathcal{C}X \quad \text{the complexification of a real normed linear space } X, \text{ with the Euclidean norm induced by the } X\text{-norm, i.e., } \| z \|_X = \sqrt{\| x \|^2_X + \| y \|^2_X} \text{ for } z = x + iy \text{ (} x, y \in X). \]
\[ T_{x^n} := T(x, \ldots, x)(n \text{ copies}), \text{ where } T \in \mathcal{L}^n[X] \text{ and } x \in X. \]
\[ T_{x_1 \cdots x_n} := T(x_1, \ldots, x_n) \text{ for } x_1, \ldots, x_n \in X \text{ and } T \in \mathcal{L}^n[X]. \]
\[ Df(a) \quad \text{For notational convenience, this notation denotes the Fréchet derivative at } a \text{ in the direction of any subspace of } \mathcal{F}^* \text{ if it exists.} \]
\[ (\cdot, \cdot) \quad \mathcal{F}^* - \mathcal{F} \text{ or } \mathcal{L}_p - \mathcal{L}_p \text{ paring} \]
\[ \langle \cdot, \cdot \rangle_p \quad \text{inner product in } \mathcal{F}_p. \]
\[ \bar{\zeta}(x) := (x, \zeta) \text{ for } \zeta \in \mathcal{F}_p(\mathcal{F}) \text{ and } x \in \mathcal{L}_p(\mathcal{F}^*). \]
\[ \langle \cdot, \cdot \rangle_{\alpha} \quad \mathcal{F}^* - \mathcal{F}_\alpha \text{ or } \mathcal{A}_\alpha^* - \mathcal{A}_\alpha \text{ pairing.} \]
\[ SF \quad \text{S-transform of } F. \text{ If } F \in (L^2), \text{ } SF(\zeta) = \mu F(\zeta) = \int_{\mathcal{F}} F(\xi + y) \mu(dy); \text{ if } F \in (\mathcal{L}_p)(p \in \mathbb{N}), \text{ } SF(\zeta) := \exp(-\frac{1}{2} \| \zeta \|^2_0) \langle F, \exp(\bar{\zeta}) \rangle, \text{ for } \zeta \in \mathcal{F}_p. \]

2. Properties of \((\mathcal{F}_p)\)

For \(p \geq 1\), the pair \((L^2, \mathcal{L}_p)\) forms an abstract Wiener space, with the Gaussian measure \(\mu\) serving as the Wiener measure. Let \(f \in L^\alpha(\mathcal{F}^*, \mu)\) \((\alpha > 1)\). Then \(f\), regarded as a function defined on \(\mathcal{L}_p\), is also in \(L^\alpha(\mathcal{L}_p, \mu)\). It follows from [17, Proposition 3.1, 3.2] that \(\mu f\) is infinitely \(L^2\)-differentiable at the origin (i.e., the function \(g(h) = \mu f(h)\), regarded as a function on \(L^2\), is infinitely Fréchet differentiable [5]), and its \(n\)th \(L^2\)-derivative at \(0\), \(D^n\mu f(0) : L^2 \times \cdots \times L^2 \to \mathbb{R}^1\), is a symmetric \(n\)-linear operator of Hilbert–Schmidt type so that \(A^nD^n\mu f(0)(x) = \)
$\int_{\mathcal{S}_-} D^n \mu f(0)(x+iy)^n \mu(dy)$ exists almost everywhere with respect to $\mu$. Here $A^n D^n \mu f(0)$ is defined as the $L^2(\mathcal{S}_-, \mu)$-limit of $\{T_k(x)\} = \int_{\mathcal{S}_-} T_k(x+iy)^n \mu(dy)$, where $\{T_k\}$ is a sequence of symmetric $n$-linear finite rank operators on $\mathcal{S}_-$ such that $T_k \rightharpoonup D^n \mu f(0)$ in $L^2_\mu$. According to [19, Corollary 2.3], a function $f \in L^2(\mathcal{S}_-, \mu)$ can be written as the orthogonal direct sum of $(1/n!) A^n D^n \mu f(0)$. Observe that, for $f \in L^2(\mathcal{S}_-, \mu)$, $A^n D^n \mu f(0)$ is also defined and independent of $p \geq 1$. From this observation and employing the argument of [19], we obtain the following.

2.1. LEMMA. For $f \in (L^2) = L^2(\mathcal{S}_-, \mu)$, the operator $Q_n: f \rightarrow (1/n!) A^n D^n \mu f(0)$ is an orthogonal projection from $(L^2)$ onto the homogeneous chaos of order $n$. Moreover, $f = \sum_{n=0}^{\infty} (1/n!) A^n D^n \mu f(0)$, where the sum is convergent in $(L^2)$.

By the uniqueness of Wiener–Itô decomposition of $(L^2)$ and applying Lemma 2.1, we immediately have

2.2. LEMMA. Suppose $\{f_n\}$ are the kernel functions that appear in the Wiener–Itô decomposition of $f \in (L^2)$. Then $I_n(f_n) = (1/n!) A^n D^n \mu f(0)$.

2.3. Remark. Lemma 2.2 implies that the kernel function $f_n$ is nothing but the kernel function of the Hilbert–Schmidt operator $(1/n!) D^n \mu f(0)$. One may regard $f_n(t_1, \ldots, t_n)$ as $(1/n!) D^n \mu f(0) \delta_{t_1} \cdots \delta_{t_n}$. 

Now suppose $\{f_n\}$ are the kernel functions associated with $f$ as given in Lemma 2.2. It follows from Remark 2.3 and the definition of $\mathcal{S}_p(\mathbb{R}^n)$ that we have

$$\|f_n\|_{\mathcal{S}_p(\mathbb{R}^n)} = \frac{1}{n!} \|D^n \mu f(0)\|_{\mathcal{S}^2_1[\mathcal{S}_-]}.$$

As a result, we have, for $p \in \mathbb{N}$,

$$\|f\|_{L^2,\mu} = \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|_{\mathcal{S}^2_1[\mathcal{S}_-]}.$$  \hspace{1cm} (1)

This proves the following.

2.4. THEOREM. Let $p \in \mathbb{N}$. Then $f \in (\mathcal{S}_p)$ if and only if $f \in (L^2)$ and

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|_{\mathcal{S}^2_1[\mathcal{S}_-]}$$

is finite.
2.5. COROLLARY. Let \( p \in \mathbb{N} \). Then \( f \in (\mathcal{S}_p) \) if and only if \( f \in (\mathcal{S})^* \) and the number

\[
K_{f,p} = \sum_{n=0}^{\infty} \frac{1}{n!} \| D^n Sf(0) \|_{\mathcal{S}_p}^2
\]

is finite. Moreover \( \| f \|_{2,-p} = \sqrt{K_{f,p}} \) for \( f \in (\mathcal{S}_p) \).

