THREE HOMOTOPY THEORIES FOR CYCLIC MODULES*

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1. Introduction

1.1. The main result. Let \( R \) be a ring with \( 1 \neq 0 \) and let \( C \) denote the category of the cyclic \( R \)-modules of Connes, i.e., \([3, 6]\) simplicial \( R \)-modules \( X \) with, in each dimension \( n \geq 0 \), an extra degeneracy map \( s_{n+1} : X_n \rightarrow X_{n+1} \) satisfying the usual identities, except that the identities \( d_0 s_{n+1} = s_0 d_0 : X_n \rightarrow X_n \) \((n \geq 0)\) are replaced by \((d_0 s_{n+1})^{n+1} = id : X_n \rightarrow X_n \) \((n \geq 0)\). Such a cyclic \( R \)-module has not only its underlying simplicial \( R \)-module, which we will denote by \( \ast X \), but also an underlying cosimplicial \( R \)-module \( k \ast X \) (with \( d^i = s_{n-i} : X_{n-i} \rightarrow X_n \) and \( s^i = d_{n-i} : X_n \rightarrow X_{n-i} \)).

As a result, there are three obvious homotopy theories which one can associate with \( C \); these correspond to three possible criteria for calling a map \( f : X \rightarrow X' \) in \( C \) a 'weak equivalence' \([6, \S 7]\). First there is the one-sided homotopy theory in which the weak equivalences are the maps \( f \) which induce isomorphisms \( \pi_i \ast X \cong \pi_i \ast X' \) \((i \geq 0)\) on the homotopy groups of the underlying simplicial modules. Next, there is a dual one-sided theory in which the weak equivalences are the maps \( f \) which induce isomorphisms \( \pi^k \ast X \cong \pi^k \ast X' \) \((i \geq 0)\) on the cohomotopy groups of the underlying cosimplicial modules. Finally, there is a strong or two-sided theory in which the weak equivalences are the maps \( f \) which induce isomorphisms on both the homotopy groups of the underlying simplicial modules and the cohomotopy groups of the underlying cosimplicial modules. The main aim of this paper is to show that each of these three homotopy theories is equivalent to a corresponding homotopy theory of differential graded modules over a graded exterior \( R \)-algebra. This implies that from any of the above three points of view the study of cyclic \( R \)-modules is equivalent to the study of the classical homological algebra of certain chain complexes. In more detail:

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Let \( \mathcal{E} \) be the category of differential graded \( R \)-modules with an exterior action, i.e., diagrams of \( R \)-modules of the form

\[
\cdots \xrightarrow{\partial} U_{-1} \xleftarrow{\partial} U_0 \xrightarrow{\partial} U_1 \xleftarrow{\partial} \cdots
\]

in which \( \partial^2 = 0, \partial^2 = 0 \) and \( \partial \delta + \delta \partial = 0 \) and call a map in \( \mathcal{E} \) a weak equivalence if it induces an isomorphism on the homology (with respect to \( \partial \)). Let \( (\mathcal{E}, \mathcal{E}) \) denote the category of maps of \( \mathcal{E} \) (which has as objects the maps \( f: U \to V \in \mathcal{E} \) and as maps \( (f': U' \to V') \to (f: U \to V) \) the pairs of maps \( u: U \to U', v: V \to V' \in \mathcal{E} \) such that \( uf = f'u \) and call a map of \( (\mathcal{E}, \mathcal{E}) \) a weak equivalence if it consists of a pair of weak equivalences in \( \mathcal{E} \). Let \( \mathcal{E}^+ \subset \mathcal{E} \) and \( \mathcal{E}^- \subset \mathcal{E} \) denote the full subcategories spanned by the objects with trivial homology in negative or positive dimensions respectively and let \( (\mathcal{E}^-, \mathcal{E}^+) \subset (\mathcal{E}, \mathcal{E}) \) be the full subcategory spanned by the maps \( U \to V \in \mathcal{E} \) with \( U \in \mathcal{E}^- \) and \( V \in \mathcal{E}^+ \). These categories \( \mathcal{E}^-, \mathcal{E}^- \) and \( (\mathcal{E}^-, \mathcal{E}^+) \) inherit, in an obvious way, notions of weak equivalences from \( \mathcal{E} \) and \( (\mathcal{E}, \mathcal{E}) \). We then show that the three homotopy theories of \( C \) mentioned above are equivalent to the homotopy theories of \( \mathcal{E}^+, \mathcal{E}^- \) and \( (\mathcal{E}^-, \mathcal{E}^+) \) respectively.

