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Problems for elliptic singular equations with a quadratic gradient term

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Abstract

We investigate the homogeneous Dirichlet problem for a class of second-order elliptic partial differential equations with a quadratic gradient term and singular data. In particular, we study the asymptotic behaviour of the solution near the boundary under suitable assumptions on the growth of the coefficients of the equation.

Keywords: Nonlinear elliptic singular equations; Boundary behaviour

1. Introduction

In this paper we investigate positive solutions of the Boundary Value Problem (BVP)

$$\Delta u + g(u)|\nabla u|^2 + f(u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{1}$$

under various conditions on the functions g and f, with the aim to get existence and uniqueness results and to describe the asymptotic behaviour near the boundary of D. We always assume $D \subset \mathbb{R}^N$, $N \geqslant 2$, and suppose $g(t)$ and $f(t)$ to be continuous for $t > 0$, with $f(t) > 0$. Moreover $f(t)$ may have a singularity at $t = 0$, as well as $g(t)$. Problems of this kind have been extensively studied in the literature, see for instance [5,6,8–10,15,18].

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In particular [15] is concerned with the semilinear equation

$$\Delta u + p(x)u^{-\alpha} = 0,$$

where $\alpha > 0$, while in [5] the equation

$$Lu + f(x,u) = 0,$$

for a linear second-order (uniformly) elliptic operator L is considered.

The influence of a nonlinear gradient term is discussed in [10]. In the case of the BVP

$$\Delta u + g(u)|\nabla u|^q + f(u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where $0 < q < 2$ and the functions f and g are decreasing, in [10] it is shown that, when g is not too "large" compared with f, the term $g(u)|\nabla u|^q$ can be viewed as a small perturbation.

Here we treat the borderline case $q = 2$. For a general g, the gradient term may have the same "size" of the other terms. Therefore, the sign of the coefficient g could be relevant. In fact, if g is negative and "large" with respect to f, we may expect to have existence of positive solutions for BVP (1) under weaker conditions on f with respect to the semilinear case

$$\Delta u + f(u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{2}$$

To show this, let us consider the model Boundary Value Problem

$$\Delta u - \frac{\beta}{u}|\nabla u|^2 + u^{-\alpha} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{3}$$

with two real parameters β and α. If $\beta = 0$, from the semilinear case (see Lemma 1 below) we have existence of positive solutions provided that $\alpha \geqslant 0$. On the other side, as we will see in the sequel, when $\beta > 0$, to have existence for BVP (3) we will only need $\alpha > \max[-1, -\beta]$.

In Section 2 of our paper, by a suitable transformation $u = h(w)$ the equation in (1) will be transformed into a semilinear equation without gradient term. It should be also outlined that the transformation $u = h(w)$ may either leave unchanged the boundary condition, i.e. $w = 0$ on ∂D, or lead to the boundary condition $w \to \infty$ as $x \to \partial D$. In the latter case, we get a boundary blow-up problem, for which a satisfactory theory has been well developed in the last years, see for instance [1,4,7,13,16,17].

In Section 3 a direct method will be employed, which is performed with the aid of a Cauchy problem for a linked ordinary differential equation. First we discuss the radial case, then we use the method of Kazdan and Kramer [12] for a general domain. The main result (Theorem 5) gives the existence of a solution in some situation in which the method of Section 2 is not applicable (see Remark 2).

In Section 4 we investigate the behaviour of the solution near the boundary. Assuming a sufficiently fast growth of $f(t)$ as $t \to 0^+$, i.e. $\int_0^1 f(t)\,dt = +\infty$, we show that, if u is a solution of the semilinear BVP (2) then

$$\lim_{x \to \partial D} \frac{u(x)}{\phi(\delta(x))} = 1,$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$ and $\phi(s)$ is defined, for s small, as

$$\int_0^{\phi(s)} \frac{dt}{\sqrt{2\int_t^1 f(\tau)\,d\tau}} = s.$$

We will also obtain similar results for solutions $u(x)$ of the quasilinear BVP (1) with a quadratic gradient term. We note that, the first approximation of $u(x)$ may be influenced by g. For instance, in the case of the Boundary Value Problem (3) with $0 \leqslant \beta$ and $\alpha > \max[-1, 1-2\beta]$, we get (see Section 4)

$$\lim_{x\to\partial D} \frac{u(x)}{\Phi(\delta(x))} = 1, \qquad \Phi(s) = \left(\frac{\alpha+1}{\sqrt{2(\alpha+2\beta-1)}} s\right)^{\frac{2}{\alpha+1}}.$$