Proof. Suppose \( f \in (\mathcal{S}_p) \). Then there exists a function \( \psi \in (\mathcal{S}_p) \) such that \( \langle f, \phi \rangle_{2,p} = \langle \psi, \phi \rangle_{2,p} \) and \( \| f \|_{2,-p} = \| \psi \|_{2,p} \), where \( \langle \cdot, \cdot \rangle_{2,p} \) denotes the inner product of \( (\mathcal{S}_p) \). Note that \( Sf(\xi) \) is infinitely Fréchet differentiable in \( \mathcal{S}_p \) and

\[
D^n Sf(0) \xi_1 \cdots \xi_n = D^n \mu \psi_f(0)(A^{2p} \xi_1) \cdots (A^{2p} \xi_n)
\]

for \( \xi_1, \ldots, \xi_n \in \mathcal{S} \), where \( A \) is the operator as given in Section 1. Applying Theorem 2.4 and the identity (2), we obtain

\[
K_{f,p} = \sum_{n=0}^{\infty} \frac{1}{n!} \| D^n \mu \psi_f(0) \|_{\mathcal{S}_p}^2 \leq \| \psi_f \|_{2,p}^2 < \infty.
\]

Thus we have, for \( f \in (\mathcal{S}_p) \),

\[
\| f \|_{2,-p} = \sum_{n=0}^{\infty} \frac{1}{n!} \| D^n Sf(0) \|_{\mathcal{S}_p}^2.
\]

Conversely, if \( f \in (\mathcal{S})^* \) and \( K_{f,p} < \infty \), we define

\[
\langle F, \phi \rangle_{2,p} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \sum_{i_1, \ldots, i_n=0} \left( D^n Sf(0) e_{i_1} \cdots e_{i_n} \right)(D^n \mu \phi(0) e_{i_1} \cdots e_{i_n}) \right\},
\]

for \( \phi \in (\mathcal{S}_p) \). Then \( F \in (\mathcal{S}_p)^* = (\mathcal{S}_p) \). Note that \( SF(\xi) = Sf(\xi) \) for all \( \xi \in \mathcal{S} \).

Hence \( F = f \) and \( f \in (\mathcal{S}_p) \).

Observe that if \( H \) is a Hilbert space, real or complex, then for all \( T \in \mathcal{L}_2[H] \), \( \| T \|_{\mathcal{S}_2[H]} \leq \| T \|_{\mathcal{S}_2[H]} \) so that \( \mathcal{L}_2[H] \subset \mathcal{S}[H] \). Using this fact and applying Theorem 2.4, we obtain the following.

2.6. PROPOSITION. Let \( f \in (\mathcal{S}_p) \), \( p \in \mathbb{N} \). We have the following,

(a) The Wiener–Ito decomposition of \( f \) is defined everywhere in \( \mathcal{C}\mathcal{S}_p \) and converges absolutely and uniformly on bounded subsets of \( \mathcal{C}\mathcal{S}_p \).

(b) Let \( V_p f(z) = \sum_{n=0}^{\infty} (1/n!) A^n D^n \mu f(0)(z) \) for \( z \in \mathcal{C}\mathcal{S}_p \). Then, for \( 0 < \varepsilon < 1 \),

\[
| V_p f(z) | \leq C_{p,\varepsilon} \| f \|_{2,-p} \exp \left[ \frac{1}{2} \left( 1 + \frac{1}{\varepsilon} \right) \| z \|_{-p}^2 \right]
\]

for all \( z \in \mathcal{C}\mathcal{S}_p \), where \( C_{p,\varepsilon} \) is a constant.
Proof. To prove (a), it suffices to verify that the series \( \sum_{n=0}^{\infty} \frac{1}{n!} |A^nD^n\mu f(0)(z)| \) converges uniformly for \( \| z \|_{-p} \leq c \), where \( c \) is an arbitrary positive real number.

Recall that \( \| \cdot \|_{\mathcal{F}_p} \leq \| \cdot \|_{\mathcal{F}_p} \). If \( f \in (\mathcal{F}_p) \) then \( D^n\mu f(0) \in \mathcal{L}^n[\mathcal{F}_p] \) so that \( A^nD^n\mu f(0)(z) = \int_{\mathcal{F}_p} D^n\mu f(0)(z+iy)^n \mu(dy) \) is defined for all \( z \in \mathcal{C}\mathcal{F}_p \), where \( D^n\mu f(0) \) is extended to an \( n \)-linear operator on \( \mathcal{C}\mathcal{F}_p \) in the natural way. It follows that

\[
\sum_{n=0}^{\infty} \frac{1}{n!} |A^nD^n\mu f(0)(z)|
\leq \sum_{n=0}^{\infty} \frac{1}{n!} \| D^n\mu f(0) \|_{\mathcal{L}^n[\mathcal{F}_p]}
\left\{ \int_{\mathcal{F}_p} (\| z \|_{-p} + \| y \|_{-p})^n \mu(dy) \right\}
\leq \sum_{n=0}^{\infty} \left\{ \int_{\mathcal{F}_p} \frac{1}{n!} (\| z \|_{-p} + \| y \|_{-p})^n \mu(dy) \right\}
\leq \| f \|_{2,p} \left\{ \int_{\mathcal{F}_p} \frac{1}{n!} (\| z \|_{-p} + \| y \|_{-p})^{2n} \mu(dy) \right\}^{1/2}
\leq \| f \|_{2,p} \exp \left[ \frac{1}{2} \left( \frac{1+\varepsilon}{\varepsilon} \right) \| z \|_{-p}^2 \right]
\left\{ \int_{\mathcal{F}_p} \exp \left[ (1+\varepsilon) \| y \|_{-p}^2 \right] \mu(dy) \right\}^{1/2}.
\]

(4)

The above estimation (i.e., (4)) clearly implies that the series \( \sum_{n=0}^{\infty} (1/n!) |A^nD^n\mu f(0)(z)| \) converges uniformly for \( \| z \|_{-p} \leq c \). Furthermore, we have

\[
\sup_{\| z \|_{-p} \leq c} \left| \sum_{n=0}^{\infty} \frac{1}{n!} A^nD^n\mu f(0)(z) \right| \leq C_{p,\varepsilon} e^{1/2(1+\varepsilon)\varepsilon^2} \| f \|_{2,p},
\]

where \( C_{p,\varepsilon} = \left\{ \int_{\mathcal{F}_p} \exp \left[ (1+\varepsilon)\| y \|_{-p}^2 \right] \mu(dy) \right\}^{1/2} \). This proves (a). (b) follows also from the estimation (4).

2.7. Corollary. Let \( f \in (\mathcal{S}_p) \) and \( V_p f \) be the same function as given in Proposition 2.6. Then \( V_p f \) is analytic on \( \mathcal{C}\mathcal{S}_p \). Moreover, if \( f \in (\mathcal{S}) \) there exists a unique continuous function \( \tilde{f} \), defined on \( \mathcal{C}\mathcal{S}_p \) and independent of \( p \), such that \( \tilde{f} = V_p f \) on \( \mathcal{C}\mathcal{S}_p \) (cf. [26]).

Proof. For each \( p \in \mathbb{N} \), since \( D^n\mu f(0) \in \mathcal{L}^n(\mathcal{F}_p) \), \( A^nD^n\mu f(0)(z) \) is analytic in \( \mathcal{C}\mathcal{S}_p \). Thus \( V_p f(z) \), as a limit of a sequence of analytic functions which converges uniformly on bounded subsets of \( \mathcal{C}\mathcal{S}_p \), must be analytic in \( \mathcal{C}\mathcal{S}_p \); hence, it is continuous there. Define the function \( \tilde{f} \) on
$\mathcal{S}^*$ such that $\bar{f} = V_p f$ on $\mathcal{S}_{-p}$. Since $\{\mathcal{S}_{-p}\}$ is monotone and $\bigcup_{p \in \mathbb{N}} \mathcal{S}_{-p} = \mathcal{S}^*$, $\bar{f}$ is well-defined. It follows immediately from the nuclearity of $\mathcal{S}$ and the continuity of $V_p f$ that $\bar{f}$ is continuous. The uniqueness of $\bar{f}$ follows from the fact that $\mathcal{S}^*_{-p}$ are dense in $\mathcal{S}^*$.  