1.2. Outline of the proof. The proof of our main result consists of three parts:

(i) Let \( \mathcal{R}(\partial, \delta) \) denote the category of duchain complexes over \( R \), i.e., diagrams of \( R \)-modules of the form

\[
X_0 \xrightarrow{\partial} X_1 \xleftarrow{\partial} X_2 \xrightarrow{\partial} \cdots
\]

in which \( \partial^2 = 0 \) and \( \delta^2 = 0 \), but in which the \( \partial \)'s and the \( \delta \)'s are otherwise independent (and in particular are not required to commute). We then recall from [6] the existence of a full subcategory \( \mathcal{D} \subset \mathcal{R}(\partial, \delta) \) of cyclic chain complexes and an equivalence of categories \( N: C \to D \) which, for every object \( X \in C \), induces natural isomorphisms

\[
\pi_j X \cong H_j N X \quad \text{and} \quad \pi^i k X \cong H^i N X \quad (i \geq 0).
\]

Thus the three homotopy theories of \( C \) are equivalent to the corresponding (under \( N \)) homotopy theories of \( D \), i.e., the theories in which a map \( X \to X' \in D \) is a weak equivalence whenever it induces isomorphisms \( H_i X \cong H_i X' \) (\( i \geq 0 \)) or isomorphisms \( H^i X \cong H^i X' \) (\( i \geq 0 \)) or both.

(ii) Next we consider the category \( \mathcal{R}(\partial, \delta) \) of extended duchain complexes, i.e. diagrams of \( R \)-modules of the form

\[
\cdots \xrightarrow{\partial} U_{-1} \xleftarrow{\partial} U_0 \xrightarrow{\partial} U_1 \xleftarrow{\partial} \cdots
\]

in which \( \partial^2 = 0 \) and \( \delta^2 = 0 \), and a corresponding full subcategory \( \mathcal{D} \subset \mathcal{R}(\partial, \delta) \) of extended cyclic chain complexes. Let \( \mathcal{D}^+ \subset \mathcal{D} \) and \( \mathcal{D}^- \subset \mathcal{D} \) be the full subcategories spanned by the objects with trivial homology in negative and positive dimensions
respectively, let $(\tilde{R}(\partial, \delta), \tilde{R}(\partial, \delta))$ be the category of maps of $\tilde{R}(\partial, \delta)$ and let $(\tilde{D}_-, \tilde{D}_+) \subset (\tilde{R}(\partial, \delta), \tilde{R}(\partial, \delta))$ be the full subcategory spanned by the maps $U \to V \in \tilde{R}(\partial, \delta)$ with $U \in \tilde{D}_-$ and $V \in \tilde{D}_+$. Then we construct a functor $F: \tilde{D} \to (\tilde{D}_-, \tilde{D}_+)$ which induces an equivalence between the two-sided homotopy theory of $\tilde{D}$ and the homotopy theory of $(\tilde{D}_-, \tilde{D}_+)$ in which the weak equivalences are the (pairs of) homology isomorphisms. Moreover, composition of $F$ with the projections $(\tilde{D}_-, \tilde{D}_+) \to \tilde{D}_+$ and $(\tilde{D}_-, \tilde{D}_+) \to \tilde{D}_-$ results in functors which induce equivalences between the one-sided homotopy theories of $\tilde{D}$ and the homotopy theories of $\tilde{D}_-$ and $\tilde{D}_+$ in which the weak equivalences are the homology isomorphisms.