2. Existence via a substitution

Let us recall a result concerning the special Boundary Value Problem

$$\Delta u + f(u) = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{4}$$

Lemma 1. *Let D be a bounded smooth domain of $\mathbb{R}^N$ and let $f(t) > 0$ be a continuous non-increasing function in $(0,\infty)$. Then BVP (4) has a unique positive classical solution.*

Proof. It can be found in [5]. Note that no conditions are imposed on $f(t)$ for $t \to 0$. In particular, we may have $f(t) \to \infty$ as $t \to 0$ (singular equations). □

Consider now the general BVP (1). Define

$$G(t) = \int_1^t g(\tau)\,d\tau. \tag{5}$$

Theorem 1. *Let D be a bounded smooth domain of $\mathbb{R}^N$. Let $f(t)$ and $g(t)$ be continuous functions in $(0,\infty)$, with $f(t) > 0$. If*

$$\int_0^1 e^{G(t)}\,dt < \infty, \tag{6}$$

and if the function $f(t)e^{G(t)}$ is non-increasing then BVP (1) has a unique positive classical solution.

Proof. For $s > 0$, define $h(s)$ so that

$$\int_0^{h(s)} e^{G(t)}\,dt = s. \tag{7}$$

Note that $h(s)$ is positive and increasing with $h(0) = 0$. Moreover we have

$$h''(s) + g\big(h(s)\big)\big(h'(s)\big)^2 = 0.$$

Therefore, by the transformation $u = h(w)$, the BVP (1) becomes:

$$\Delta w + \tilde{f}(w) = 0 \quad \text{in } D, \qquad w = 0 \quad \text{on } \partial D,$$

where

$$\tilde{f}(s) = f\big(h(s)\big)e^{G(h(s))}.$$

Since $h(s)$ is increasing and $f(t)e^{G(t)}$ is non-increasing, the function $\tilde{f}(s)$ is non-increasing. The assertion of the theorem follows by Lemma 1. $\square$

An example which satisfies the assumptions of Theorem 1 is the following:

$$\Delta u - \frac{\beta}{u}|\nabla u|^2 + u^{-\alpha} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

with $0 < \beta < 1$ and $\alpha > -\beta$. We find $G(t) = \log(t^{-\beta})$ and $f(t)e^{G(t)} = t^{-\alpha-\beta}$.

To discuss the case condition (6) fails to hold, we recall the following result on boundary blow-up problems.

Lemma 2. *Let $D \subset \mathbb{R}^N$ be a bounded smooth domain. Suppose that, either*

(i) *$f(t)$ is continuous, increasing in $[0, \infty)$ and satisfying $f(0) = 0$; or*
(ii) *$f(t)$ is non-negative and continuous in $(-\infty, \infty)$, increasing in $(0, \infty)$, and satisfying*

$$\int_{-\infty}^{0} f(t)\,dt < \infty.$$

In addition, assume $f(t)$ satisfies the Keller–Osserman condition

$$\int_{1}^{\infty} \frac{1}{\sqrt{F(t)}}\,dt < \infty, \qquad F'(t) = f(t). \tag{8}$$

Then the problem

$$\Delta u = f(u) \quad \text{in } D, \qquad u \to \infty \quad \text{as } x \to \partial D, \tag{9}$$

has a classical solution.

Proof. See [13,17]. $\square$

Theorem 2. *Let $D \subset \mathbb{R}^N$ be a bounded smooth domain. Let $f(t)$ and $g(t)$ be continuous functions in $(0, +\infty)$, with $f(t) > 0$. With $G(t)$ defined as in (5), let*

$$\int_{0}^{1} e^{G(t)}\,dt = \infty, \qquad \int_{1}^{\infty} e^{G(t)}\,dt < \infty. \tag{10}$$