As a consequence of the above results, we have the following.

2.8. Theorem. Let $f \in (\mathcal{S})$ and $\bar{f}$ be the continuous version of $f$ defined in Corollary 2.7.

(a) $\bar{f}$ is analytic in $\mathcal{S}^*$. (For the definition of an analytic function on $\mathcal{S}^*$ and its basic property we refer the reader to the Appendix given at the end of the present paper.)

(b) For $p \in \mathbb{N}$,

$$\sup_{z \in \mathcal{S}^*_{-(p-1)}} \{ |\bar{f}(z)| e^{(-1/2\|z\|_2^2)} \} \leq C_p \| f \|_{2,p},$$

where $C_p$ is a constant.

Proof. Recall that $(\mathcal{S}) = \bigcap_{p \in \mathbb{N}} (\mathcal{S}_p)$. If $f \in (\mathcal{S})$, then $f \in (\mathcal{S}_p)$ for all $p \in \mathbb{N}$. It follows from Corollary 2.7 that $\bar{f}$, as a continuous version of the analytic function $V_p f$, is analytic in $\mathcal{S}^*_{-p}$ for each $p$. Apply Theorem A.2 from the Appendix: we conclude that $\bar{f}$ is analytic in $\mathcal{S}^*$. This proves (a). (b) follows clearly from Proposition 2.6(b) by choosing $\varepsilon = \frac{1}{2}$.  

To conclude this section, we shall show that $(\mathcal{S}_p)$ may be characterized by the space of Bargman–Segal analytic functions on $\mathcal{S}^*_{-p}$, $p \in \mathbb{Z}$.

Let $H$ be a real separable Hilbert space with norm $|.| = \sqrt{\langle \cdot, \cdot \rangle}$ and $n_t$ the Gauss cylinder set measure with variance parameter $t > 0$.

2.9. Definition (Bargmann–Segal analytic function). A single-valued function $f$ defined on $\mathcal{S} H$ is called a Bargmann–Segal analytic (or entire) function if it satisfies the following conditions:

(i) $f$ is analytic in $\mathcal{S} H$ (see Appendix);

(ii) $\sup_p \int_H \int_H |f(Px + iPy)|^2 n_t(dx) n_t(dy) < \infty$,

where the supremum is taken over the set of all finite rank orthogonal projections on $H$.

Denote the class of Bargman–Segal analytic functions defined above by $K[H, n_t]$. Then $K[H, n_t]$ is a separable Hilbert space with inner product given by

$$\{ f, g \}_{H, t} = \lim_{n \to \infty} \int_H \int_H f(P_n x + iP_n y) \overline{g(P_n x + iP_n y)} n_t(dx) n_t(dy).$$
The Definition 2.9 is essentially due to Segal [22, 23]. When $H$ is of finite dimension, $K[H, \mu_r]$ reduces to the Bargmann analytic functions introduced in [1].

We shall show that $K[H, n_r]$ is a Foch-type space given as follows.

2.10. **Definition** [2]. For $\lambda > 0$, denote by $F^\lambda(H)$ the class of analytic functions on $\mathcal{C}H$ such that

$$
\| f \|_{F^\lambda(H)} = \left\{ \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \| D^k f(0) \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/2} < \infty.
$$

Suppose $f \in K[H, n_r]$ and $P$ is a finite rank orthogonal projection on $H$. Then it follows immediately from [23, Corollary 3.2] that we have

$$
\int_H \int_H |f(Px + iPy)|^2 n_r(dx) n_r(dy)
$$

$$
= \sum_{k=0}^{\infty} \left( \frac{2t}{k!} \right)^k \left( \sum_{i_1, \ldots, i_k = 1}^{N} |D^k f(0) e_{i_1} \cdots e_{i_k}|^2 \right),
$$

where \( \{e_1, \ldots, e_N\} \) is an ONB in $P(H)$.

As an immediate consequence of (6), we have the following.

2.11. **Theorem.** $K[H, n_r] = F^{2t}(H)$ and $\| f \|_{F^{2t}(H)}^2 = \{f, f\}_{H, l}$ for $f \in K[H, n_r]$.

2.12. **Corollary.** The $S$-transform is an isomorphism from $(\mathcal{S}_p)$ onto $F^1(\mathcal{S}_{-p})$ (cf. [11]).

**Proof.** Applying Theorem 2.4, Corollary 2.5, and Theorem 2.11, one sees that $S$-transform is an isometry from $(\mathcal{S}_p)$ into $F^1(\mathcal{S}_{-p})$. Next suppose that $f$ is an arbitrary element in $F^1(\mathcal{S}_{-p})$; we define

$$
F(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, \ldots, i_n = 0}^{\infty} (D^\varphi f(0) e_{i_1} \cdots e_{i_n})(D^\varphi \mu f(0) e_{i_1} \cdots e_{i_n})
$$

for $\varphi \in (\mathcal{S}_p)$, where \( \{e_i\} \) is the ONB of $L^2$ as given in Section 1. Then $|F(\varphi)| \leq \| f \|_{F^1(\mathcal{S}_{-p})} \| \varphi \|_{2, -p}$ so that $F \in (\mathcal{S}_p)$. Clearly $SF(\xi) = f(\xi)$ for all $\xi \in \mathcal{S}_p$. Thus $S$ is onto.

2.13. **Corollary.** For $f \in (\mathcal{S}_p)$, we have

$$
\sup_{z \in \mathcal{S}_{-p}} \{ |SF(z)| e^{-1/2} \| z \|_{L^2} \} \leq \| f \|_{2, -p} = \| SF \|_{F^1(\mathcal{S}_{-p})}.
$$
Proof. Write $Sf(z) = \sum_{n=0}^{\infty} (1/n!) D^n Sf(0) z^n$, $z \in \mathfrak{C}_{-p}$. Then we have

$$|Sf(z)| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n Sf(0)\|_{\mathfrak{L}^p[0,1]} \leq \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \|D^n Sf(0)\|_{\mathfrak{L}^p[0,1]} \cdot \frac{1}{\sqrt{n!}} \|z\|^{-p} \leq \|f\|_{2,p} \exp \left( \frac{1}{2} \|z\|^{-p} \right),$$

for all $z \in \mathfrak{C}_{-p}$. Consequently, we obtain

$$\sup_{z \in \mathfrak{C}_{-p}} \{ |Sf(z)| e^{-1/2}\|z\|^{2-p} \} \leq \|f\|_{2,p} = \|Sf\|_{F_1[0,1]}.$$

3. $\mathcal{A}_{\infty}$—The Space of Analytic Versions of $(\mathfrak{C})$

For $p \in \mathbb{Z}$, denote by $\mathcal{A}_p$ the space of functions $f$ defined on $\mathfrak{C}_{-p}$ satisfying the following conditions:

(i) $f$ is analytic on $\mathfrak{C}_{-p}$.