(iii) The last step consists of the observation that a simple change in the coboundary maps yields a functor $E: \tilde{D} \to \tilde{E}$ (1.1) which is an equivalence of homotopy theories (with respect to the homology isomorphisms) and which therefore induces equivalences of homotopy theories $\tilde{D}_+ \to \tilde{E}_+$, $\tilde{D}_- \to \tilde{E}_-$ and $(\tilde{D}_-, \tilde{D}_+) \to (\tilde{E}_-, \tilde{E}_+)$.  

1.3. A generalization. The category $\tilde{D}$ of cyclic chain complexes is (in Section 2) defined by means of a sequence $\{f_i = (1 + (-1)^i t)^{i+1}\}$ of polynomials in one variable $t$. However, it is not difficult to verify that the only properties of these polynomials which are used in the proofs of the results mentioned in 1.2(ii) and 1.2(iii) are that the $f_i$ have their coefficients in the center of $R$, that they are not constant, that their constant term is 1 and that each $f_i$ has a unit of $R$ as leading coefficient. Hence these results remain valid if the sequence $\{f_i\}$ is replaced by any sequence of polynomials with these properties. Any such sequence $\{g_i\}$ thus gives rise to full subcategories $D' \subset R(\partial, \delta)$ and $\tilde{D}' \subset \tilde{R}(\partial, \delta)$ with the same properties as $\tilde{D}$ and $\tilde{D}$.

The simplest case is when $g_i = 1 + t$ for all $i \geq 0$. Then $\tilde{D}' = \tilde{E}$ (1.1) and $\tilde{D}'$ (which we therefore will denote by $E$) is the category of the commutative duchain complexes, i.e., the diagrams of $R$-modules of the form

$$X_0 \xrightarrow{\delta} X_1 \xrightarrow{\delta} X_2 \xrightarrow{\delta} \cdots$$

in which $\partial^2 = 0$, $\delta^2 = 0$ and $\partial \delta + \delta \partial = 0$ (C. Kassel [7] calls these ‘mixed complexes’ and D. Burghelea [2] calls them ‘complexes with an algebraic circle action’). Thus the category $C$ of the cyclic modules of Connes and the category $E$ of the commutative duchain complexes are equivalent, not just from the one-sided homotopy points of view, but also from the two-sided one.

1.4. Summary. All this can be summarized by saying that (in the above notation) there is a commutative diagram (see next page) in which the horizontal functors induce equivalences of homotopy theories, if the middle $C$, $D$ and $E$ are endowed with their two-sided homotopy theories and the upper and lower ones with the appropriate one-sided theories. For the first three columns this will be proved in Sections 2, 4 and 3 respectively. For the last column this follows from the results of Section 4 and the above (1.3) comments.
1.5. Notation, terminology, etc. We freely use the notation, terminology and results of [6]. In particular we refer to [6, § 7] for the precise meaning of 'the homotopy theory of a category with respect to a subcategory of weak equivalences' and 'a functor which induces an equivalence between two such homotopy theories'.

Note that our description above of a cyclic object $X_*$ is slightly different from Connes' [3]. One approach is translated into the other by setting the cyclic operator $t_{n+1}: X_n \to X_n$ equal to $(d_0 s_{n+1})^a = (d_0 s_{n+1})^{-1}$ or equivalently $s_n : X_{n-1} \to X_n$ equal to $(t_{n+1})^{-1} s_0$.