Moreover, suppose that $f(t)e^{G(t)}$ is decreasing and that $f(t)e^{G(t)} \to 0$ as $t \to \infty$. Finally, if $h(s)$ is defined, for $s > 0$, as

$$\int_{h(s)}^{\infty} e^{G(t)}\,dt = s, \tag{11}$$

we suppose that the function

$$F(t) = \int_{0}^{t} f\big(h(s)\big)e^{G(h(s))}\,ds$$

satisfies the condition

$$\int_1^{\infty} \frac{1}{\sqrt{F(t)}}\, dt < \infty. \tag{12}$$

Then BVP (1) *has a classical solution.*

Proof. The function $h(s)$ is decreasing and $h(s) \to 0$ as $s \to \infty$. Moreover,

$$h''(s) + g\big(h(s)\big)\big(h'(s)\big)^2 = 0.$$

Therefore, putting $u = h(w)$, problem (1) reads as

$$\Delta w = \tilde{f}(w) \quad \text{in } D, \qquad w \to \infty \quad \text{as } x \to \partial D, \tag{13}$$

with

$$\tilde{f}(s) = f\big(h(s)\big)e^{G(h(s))}.$$

We have found a boundary blow-up problem. Let us show that the assumptions of Lemma 2 are fulfilled. Since $h(s)$ and $f(t)e^{G(t)}$ are decreasing, the function $\tilde{f}(s)$ is increasing. Since $h(s) \to \infty$ as $s \to 0$ and $f(t)e^{G(t)} \to 0$ as $t \to \infty$ we have that $\tilde{f}(s) \to 0$ as $s \to 0$. Therefore, condition (i) of Lemma 2 holds. Moreover, $F'(s) = \tilde{f}(s)$, and assumption (12) yields the Keller–Osserman condition of Lemma 2, from which the assertion of our theorem follows. □

Consider the following example:

$$\Delta u - \frac{\beta}{u}|\nabla u|^2 + u^{-\alpha} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

with $\beta > 1$ and $\alpha > -1$. We find $G(t) = \log(t^{-\beta})$, $h(s) = ((\beta - 1)s)^{\frac{1}{1-\beta}}$ and

$$F(t) = \frac{1}{2\beta + \alpha - 1}\big((\beta - 1)t\big)^{\frac{2\beta+\alpha-1}{\beta-1}}.$$

All the assumptions of Theorem 2 are fulfilled.

Theorem 3. *Let $D \subset \mathbb{R}^N$ be a bounded smooth domain. Let $f(t)$ and $g(t)$ be continuous functions in* $(0, +\infty)$*, with $f(t) > 0$. With $G(t)$ defined as in* (5)*, let*

$$\int_0^1 e^{G(t)}\, dt = \infty, \qquad \int_1^{\infty} e^{G(t)}\, dt = \infty. \tag{14}$$

Moreover, suppose that $f(t)e^{G(t)}$ is decreasing and that $\int_1^{\infty} f(t)e^{2G(t)}\, dt < \infty$. Finally, if $h(s)$ is defined as

$$\int_{e^{-h(s)}}^{1} e^{G(t)}\, dt = s, \tag{15}$$

we suppose that the function

$$F(t) = \int_{-\infty}^{t} f\big(e^{-h(s)}\big)e^{G(e^{-h(s)})}\, ds \tag{16}$$

satisfies the condition

$$\int_1^\infty \frac{1}{\sqrt{F(t)}}\,dt < \infty. \tag{17}$$

Then BVP (1) *has a classical solution.*

Proof. Note that $h(s)$ is increasing, $h(0)=0$, $h(s)\to+\infty$ as $s\to+\infty$ and $h(s)\to-\infty$ as $s\to-\infty$. Moreover,

$$\frac{h''(s)}{h'(s)} - h'(s) - g\bigl(e^{-h(s)}\bigr)e^{-h(s)}h'(s) = 0.$$

Hence, putting $u = e^{-h(w)}$ we find

$$\Delta w = \tilde f(w) \quad \text{in } D, \qquad w\to\infty \quad \text{as } x\to\partial D,$$

with

$$\tilde f(s) = f\bigl(e^{-h(s)}\bigr)e^{G(e^{-h(s)})}.$$

We have found again a boundary blow-up problem. Let us show that the assumptions of Lemma 2 are fulfilled. Since $e^{-h(s)}$ and $f(t)e^{G(t)}$ are decreasing, the function $\tilde f(s)$ is increasing. Furthermore, since $h'(s) = e^{h(s)}e^{-G(e^{-h(s)})}$, recalling the assumptions of the theorem we find

$$\int_{-\infty}^0 \tilde f(s)\,ds = \int_{-\infty}^0 f\bigl(e^{-h(s)}\bigr)e^{G(e^{-h(s)})}\,ds = \int_1^\infty f(t)e^{2G(t)}\,dt < \infty.$$

Therefore, condition (ii) of Lemma 2 holds. Finally, since $F'(t) = \tilde f(t)$, by assumption (17), also the Keller–Osserman condition of Lemma 2 is satisfied. The assertion of our theorem follows by Lemma 2. □

Consider the following example:

$$\Delta u - \frac{1}{u}|\nabla u|^2 + u^{-\alpha} = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

with $\alpha > -1$. We find $G(t) = \log(t^{-1})$, $h(s) = s$ and $F(t) = (1+\alpha)^{-1}e^{(1+\alpha)t}$. All the assumptions of Theorem 3 are fulfilled.