(ii) There exists a constant $C_f$, depending only on $f$, such that

$$|f(z)| \leq C_f \exp \left( \frac{1}{2} \|z\|^{2-p} \right)$$

for all $z \in \mathfrak{C}_{-p}$.

For $f \in \mathcal{A}_p$, $p \in \mathbb{Z}$ define

$$\|f\|_{\mathcal{A}_p} = \sup_{z \in \mathfrak{C}_{-p}} \{ |f(z)| \exp \left( -\frac{1}{2} \|z\|^{2-p} \right) \}.$$

Then we have the following.

3.1. Proposition. Let $p \in \mathbb{Z}$.

(a) $\|\cdot\|_{\mathcal{A}_p}$ is a norm.

(b) Convergence with respect to the norm $\|\cdot\|_{\mathcal{A}_p}$ implies uniform convergence on bounded subsets of $\mathfrak{C}_{-p}$.

(c) $(\mathcal{A}_p, \|\cdot\|_{\mathcal{A}_p})$ is a Banach space.

(d) If $p \geq q$, then $\|\cdot\|_{\mathcal{A}_q} \leq \|\cdot\|_{\mathcal{A}_p}$ so that $\mathcal{A}_p \subset \mathcal{A}_q$. Moreover, the embedding $i_{pq}: \mathcal{A}_p \subset \mathcal{A}_q$ is continuous. (See also [27].)

Proof. Clear.
3.2. Proposition. (a) Let \( p \in \mathbb{Z} \) and \( f \in \mathcal{A}_p \). Then, for \( z, h_1, \ldots, h_n \in \mathcal{S}_{-p} \) and for positive real numbers \( r_1, \ldots, r_n \), we have

\[
\left| D^n f(z) \right| h_1, \ldots, h_n \| \leq (r_1, \ldots, r_n)^{-1} \| f \|_{\mathcal{A}_p} \exp \left[ \frac{1}{2} \left( \sum_{j=1}^{n} r_j \| h_j \|_{-p} \right)^2 \right].
\]

(b) If \( p \in \mathbb{Z} \) and \( f \in \mathcal{A}_p \), we have

\[
\| D^n f(0) \|_{L^n[\mathcal{S}_{-p}]} \leq \left( \frac{\sqrt{n}}{e} \right)^n e^{n e^2} \| f \|_{\mathcal{A}_p}.
\]

(c) Let \( q \) be the smallest even positive integer such that \( q > 2e^2 (\ln 2)^{-1} \) and let \( p - \frac{1}{2} q = r \). If \( f \in \mathcal{A}_p \), then \( \sum_{n=0}^{\infty} (1/n!) D^n f(0) z^n \) converges to \( f \) in \( \mathcal{A}_r \).

Proof:

(a)

\[
\left| D^n f(z) \right| h_1, \ldots, h_n \|
\leq (r_1, \ldots, r_n)^{-1} \| f \|_{\mathcal{A}_p} \exp \left[ \frac{1}{2} \left( \sum_{j=1}^{n} r_j \| h_j \|_{-p} \right)^2 \right].
\]

(b) Take \( r_j = e/\sqrt{n} \) for \( j = 1, \ldots, n \), \( z = 0 \), and then apply (a); (b) follows.

(c) For \( z \in \mathcal{S}_{-r} \), we have

\[
\left| \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n \right|
\leq \sum_{n=0}^{\infty} \frac{1}{n!} \left| D^n f(0) z^n \right|
\leq \sum_{n=0}^{\infty} \frac{\left( \frac{\sqrt{n}}{e} \right)^n}{n!} e^{n e^2} \| z \|_{-p} \cdot \| f \|_{\mathcal{A}_p} \quad \text{(by (b))}
\leq \left( \sum_{n=0}^{\infty} \frac{n^n}{n!} e^{-2n} \right)^{1/2} e^{(1/2)M \| z \|_{-p}^2} \| f \|_{\mathcal{A}_p} \quad \text{(M = e^{2e^2})}
\leq C \cdot e^{(1/2)\| z \|_{-p}^2} \| f \|_{\mathcal{A}_p}
\quad \text{\( C = \left( \sum_{n=0}^{\infty} \frac{n^n}{n!} e^{-2n} \right)^{1/2} \)}
so that
\[
\left\| \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(0) z^n \right\|_{\mathcal{A}_p} \leq C \| f \|_{\mathcal{A}_p} \tag{9}
\]
which implies (c). \(\square\)

3.3. Theorem. Let \(p \in \mathbb{N}\).

(a) For \(f \in (\mathcal{S}_p)\), we have
\[
\| f \|_{\mathcal{A}_{p-1}} \leq C_p \| f \|_{2,p},
\tag{10}
\]
where \(C_p\) is a constant, depending only on \(p\), and \(f\) is the analytic version of \(f\) introduced in Theorem 2.8.

(b) Let \(p \geq 3\). Then there exists a constant \(C_p\), depending only on \(p\), such that
\[
\| f \|_{2,p-3} \leq C_p \| f \|_{\mathcal{A}_p}
\tag{11}
\]
for all \(f \in \mathcal{A}_p\). Thus \(\mathcal{A}_p = (\mathcal{S}_{p-3})\).

Proof. (a) follows from (5) in Theorem 2.8(b).

(b) We divide the proof of (b) into two steps.

(Step 1). Claim that, for \(f \in \mathcal{A}_p\) and \(x \in \mathcal{S}_{p-3}\), we have
\[
\| D^n f(x) \|_{\mathcal{S}^n[\mathcal{S}_{p-3}+2]} \leq \left(\frac{\sqrt{n}}{e}\right)^n e^{(1/16) n^2} \| f \|_{\mathcal{A}_p} e^{\| x \|^2_{p-3}}
\tag{12}
\]
and
\[
\| D^n f(0) \|_{\mathcal{S}^n[\mathcal{S}_{p-3}+2]} \leq C_p \left(\frac{\sqrt{n}}{e}\right)^n e^{(1/16) n^2} \| f \|_{\mathcal{A}_p},
\tag{13}
\]
where \(C_p = \int_{\mathcal{S}_{p-3}} \exp(\| x \|^2_{2-p}) \mu(dx)\).

In fact, if one applies Proposition 3.2(a) with \(r_1 = r_2 = \cdots = r_n = e/\sqrt{n}\) and use the fact that \(4 \| y \|_{p-3} \leq \| y \|_{p-3+2}\) for \(y \in \mathcal{S}_{p-3+2}\), we have
\[
| D^n f(x) h_1 \cdots h_n | \leq \left(\frac{\sqrt{n}}{e}\right)^n \| f \|_{\mathcal{A}_p} \exp \left[ \| x \|^2_{p-3} + \frac{e^2}{16n} \left( \sum_{j=0}^{\infty} \| h_j \|_{p-3+2} \right)^2 \right],
\]
for all \(h_1, \ldots, h_n \in \mathcal{S}_{p-3+2} \subset \mathcal{S}_{p-3}\). The inequality (12) then follows. Inequality (13) clearly follows from inequality (12).