2. Cyclic chain complexes

In this section we

(i) define (2.1) cyclic chain complexes,

(ii) recall (2.2) from [6] the existence of an equivalence $N: C \to D$ between the category $C$ of the cyclic modules of Connes and the category $D$ of these cyclic chain complexes,

(iii) construct (2.3) three closed model category structures on $D$ with as weak equivalences, respectively, the homology isomorphisms, the cohomology isomorphisms and the maps which are both, and

(iv) observe that (2.4) the corresponding (under $N$) closed model category structures on $C$ have as weak equivalences the maps considered in 1.1 and that it thus [6, § 7] makes sense to talk of 'the homotopy theories of $C$ with respect to these subcategories of weak equivalences'.

2.1. Cyclic chain complexes. Let $R$ be a ring with $1 \neq 0$ and let $R(\delta, \delta)$ be the category of duchain complexes over $R$ (1.2(i)). Then we denote by $D \subset R(\delta, \delta)$ the full subcategory spanned by the cyclic chain complexes, i.e., the objects $X \in R(\delta, \delta)$ such that

$$f_{n-1}(\delta \delta) f_n(\delta \delta) x = x \quad \text{for all } x \in X_n, \quad n > 0,$$
where \( f_i(t) = (1 + (-1)^i t)^{i+1} \) for \( i \geq 0 \).

This definition and terminology is justified by [6, 6.6] which states:

2.2. Proposition. Let \( C \) be the category of cyclic \( R \)-modules (1.1). Then the normalization functor of \([6, 3.3]\) induces an equivalence of categories \( N: C \to D \). Moreover, for every object \( X \in C \), this functor induces natural isomorphisms (see 1.1)

\[
\pi_i j^* X = H_1 NX \quad \text{and} \quad \pi_i^k X = H_i^1 NX \quad (i \geq 0).
\]

Now we construct the

2.3. Three model category structures for \( D \). (i) \( D \) admits a closed model category structure \([4, \S \, 3]\) in which a map is a weak equivalence iff it induces isomorphisms on the homology groups and a fibration iff it is onto in dimensions \( > 0 \).

(ii) \( D \) admits a closed model category structure in which a map is a weak equivalence iff it induces isomorphisms on the cohomology groups and a fibration iff it is onto.

(iii) \( D \) admits a closed model category structure in which a map is a weak equivalence iff it induces isomorphisms on the homology groups and the cohomology groups and a fibration iff it is onto in dimensions \( > 0 \).

In view of 2.2 this implies

2.4. Corollary. The category \( C \) admits three corresponding closed model category structures, with as weak equivalences the maps which induce isomorphisms on the homotopy groups of the underlying simplicial modules, the cohomotopy groups of the underlying cosimplicial modules, or both.

To prove 2.3 we need the various

2.5. Spheres and balls in \( D \). (i) For every integer \( k \geq 0 \), let \( B_k \in D \) be the object freely generated by one element \( w_k \) in dimension \( k \) and let \( B^{-1} = 0 \).

(ii) For every integer \( k \geq 0 \), let \( S^k_\partial \in D \) be the object generated by an element \( u_k \) in dimension \( k \), subject to the relation \( \partial u_k = 0 \) and let \( S^{-1}_\partial = 0 \).

(iii) For every integer \( k \geq 0 \), let \( S_k \in D \) be the object generated by an element \( v_k \) in dimension \( k \), subject to the relation \( \partial v_k = 0 \) and let \( S^{-1}_\delta = 0 \).

There are obvious maps

\[
S^k_\partial \to B^{k+1} \quad \text{and} \quad S^{k+1}_\delta \to B^{k+1}
\]
given by \( u_k \to \partial w_{k+1} \) and \( v_k \to \delta w_{k-1} \). Moreover a straightforward calculation yields.
2.6. Proposition

\[ H_k B_k = 0 \quad \text{for } k \geq 1, \]
\[ H_k B_k = 0 \quad \text{for } k \geq 0. \]

2.7. Proof of 2.3. This is the same as the proof of [4, 3.11] using the fact that a map in \( D \) is a fibration (resp. a trivial fibration) iff it has the right lifting property with respect to the maps

(i) \( 0 \rightarrow B_k^k (k \geq 1) \) (resp. \( S_k \rightarrow B_{k+1}^k (k \geq -1) \)),
(ii) \( 0 \rightarrow B_k^k (k \geq 0) \) (resp. \( S_k \rightarrow B_{k-1}^k (k \geq 0) \)), or
(iii) \( 0 \rightarrow B_k^k (k \geq 1) \) (resp. \( S_k \rightarrow B_{k+1}^k (k \geq -1) \) and \( S_k \rightarrow B_{k-1}^k (k \geq 0) \)).