(Step 2). Claim that inequality (11) holds.
Let \( \{e_n\} \) be the CONS of \( L^2(\mathbb{R}^1) \), consisting of Hermite functions. Then \( \{g_n = (2n + 2)^{-\frac{3}{2}} e_n\} \) forms a CONS of \( \mathscr{L}_{-p + 3} \) and we have

\[
\|f\|_{2, p - \frac{3}{2}}^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|_{\mathscr{L}_{-p + 3}}^2
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \sum_{k_1, \ldots, k_n = 0}^{\infty} (D^n \mu f(0) g_{k_1} \cdots g_{k_n})^2 \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|_{\mathscr{L}_{-p + 2}}^2 \left( \sum_{k=0}^{\infty} \|g_k\|_{-p + 2}^2 \right)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n \mu f(0)\|_{\mathscr{L}_{-p + 2}}^2 \left( \sum_{k=0}^{\infty} (2k + 2)^{-2} \right)^n
\]

\[
\leq C_p \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} n^n e^{-2n e^{(1/8)n^2}} \right\} \|f\|_{\mathscr{A}_p}^2 \quad \text{(by Inequality (13))}
\]

\[
\leq C_p^2 \cdot \|f\|_{\mathscr{A}_p}^2
\]

where \( C_p = C_p \left\{ \sum_{n=0}^{\infty} (1/n!) n^n e^{-2n e^{(1/8)n^2}} \right\}^{1/2} < \infty \).

This verifies inequality (11) and completes the proof of (b).

It is worth noting that the following fact can also be obtained in the course of the proof of Theorem 3.3(b). A similar result was also obtained recently in [21].

3.4. Corollary. For \( p \in \mathbb{Z} \), and for \( f \in (\mathcal{F}_p) \), there exists a constant \( \alpha_p > 0 \) such that

\[
\alpha_p \|f\|_{\mathcal{F}_{p + 1}} \leq \|f\|_{\mathcal{A}_p} \leq \|f\|_{2, p}.
\]

Proof. The first inequality in fact follows from the observation that inequality (14) is true not only for positive \( p \) but also for any \( p \in \mathbb{Z} \). The second inequality follows from Corollary 2.13.

Now define \( \mathcal{A}_\infty = \bigcap_{p \in \mathbb{N}} \mathcal{A}_p \), and endow \( \mathcal{A}_\infty \) with the projective limit topology, i.e., \( \mathcal{A}_\infty = \varprojlim \mathcal{A}_p \). Then \( \mathcal{A}_\infty \) is a complete topological linear space. Applying Theorem 3.3 and Corollary 3.4, we have the following.

3.5. Corollary. (a) \( \mathcal{A}_\infty \subset (\mathcal{F}) \) and \( \mathcal{A}_\infty = (\mathcal{F})^\sim \), the continuous version of \( (\mathcal{F}) \). Hence \( \mathcal{A}_\infty \) is the analytic version of \( (\mathcal{F}) \).

(b) The families of norms \( \{\|\cdot\|_{\mathcal{A}_p}, p \in \mathbb{N}\} \), \( \{\|S(\cdot)\|_{\mathcal{A}_p}, p \in \mathbb{N}\} \), and \( \{\|\cdot\|_{2, p}, p \in \mathbb{N}\} \) are equivalent on \( \mathcal{A}_\infty \).
(c) Let the dual spaces of $\mathcal{A}_\infty$ and $\mathcal{A}_p$ be respectively denoted by $\mathcal{A}_\infty^*$ and $\mathcal{A}_p^*$. Then we have $\mathcal{A}_\infty^* = \bigcup_{p \in \mathbb{N}} \mathcal{A}_p^*$.

Next, we show that the space $\mathcal{A}_\infty$, under pointwise multiplication, is an algebra. Then, applying Theorem 3.3 and Corollary 3.5, one proves easily the known fact that $(\mathcal{S})$ is an algebra (see [7, 13, 20, 26]).

3.6. **Proposition.** $\mathcal{A}_\infty$ is an algebra.

**Proof.** Suppose $f, g \in \mathcal{A}_\infty$. Clearly, $f \cdot g$ is analytic in $\mathcal{C}\mathcal{S}^*$. Moreover, for $p \geq 2$, $z \in \mathcal{S}_{-p+1}$,

$$|f(z) \cdot g(z)| \leq \|f\|_{\mathcal{A}_p} \|g\|_{\mathcal{A}_p} \exp(\|z\|_{-p}^2)$$

$$\leq \|f\|_{\mathcal{A}_p} \|g\|_{\mathcal{A}_p} \exp \left( \frac{1}{2} \|z\|_{-p}^2 \right).$$

Which, in turns, implies that $f \cdot g \in \mathcal{A}_\infty$ and

$$\|f \cdot g\|_{\mathcal{A}_{p-1}} \leq \|f\|_{\mathcal{A}_p} \|g\|_{\mathcal{A}_p}.$$  \hfill (15)

This completes the proof of Proposition 3.6. \hfill \Box

3.7. **Corollary.** $(\mathcal{S})$ is an algebra.

**Proof.** Let $f, g \in (\mathcal{S})$. Then $f = \tilde{f}$ a.e.($\mu$) and $g = \tilde{g}$ a.e.($\mu$), so that $f \cdot g = \tilde{f} \cdot \tilde{g}$ a.e.($\mu$). Since $\tilde{f} \cdot \tilde{g} \in \mathcal{A}_\infty \subset (\mathcal{S})$, $f \cdot g \in (\mathcal{S})$ and, moreover, for $p \in \mathbb{N}$ we have

$$\|f \cdot g\|_{2,p} = \|\tilde{f} \cdot \tilde{g}\|_{2,p}$$

$$\leq C_{p+3} \|\tilde{f} \cdot \tilde{g}\|_{\mathcal{A}_{p+3}} \quad \text{(by (10))}$$

$$\leq C_{p+3} \|\tilde{f}\|_{\mathcal{A}_{p+4}} \|\tilde{g}\|_{\mathcal{A}_{p+4}} \quad \text{(by (15))}$$

$$\leq C_{p+3} (C_{p+5})^2 \|\tilde{f}\|_{2,p+5} \|\tilde{g}\|_{2,p+5} \quad \text{(by (9))}.$$

This proves the corollary. \hfill \Box

**Equivalence of Calculus on $\mathcal{A}_\infty^*$ and $(\mathcal{S})^*$**

Recall that any function $f \in L^2(\mathcal{S}^*, \mu)$ is identified with the equivalent class of functions $[f]$ in which any function is equal to $f$ almost everywhere with respect to $\mu$. In this sense, we identify $(\mathcal{S})$ with $(\mathcal{S})^- = \mathcal{A}_\infty$. We further identify $(\mathcal{S})^*$ with $\mathcal{A}_\infty^*$ in the following way.

Denote the $(\mathcal{S})^*-(\mathcal{S})$ pairing by $\langle \cdot, \cdot \rangle$ and the $\mathcal{A}_\infty^* - \mathcal{A}_\infty$ pairing by $\langle \cdot, \cdot \rangle_a$.

For any $F \in \mathcal{A}_\infty^*$, define $\tilde{F}$ by

$$\langle \tilde{F}, \varphi \rangle_a := \langle F, \varphi \rangle$$ \hfill (16)

for all $\varphi \in (\mathcal{S})$. Then $\tilde{F} \in (\mathcal{S})^*$. 

Conversely, since the imbedding \(j: \mathcal{A}_\infty \subset (\mathcal{S})^*\) is continuous, so is \(j^*: (\mathcal{S})^* \subset \mathcal{A}_\infty^*\). Thus \((\mathcal{S})^*\) is identified as a subspace of \(\mathcal{A}_\infty^*\).