2.8. Corollary. In the closed module category structures for \( D \) of 2.3, the cofibrations are the retracts of the (possibly transfinite) compositions of cobase extensions of the maps

(i) \( S_k \rightarrow B_{k+1}^k (k \geq -1) \),
(ii) \( S_k \rightarrow B_{k-1}^k (k \geq 0) \), or
(iii) \( S_k \rightarrow B_{k+1}^k (k \geq -1) \) and \( S_k \rightarrow B_{k-1}^k (k \geq 0) \).

2.9. Corollary. In the closed model category structures for \( D \) (2.3) the factorizations can be made functorial, i.e., every map \( f \in D \) admits functorial factorizations:

(i) \( f = pi, \) where \( i \) is a cofibration and \( p \) is a trivial fibration (i.e., a fibration as well as a weak equivalence), and
(ii) \( f = pi, \) where \( p \) is a fibration and \( i \) is a trivial cofibration (i.e., a cofibration as well as a weak equivalence).

Another easy consequence is

2.10. Corollary. In the first closed model category structure for \( D \) (2.3(i)), every cofibration is \( 1 \)-\( 1 \) and induces isomorphisms on the cohomology groups. Hence every cofibrant object has trivial cohomology.

3. Extended cyclic chain complexes

Next we

(i) define (3.1) extended cyclic chain complexes,
(ii) obtain (3.2) closed model category structures for the category \( D \) of these extended cyclic chain complexes and the category \( \mathcal{E} \) (1.1) of differential graded modules with an exterior action with, in both, the homology isomorphisms as weak equivalences, and
(iii) construct (3.3) a functor \( E : D \rightarrow \mathcal{E} \) which preserves these weak equivalences
and which induces equivalences of homotopy theories \( D_+ \to \tilde{E}_+, \tilde{D}_- \to \tilde{E}_- \) and \( (\tilde{D}_-, \tilde{D}_+) \to (\tilde{E}_-, \tilde{E}_+) \).

3.1. Extended cyclic chain complexes. Let \( \tilde{R}(\partial, \delta) \) be the category of extended duchain complexes (1.2(ii)) and denote by \( \tilde{D} \subset \tilde{R}(\partial, \delta) \) the full subcategory spanned by the extended cyclic chain complexes, i.e., the objects \( U \in \tilde{R}(\partial, \delta) \) such that

\[
f_{n-1}(\partial \delta)f_n(\partial \delta)u = u \quad \text{for all } u \in U_n \text{ and } n,
\]

where \( f_i(t) \) is as in 2.1 for \( i \geq 0 \) and \( f_i(t) = f_{-i-1}(t) \) for \( i < 0 \).

The arguments of 2.7 then yield the following

3.2. Closed model category structures for \( \tilde{D} \) and \( \tilde{E} \). The categories \( \tilde{D} (3.1) \) and \( \tilde{E} (1.1) \) each admit a closed model category structure with functorial factorizations (2.9), in which

(i) a map is a weak equivalence iff it induces isomorphisms on the homology groups,

(ii) a map is a fibration iff it is onto, and

(iii) a map is a cofibration iff it is a retract of a (possibly transfinite) composition of cobase extensions of the inclusions \( S^k \to B^{k+1} \), where \( B^{k+1} \) denotes the free object on a single generator \( w_{k+1} \) in dimension \( k+1 \) and \( S^k \) is its sub duchain complex generated by the element \( \partial w_{k+1} \).