Next, we remark that the calculi on \(\mathcal{A}_\infty^*\) and \((\mathcal{S})^*\) are also equivalent. In fact, if \(T: \mathcal{A}_\infty \to \mathcal{A}_\infty\) is linear and continuous and \(F \in (\mathcal{S})^*\), then the element \(T^*F\) is defined by

\[
\langle T^*F, \varphi \rangle_a := \langle j^*F, T\hat{\varphi} \rangle_a.
\]  

(17)

Conversely, if \(S: (\mathcal{S}) \to (\mathcal{S})\) is linear and continuous and \(F \in \mathcal{A}_\infty^*\), then \(S^*F\) is defined as an element in \(\mathcal{A}_\infty^*\) in the sense that

\[
\langle S^*F, \varphi \rangle_a := \langle \hat{F}, S(j\varphi) \rangle_a.
\]  

(18)

for \(\varphi \in \mathcal{A}_\infty\).

For notational convenience, we shall omit \(j\) and \(j^*\) above. For example, we write \(\langle T^*F, \varphi \rangle_a = \langle F, T\hat{\varphi} \rangle_a\) for (17) and write \(\langle S^*F, \varphi \rangle_a = \langle \hat{F}, S\varphi \rangle_a\), for (18).

To close this section we list below some examples of operations for generalized white noise functionals in \((\mathcal{S})^*\). These definitions, which were introduced in [18], still make sense in the present case.

3.8. Examples.

(a) Differentiation. For \(F \in (\mathcal{S})^*\), \(h \in \mathcal{S}\), define

\[
\langle D_h F, \varphi \rangle_a := \langle F, \hat{h}\hat{\varphi} \rangle_a - \langle F, D_h \varphi \rangle_a,
\]

where \(\hat{h}(x) = (x, h)\) and \((\cdot, \cdot)\) denotes the \(\mathcal{S}^* - \mathcal{S}\) pairing, and where \(\varphi \in (\mathcal{S})\). \(\partial_s F\) may be defined in the same way as given in [18].

(b) The operator \(D_s^*\). For \(F \in (\mathcal{S})^*\), \(x \in \mathcal{S}^*\), we have

\[
\langle D_s^* F, \varphi \rangle_a := \langle F, D_s \varphi \rangle_a.
\]

(c) The number operator. For \(F \in (\mathcal{S})^*\), define

\[
\langle NF, \varphi \rangle_a := \langle F, N\hat{\varphi} \rangle_a.
\]

(d) The operator \(A^*\). For \(F \in (\mathcal{S})^*\), define

\[
\langle A^* F, \varphi \rangle_a := \langle F, A\hat{\varphi} \rangle_a.
\]

(e) Translation. For \(\tau \in \mathcal{S}\), \(F \in (\mathcal{S})^*\), we define

\[
\langle \tau \varphi \rangle \varphi := \langle F, e^{-\tau \varphi} \rangle_a e^{(\cdot - 1/2)|z|^2},
\]

where \(\tau \varphi(x) = \varphi(x - y)\) for \(x, y \in \mathcal{S}^*\).
(f) Multiplication. For \( F \in (\mathcal{S})^* \), \( \eta \in (\mathcal{S}) \), we define

\[
\langle \eta F, \varphi \rangle_s := \langle F, \eta \varphi \rangle_s = \langle F, \eta \varphi \rangle_a.
\]

The Fourier transform is discussed in more detail in the next section, and the composite of the tempered distribution with Brownian motion, which can be defined in the same way as given in [18], will be investigated later in another paper.

4. FOURIER TRANSFORM

After our previous paper [13], we intend to define the Fourier transform \( \mathcal{F} F \) of a generalized white noise functional \( F \in (\mathcal{S})^* \) by the relation

\[
\langle \mathcal{F} F, \varphi \rangle_s := \langle F, \mathcal{F}_{L^{-1}} \varphi \rangle_a
\]

for \( \varphi \in (\mathcal{S}) \), where \( \mathcal{F}_{L^{-1}} \varphi(y) = \int_{\mathcal{S}^*} \tilde{\varphi}(ax + \beta y) \mu(dx) \), \( a, \beta \in \mathbb{C} \) (cf. [10]).

To prove that definition (19) makes sense for \( F \in (\mathcal{S})^* \), it is necessary to verify the continuity of \( \mathcal{F}_{L^{-1}} \) on \( \mathcal{A}_\infty \). This requirement proceeds from the following.

4.1. LEMMA. \( \mathcal{F} \mathcal{A}_\beta (\mathcal{A}_\infty) \subset \mathcal{A}_\infty \) and \( \mathcal{F} \mathcal{A}_\beta \) is continuous on \( \mathcal{A}_\infty \).

Proof: Recall that \( \mathcal{A}_\infty = \cap_{p \in \mathbb{N}} \mathcal{A}_p \). Choose \( q \in \mathbb{N} \) such that \( \log_2 |\beta| \leq q - \frac{1}{2}, \log_2 |\alpha| < p - \frac{1}{2}, \) and \( p - q \geq 1 \). Then, for \( \varphi \in \mathcal{A}_\infty \) and for \( z \in \mathcal{S}_{q-p} \), we have

\[
|\mathcal{F}_{a,\beta} \varphi(z)| \leq \int_{\mathcal{S}^*} |\varphi(ax + \beta z)| \mu(dx)
\]

\[
\leq \| \varphi \|_{a,p} \left\{ \int_{\mathcal{S}_{q-p}} \exp \left[ \frac{1}{2} (|\alpha| \|x\|_{-p} + |\beta| \|z\|_{-p})^2 \right] \mu(dx) \right\}
\]

\[
\leq C_{r,p} \| \varphi \|_{a,p} \exp[|\beta|^2 \|z\|^2_{-p}]
\]

\[
\leq C_{r,p} \| \varphi \|_{a,p} \exp[\frac{1}{2} \|z\|^2_{-p+q}]
\]

so that

\[
\| \mathcal{F}_{a,\beta} \varphi \|_{a,q} \leq C_{a,p} \| \varphi \|_{a,p},
\]

where \( C_{a,p} = \int_{\mathcal{S}_{q-p}} \exp \left[ |\alpha|^2 \|x\|^2_{-p} \right] \mu(dx) < \infty \). The lemma follows from (20).
4.2. PROPOSITION. (a) The Fourier transform defined by (19) is well-defined for any \( F \in (\mathcal{F})^* \).

(b) \( \mathcal{F} F \in \mathcal{A}_p^* \) for \( F \in \mathcal{A}_p^* \) and for \( p \geq 2 \).

Proof. Apply Lemma 4.1 with \( \alpha = 1, \beta = -i \); (a) follows. Furthermore, if one takes \( p \geq 2, q = 1 \) and applies inequality (11), one obtains

\[
\| \mathcal{F}_{1, -i} \phi \|_{\mathcal{A}_p} \leq C_{1, p} \| \phi \|_{\mathcal{A}_p}.
\]

Consequently, we have \( \mathcal{F} (\mathcal{A}_p^*) \subset \mathcal{A}_p^* \).