Moreover, every cofibration is 1–1 and a cohomology isomorphism and hence every cofibrant object has trivial cohomology.

We end with considering

3.3. The functor \( E : \tilde{D} \to \tilde{E} \). This is the functor which sends an object \( U \in \tilde{D} \) to the object \( EU \in \tilde{E} \) such that \( EU_n = U_n \) for all \( n \) and with \( \partial \) and \( \delta \) defined as follows. For every integer \( n \), let \( h_n(t) \) denote the polynomial such that (in the notation of 3.1) \( 1 + t h_n(t) = f_n(t) \) and, for every element \( u \in U_n \), let \( Eu \) denote the corresponding element of \( EU_n \). Then

\[
\partial Eu = E\partial u \quad \text{and} \quad \delta Eu = E\delta h_n(\partial \delta)u = Eh_n(\partial \delta)\delta u
\]

for all \( u \in U_n \) and \( n \).

This functor has the property:

3.4. Proposition. The functor \( E : \tilde{D} \to \tilde{E} \) induces an equivalence of homotopy theories (i.e. [6, 7.8], \( E \) preserves weak equivalences and induces a weak equivalence between the simplicial localizations of \( \tilde{D} \) and \( \tilde{E} \) with respect to the weak equivalences) and so do the induced functors \( \tilde{D}_+ \to \tilde{E}_+, \tilde{D}_- \to \tilde{E}_- \) and \( (\tilde{D}_-, \tilde{D}_+) \to (\tilde{E}_-, \tilde{E}_+) \) (where the weak equivalences in the last two categories are as in 1.2(iii) and 1.1).
Proof. Let $D : \mathcal{E} \to \mathcal{D}$ be the left adjoint of $E$. As $E$ preserves fibrations and as a map $U \to U' \in \mathcal{D}$ is a weak equivalence iff the induced map $EU \to EU' \in \mathcal{E}$ is so, the functor $D$ preserves cofibrations and trivial cofibrations and hence [1, 1.2 and 1.3] weak equivalences between cofibrant objects. It is not difficult to see [5, 5.4] that, in order to prove the first part of 3.4, it now suffices to show that, for every cofibrant object $V \in \mathcal{E}$, the adjunction map $V \to EDV \in \mathcal{E}$ is a weak equivalence. This is done by combining 3.2(iii) with the observation that

(i) this statement is true if $V = S^k$ and $V = B^{k+1}$, and

(ii) the functors $D$ and $E$ both preserve push outs.

3.5. Remark. One might wonder why we only considered the closed model category structure on $\mathcal{D}$ (or $\mathcal{E}$) in which the weak equivalences were the homology isomorphisms and not the cohomology isomorphisms or both. The reason is that this would not really have produced anything new, as 4.1 readily implies that

(i) the homotopy theory of $\mathcal{D}$ (or $\mathcal{E}$) with respect to the cohomology isomorphisms is equivalent to the one with respect to the homology isomorphisms considered above, and

(ii) the homotopy theory of $\mathcal{D}$ (or $\mathcal{E}$) with respect to the maps which are both homology isomorphisms and cohomology isomorphisms is equivalent to the homotopy theory of $\mathcal{D} \times \mathcal{D}$ (or $\mathcal{E} \times \mathcal{E}$) with respect to the 'pairs of homology isomorphisms'.

4. The functor $F : \mathcal{D} \to (\mathcal{D}_-, \mathcal{D}_+)$

Finally we discuss the functor $F : \mathcal{D} \to (\mathcal{D}_-, \mathcal{D}_+)$ which was mentioned in 1.2(ii). Throughout this section the category $\mathcal{D}$ will be considered as a full subcategory of the category $\mathcal{D}$.

We start with some preliminaries.