The results of [18] have also shown that the inverse Fourier transform may be defined by

\[
\langle \mathcal{F}^{-1} F, \phi \rangle = \langle F, \mathcal{F}_{1, i} \phi \rangle.
\]  

To verify that \( \mathcal{F}^{-1} F \) given by (21) is the inverse Fourier transform, one needs to prove that

\[
\mathcal{F}_{1, i}(\mathcal{F}_{1, -i} \phi) = \mathcal{F}_{1, -i}(\mathcal{F}_{1, i} \phi) = \phi
\]

holds for \( \phi \in \mathcal{A}_\infty \). This is in fact a consequence of the following.

4.3. LEMMA. Let \( \alpha, \beta, \alpha', \beta' \) be nonzero complex numbers. Then in order that \( \mathcal{F}_{\alpha, \beta} (\mathcal{F}_{\alpha', \beta'} f) = f \) for all \( f \in \mathcal{A}_\infty \) it is necessary and sufficient that \( \beta \beta' = 1 \) and \( (\beta \alpha')^2 + \alpha^2 = 0 \) (cf. [16]).

Proof. First we show that, for \( \lambda, \gamma \in \mathbb{C} \),

\[
\int_{\mathcal{F}^*} \int_{\mathcal{F}^*} f(\lambda x + \gamma y) \mu(dx) \mu(dy) = \int_{\mathcal{F}^*} f(\sqrt{\lambda^2 + \gamma^2} z) \mu(dz). \tag{22}
\]

It follows from Proposition 3.2(c) that the series \( \sum_{n=0}^{\infty} (1/n!) D^n f(0) z^n \) converges to \( f(z) \) in \( \mathcal{A}_\infty \) which, in turn, implies that

\[
\int_{\mathcal{F}^*} \int_{\mathcal{F}^*} f(\lambda x + \gamma y) \mu(dx) \mu(dy)
\]

\[= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{F}^*} \int_{\mathcal{F}^*} D^n f(0)(\lambda x + \gamma y)^n \mu(dx) \mu(dy)
\]

\[= \sum_{n=0}^{\infty} \frac{1}{(2k)!} \int_{\mathcal{F}^*} (\lambda^2 + \gamma^2)^k D^{2k} f(0) z^{2k} \mu(dz).
\]

The last equality and the rest of the proof for (22) follows in exactly the same way as given in [16].
Now, applying the identity (22), we obtain
\[ \mathcal{F}_{\alpha, \beta} (\mathcal{F}_{\alpha, \beta} f)(z) = \int_{\mathcal{F}^*} \int f(\beta \beta' z + \sqrt{x^2 + (\beta x)^2} y) \mu(dy). \]

The lemma then follows from the above identity. \[ \square \]

In addition to the Fourier transform discussed above, we remark here one more important application of the transform \( \mathcal{F}_{\alpha, \beta} \) as follows.

4.4. PROPOSITION. Let \( f \in \mathcal{A}_\infty \). Then the Wiener–Ito decomposition of \( f \) converges in the following senses: (i) in \( \mathcal{A}_\infty \); (ii) uniformly on bounded subsets of \( \mathcal{F}^* \); (iii) in \( L^p(\mathcal{F}^*, \mu) \) for \( p \geq 1 \). (See also [19].)

Proof. The convergence in the sense of (i) clearly implies (ii) and (iii). It suffices to prove the convergence of the Wiener–Ito decomposition in \( \mathcal{A}_\infty \) (by Lemma 4.1). According Proposition 3.2(c), \( \mu f(z) = \sum_{n=0}^{\infty} (1/n!) D^n \mu f(0) \cdot z^n \) which converges in \( \mathcal{A}_\infty \). Then, it follows form Lemmas 4.1 and 4.3 that \( \mathcal{F}_{\alpha, \beta} \mu f(z) = \mathcal{F}_{\alpha, \beta} \mu f(\cdot)(z) = f(z) \) and that \( \mathcal{F}_{\alpha, \beta} (D^n \mu f(0)(\cdot)))(z) = A^n D^n \mu f(0)(z) \); the present proposition follows. \[ \square \]

5. A CHARACTERIZATION OF MEASURES IN \( (\mathcal{F})^* \)

It has been proved by Yokoi [26] and Kondrat'ev [27] that, for any positive white noise functional \( F \) in \( (\mathcal{F})^* \), there always exists a (positive) measure \( \eta \) such that \( \langle F, \varphi \rangle = \int \varphi(x) \eta(dx) \) for all \( \varphi \in (\mathcal{F}) \), where \( \varphi \) is the continuous version of \( \varphi \). According to the conclusion of Sections 2 and 3, \( \varphi \in \mathcal{A}_\infty \) and \( \eta \in \mathcal{A}_*^\infty \).

In this section, we are interested in the behavior of measures \( \eta \) in \( \mathcal{A}_*^\infty \). Since \( \mathcal{A}_*^\infty = \bigcup_{p \in \mathbb{N}} \mathcal{A}_p^\infty \), it is sufficient to consider the measures in \( \mathcal{A}_p^\infty \). For notational convenience, we shall also use \( \langle \cdot, \cdot \rangle \_a (\langle \cdot, \cdot \rangle \_s) \) to denote the pairing \( \mathcal{A}_p^\infty \_a (\mathcal{A}_p^\infty \_s) \).

Our main result is given as follows.

5.1. THEOREM. Suppose \( \eta \) is a Borel measure defined on \( \mathcal{F}^* \). Then \( \eta \in \mathcal{A}_p^\infty \) if and only if (i) the measurable support of \( \eta \) is contained in \( \mathcal{F}_{-p} \), and (ii) \( \int_{\mathcal{F}_{-p}} \exp(\frac{1}{2} \| x \|_{-p}^2) \eta(dx) < \infty \).

Proof. Sufficiency. Suppose the measure \( \eta \) satisfies conditions (i) and (ii). Then we have
\[ |\langle \eta, \varphi \rangle \_a| \leq \int_{\mathcal{F}_{-p}} |\varphi(x)| \eta(dx) \leq \left[ \int_{\mathcal{F}_{-p}} \exp(\frac{1}{2} \| x \|_{-p}^2) \eta(dx) \right] \| \varphi \|_{\mathcal{A}_p} \]
so that \( \eta \) is continuous on \( \mathcal{A}_p \) and the \( \mathcal{A}_p \)-norm of \( \eta \) satisfies
\[
\| \eta \|_{\mathcal{A}_p} \leq \int_{\mathcal{A}_p} \exp(\frac{1}{2} \| x \|_p^2) \eta(dx).
\]

Necessity. For each \( n \), define the function \( g_n \) by
\[
g_n(x) = \exp\left(\frac{1}{2} \sum_{j=0}^{n} (2j+2)^{-2p} (x, e_j)^2\right) \text{ for } x \in \mathcal{F}^*.
\]
Clearly, \( g_n \in \mathcal{A}_p \) with \( \| g_n \|_{\mathcal{A}_p} \leq 1 \) and \( g_n(x) \sim \exp(\frac{1}{2} \| x \|_p^2) \) as \( n \to \infty \). Since the measure \( \eta \) is positive, we have
\[
\int_{\mathcal{F}^*} \exp(\frac{1}{2} \| x \|_p^2) \eta(dx)
\]
\[
= \lim_{n \to \infty} \int_{\mathcal{F}^*} g_n(x) \eta(dx) \quad \text{(by monotone convergence theorem)}
\]
\[
\leq \lim_{n \to \infty} \langle \eta, g_n \rangle_{\mathcal{A}_p}
\]
\[
\leq \| \eta \|_{\mathcal{A}_p} < \infty.
\]

Note that \( \| x \|_p = \infty \) for \( x \in \mathcal{F}^* \setminus \mathcal{F}_p \). It follows from (23) that we must have \( \eta(\mathcal{F}^* \setminus \mathcal{F}_p) = 0 \). This completes the proof.

The following corollaries follows immediately from Theorem 5.1.

5.2. COROLLARY. In order for \( \eta \) to be a Borel measure in \( (\mathcal{F})^* \) it is necessary and sufficient that there exists \( p \in \mathbb{N} \) such that conditions (i) and (ii) of Theorem 5.1 are satisfied.

5.3. COROLLARY. If \( \eta \) is a measure in \( \mathcal{A}_p \), then
\[
\| \eta \|_{\mathcal{A}_p} = \int_{\mathcal{F}_p} \exp(\frac{1}{2} \| x \|_p^2) \eta(dx).
\]

5.4. COROLLARY. If \( \eta \) is a measure in \( \mathcal{A}_p \), then
\[
\eta(\| x \|_p > t) \leq e^{(-1/2)t^2} \| \eta \|_{\mathcal{A}_p}.
\]

5.5. EXAMPLE. The generalized white noise functional
\[
\exp[(\lambda/2)\int_{-\infty}^{\infty} \hat{B}(t)^2 dt]:
\]
defined by the linear functional \( \varphi \to \int_{\mathcal{F}^*} f(\sqrt{(1/(1-\lambda))} x) \mu(dx) \) is clearly positive for every \( 0 < \lambda < 1 \). As a matter of fact, \( \exp[(\lambda/2)\int_{-\infty}^{\infty} \hat{B}(t)^2 dt] \) is nothing but the Gaussian measure \( \mu_{1/(1-\lambda)} \) with variance parameter \( 1/(1-\lambda) \) which clearly satisfies conditions (i) and (ii) given in Theorem 5.1.
5.6. EXAMPLE. The Donsker's delta function \( \delta_a(B(t)) \) may be defined as

\[
\langle \delta_a(B(t)), \varphi \rangle_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iax - \frac{1}{2}a^2t} \mathcal{F}_{1,t} \varphi(sh_t) \, ds
\]

for \( \varphi \in \mathcal{S} \), where \( h_t = 1_{(0,t]}, \ t > 0 \), and \( \{B(t)\} \) denotes the Brownian motion.

It is easy to see that

\[
| \langle \delta_a(B(t)), \varphi \rangle_s | \leq k_t \| \varphi \|_{\mathcal{A}}
\]

\[
\leq k_t C_2 \| \varphi \|_{L^2}, \quad (24)
\]

where \( k_t = \sqrt{\frac{2}{3\pi t}} \int_{\mathcal{S}_E} e^{(1/2)\|x\|^2} \mu(dx) \) and \( C_2 \) is the constant as given in (9). The estimation (24) implies that \( \delta_a(B(t)) \in \mathcal{A}_1 \subset (\mathcal{S}_E)^* \).

Furthermore, it can also be shown that

\[
\langle \delta_a(B(t)), \varphi \rangle = \frac{1}{\sqrt{2\pi t}} e^{-a^2/2t} \int_{\mathcal{S}_E} \varphi \left( y - \frac{1}{t} \langle y, h_t \rangle - a \right) h_t \mu(dy). \quad (25)
\]

It follows from (25) that \( \delta_a(B(t)) \) is positive and the corresponding measure lies in \( \mathcal{A}_1^* \).

APPENDIX

Let \( E \) be a countably complex normed linear space [3] which is topologized by compatible norms \( \{ \| \cdot \|_n : n \in \mathbb{N} \} \). Assume that the norms are nondecreasing, i.e., \( \| \cdot \|_1 \leq \| \cdot \|_2 \leq \cdots \). Completing the space \( E \) with respect to each norm \( \| \cdot \|_n \), we obtain a system of Banach space \( E_1, E_2, \ldots \).

Since the norms are compatible and nondecreasing, we have \( E_1 \supseteq E_2 \supseteq \cdots \supseteq E \). Assume that \( E = \bigcap E_n \). Then \( E \) becomes a complete topological linear space, with the projective limit topology induced by the spaces \( \{ E_n, \| \cdot \|_n \} \). Let \( E^* (E_n^*) \) denote the dual space of \( E (E_n) \).

A.1. DEFINITION. A single-valued function \( f: E^* \to \mathbb{C} \) is called analytic in \( E^* \) if

(i) \( f \) is locally bounded;

(ii) for \( x, y \in E^* \), the function \( g(\lambda) = f(x + \lambda y) \) is analytic in \( \mathbb{C} \).

If \( E^* \) is itself a Banach space, Definition A.1 coincides with Definition 3.17.2 from [8].

Recall that a set \( G \subseteq E^* \) is bounded if and only if there exist \( n \in \mathbb{N} \) such that \( G \subseteq E_n^* \) and \( G \) is bounded in \( E_n^* \) [3]. Using this fact, we obtain the following.
A.2. Theorem. \( f \) is analytic in \( E^* \) if and only if \( f: E_n^* \to \mathbb{C} \) is analytic in \( E_n^* \) for all \( n \).

A.3. Corollary. If \( f \) is analytic in \( E^* \), then \( f \) is infinitely Fréchet differentiable.

Proof. It follows from [8, Theorem 3.17.1] that \( f \) is infinitely Fréchet differentiable in \( E_n^* \) for each \( n \). Consequently, we have the following,

A.4. Fact. For each \( a \in E_n^* \) and for any bounded set \( F \subset E_n^* \),

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} [ f(a + \varepsilon x) - f(a) - Df(a)x ] = 0 \text{ uniformly for } x \in F.
\]

Since every bounded set of \( E^* \) is a bounded subset of some \( E_n^* \) and every bounded subset of \( E_n^* \) is, in turn, bounded in \( E_{n+m}^* \) for \( m \in \mathbb{N} \), one sees that the Fact A.4 remains true for \( a \in E_n^* \) and \( F \subset E_n^* \) for \( n, m \in \mathbb{N} \). This implies that for any \( a \in E^* \) and for any bounded set \( F \subset E^* \), there exists a function \( f'(x; a) \) (\( = Df(a)x \), if \( a, x \) are in the same space \( E_n^* \)) such that the limit \( \lim_{\varepsilon \to 0} \varepsilon^{-1} [ f(a + \varepsilon x) - f(a) - f'(x; a) ] = 0 \) uniformly for all \( x \in F \). Thus \( f \) is Fréchet differentiable in \( E^* \) (see [25]). The higher order Fréchet differentiability of \( f \) can be proved in the same way.

Final remark. The results concerning the connection between Bargmann–Segal analytic functions and the space \( \mathcal{S}_p \) for \( p \in \mathbb{Z} \) were added during the revision of the paper. When the revision was nearly complete, we learned that a result similar to Corollary 3.4, for the case \( p \in \mathbb{Z} \setminus \mathbb{N}_0 \), was also done independently by Potthoff and Streit in their very recent paper [21]. In view of our proof in this paper, we also discovered that the results depend essentially on the operator \( A \) and the nuclearity of \( \mathcal{S} \) induced by the operator \( A \). Thus it is possible to extend the results of the present paper to a more general nuclear space, such as the Gelfand triplet introduced in [12].

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References