4.1. A duality functor. This will be the functor which sends an object $U \in \mathcal{D}$ to the object $U* \in \mathcal{D}$ with $U_n* = U_{-n}$ for all $n$ and $\partial$ defined as follows: if, for $u \in U_{-n}$, $u*$ denotes the corresponding element of $U_n*$, then

$$\partial u* = (\delta u)*$$ and $$\delta u* = (\partial u)*.$$  

Clearly $U** = U$ and $H^n U* = H_{-n} U$ for all $n$.

4.2. The folding functor. This is the functor $v : \mathcal{D} \to \mathcal{D}$ which sends an object $U \in \mathcal{D}$ to the object $vU \in \mathcal{D}$ given by

$$vU_n = U_n \oplus U_{-n}, \quad n > 0,$$

$$= U_0, \quad n = 0$$

with the obvious $\partial$'s and $\delta$'s, i.e., if $u_i \in U_i$, then
\[ \partial(u_n, u_{-n}) = (\partial u_n, \partial u_{-n}), \quad n > 1, \]
\[ \partial(u_1, u_{-1}) = \partial u_1 + \partial u_{-1}, \]
\[ \partial(u_n, u_{-n}) = (\partial u_n, \partial u_{-n}), \quad n > 0, \]
\[ \partial u_0 = (\partial u_0, \partial u_0). \]

Note that there is an obvious isomorphism \( vU^* \approx vU \).

4.3. The adjoints of the inclusion \( D \to \tilde{D} \). The left adjoint \( c : \tilde{D} \to D \) and the right adjoint \( c' : \tilde{D} \to D \) of the inclusion functor \( D \to \tilde{D} \) are given by
\[ cU_n = U_n, \quad c'U_n = U_n, \quad n > 0, \]
\[ cU_0 = U_0/(\text{im } \delta), \quad c'U_0 = U_0 \cap (\ker \delta) \]
for every object \( U \in \tilde{D} \). They clearly have the properties:

(i) If \( H^iU = 0 \) for \( i < 0 \), then the projection \( U \to cU \) induces isomorphisms on the cohomology groups.

(ii) If \( H_iU = 0 \) for \( i < 0 \), then the inclusion \( c'U \to U \) induces isomorphisms on the homology groups.

(iii) If \( H^iU = 0 \) for \( i < 0 \), then the obvious map \( vU \to cU \) induces isomorphisms on the homology groups.

Now we are ready to describe

4.4. The functor \( F : D \to (\tilde{D}_, \tilde{D}+) \). Endow \( D \) and \( \tilde{D} \) with the closed model category structures in which the weak equivalences are the homology isomorphisms (2.3(i) and 3.2) and denote fibrations, trivial fibrations, cofibrations, trivial cofibrations and weak equivalences by \( \to \) and \( \sim \) respectively. Given \( X \in D \), let \( 0 \to U \sim X^* \) be a functorial factorization (3.2) of the map \( 0 \to X^* \), let \( vU^* \sim Y \to X \) be a functorial factorization (2.9) of the resulting composition \( vU^* \to vU \to cX = X \), and define \( FX \in (\tilde{D}_, \tilde{D}+) \) as the composition \( U \to cU \to Z \), where \( cU \to Z \) is the push out of \( vU^* \to vU \) along the composition \( vU^* = vU \sim cU \). Our main result then is

4.5. Theorem. The functor \( F : D \to (\tilde{D}_, \tilde{D}+) \) induces an equivalence between the two-sided homotopy theory of \( D \) (2.3(iii)) and the homotopy theory of \( (\tilde{D}_, \tilde{D}+) \) (3.4), i.e. \([6, 7.8]\), \( F \) preserves the appropriate weak equivalences and induces a weak equivalence between the simplicial localizations of \( D \) and \( (\tilde{D}_, \tilde{D}+) \) with respect to their respective weak equivalences.

This theorem readily implies the following result which is also not difficult to prove directly:
4.6. Corollary. The compositions
\[ D \xrightarrow{F} (\tilde{D}_-, \tilde{D}_+) \xrightarrow{\text{proj.}} \tilde{D}_+ \quad \text{and} \quad D \xrightarrow{F} (\tilde{D}_-, \tilde{D}_+) \xrightarrow{\text{proj.}} \tilde{D}_- \]
induce equivalences between the one-sided homotopy theories of $D$ (2.3(i) and 2.3(ii)) and the homotopy theories of $\tilde{D}_+$ and $\tilde{D}_-$ respectively.

It thus remains to give a

4.7. Proof of 4.5. We first note that the functor $F$ admits a factorization
\[ D \xrightarrow{F_1} A \xrightarrow{F_2} B \xrightarrow{F_3} (\tilde{D}_-, \tilde{D}_+) \]
where $A$ and $B$ are as follows. The category $A$ has objects the pairs $(U, cU^*: X)$, where $U \in \tilde{D}$ is a cofibrant object such that $H_iU = 0$ for $i > 0$ and $cU^*: X \in D$. A map in $A$ consists of a pair of maps $(f: U \to V \in \tilde{D}, g: X \to Y \in D)$ such that the resulting diagram
\[
\begin{array}{ccc}
  cU^* & \longrightarrow & X \\
  \downarrow^{cf^*} & & \downarrow^g \\
  cV^* & \longrightarrow & Y
\end{array}
\]
commutes and such a map will be called a weak equivalence if $f$ and $g$ are weak equivalences (i.e., homology isomorphisms). The category $B$ is similar with as objects the pairs $(U, cU^*: X)$, where $U \in \tilde{D}$ is a cofibrant object such that $H_iU = 0$ for $i > 0$ and $cU^*: X \in D$.

Next we consider functors
\[ D \leftarrow G_1 \quad A \leftarrow G_2 \quad B \leftarrow G_3 \quad (\tilde{D}_-, \tilde{D}_+) \]
where $G_2$ is similar to $F_2$ and $G_3$ and $G_1$ are defined as follows. Given an object $U \xrightarrow{W} \in (\tilde{D}_-, \tilde{D}_+)$, let $U \xrightarrow{\sim} V \xrightarrow{W}$ be a functorial factorization of the map $U \xrightarrow{W}$, let $T \xrightarrow{c'W}$ be the pull back of $V \xrightarrow{W}$ along $c'W \xrightarrow{\sim} W$ and let $0 \xrightarrow{S} T$ be a functorial factorization of the map $0 \xrightarrow{T}$. Then $G_3(U \xrightarrow{W} = (S, cS \xrightarrow{c'W}) \in B$, where $cS \xrightarrow{c'W}$ is the composition $cS \xrightarrow{c} T \xrightarrow{cc'W} = c'W$. Given an object $(U, cU^*: Y) \in A$, let $0 \xrightarrow{X'} \xrightarrow{cU^*}$ be a functorial factorization of the map $0 \xrightarrow{cU^*} Y$ and let $X' \xrightarrow{\sim} Y' \xrightarrow{Y}$ be a functorial factorization of the composition $X' \xrightarrow{\sim} cU^* \xrightarrow{\sim} Y$. Then we define $G_1(U, cU^*: Y)$ as the push out of the diagram $cU^* \xrightarrow{\sim} X' \xrightarrow{\sim} Y'$.

Finally one has, of course, that all these constructions make sense and have the desired properties, i.e.

(i) the functors $F_i$ and $G_i$ ($i = 1, 2, 3$) are well defined,
(ii) the functors $F_i$ and $G_i$ $(i = 1, 2, 3)$ preserve the appropriate weak equivalences, and
(iii) the compositions $F_i G_i$ and $G_i F_i$ $(i = 1, 2, 3)$ are naturally weakly equivalent to the appropriate identity functors.

This is a lengthy but straightforward calculation (combining standard model category arguments with 2.10, the corresponding statement in 3.2 and 4.1–3) and will be left to the reader.

References