In all the previous examples, we note that the presence of the (negative) term $-\frac{\beta}{u}|\nabla u|^2$ with $0<\beta$ in the equation enlarges the class of admissible functions ($u^{-\alpha}$ with $\alpha > \max[-\beta, -1]$ instead of $\alpha \geqslant 0$). We may ask if this fact is general. We have the following results. Consider the problem (1) with $g(t)\leqslant 0$, and let $G(t)$ be defined as in (5).

(i) If condition (6) holds, then by Theorem 1 we have existence provided $f(t)e^{G(t)}$ is decreasing. Since $e^{G(t)}$ is decreasing (recall that we are assuming $g(t)\leqslant 0$), we see that the presence of the term $g(u)|\nabla u|^2$ in the equation enlarges the class of admissible functions f.

(ii) If (10) holds, then we use Theorem 2. The function $f(t)e^{G(t)}$ is non-increasing when $f(t)$ is decreasing, and

$$\lim_{t\to\infty} f(t)e^{G(t)} = 0 \quad \text{when } \lim_{t\to\infty} f(t) = 0.$$

Moreover, concerning the Keller–Osserman condition (12), we can evaluate the integral

$$I = \int_{h^{-1}(1)}^{\infty} \frac{dt}{\sqrt{\int_{h^{-1}(1)}^{t} f(h(s))e^{G(h(s))}\, ds}}$$
$$= \int_{h^{-1}(1)}^{\infty} \frac{dt}{\sqrt{\int_{h(t)}^{1} f(\sigma)e^{2G(\sigma)}\, d\sigma}} = \int_{0}^{1} \frac{e^{G(\rho)}\, d\rho}{\sqrt{\int_{\rho}^{1} f(\sigma)e^{2G(\sigma)}\, d\sigma}}.$$

If $f(t)$ is decreasing then we have

$$I \leqslant \int_{0}^{1} \frac{e^{G(\rho)}\, d\rho}{\sqrt{\int_{\rho}^{1} f(1)e^{2G(\sigma)}\, d\sigma}},$$

and we have to check the boundedness of the integral

$$\int_{0}^{1} \sqrt{A(\rho)}\, d\rho, \tag{18}$$

with

$$A(\rho) = \frac{e^{2G(\rho)}}{\int_{\rho}^{1} e^{2G(\sigma)}\, d\sigma}.$$

We use the following condition:

$$\exists\gamma \in [1, 2) \text{ such that } \lim_{\rho\to 0} \rho^{\gamma} g(\rho) = -L, \tag{19}$$

with $L > 0$ if $1 < \gamma < 2$ and with $L > 1/2$ if $\gamma = 1$. Then we have

$$\lim_{\rho\to 0} A(\rho)\rho^{\gamma} = \lim_{\rho\to 0} \frac{\rho^{\gamma} e^{2G(\rho)}}{\int_{\rho}^{1} e^{2G(\sigma)}\, d\sigma}$$
$$= \lim_{\rho\to 0} \frac{\gamma\rho^{\gamma-1}e^{2G(\rho)} + \rho^{\gamma}e^{2G(\rho)}2g(\rho)}{-e^{2G(\rho)}} = M,$$

with $M = 2L$ if $\gamma > 1$ and $M = 2L - 1$ if $\gamma = 1$. Therefore, the integral (18) is finite in this situation. Hence, if condition (19) holds, the presence of the term $g(u)|\nabla u|^2$ in the equation enlarges the class of admissible functions f.

(iii) If (14) holds, then we use Theorem 3. Suppose that condition (19) holds and that $\int_1^{\infty} f(t)\, dt < \infty$. Then, if $f(t)$ is decreasing, also $f(t)e^{G(t)}$ is decreasing, and the integral $\int_1^{\infty} f(t)e^{2G(t)}\, dt$ is finite. Concerning the condition (17), we can evaluate the integral

$$I = \int_{0}^{\infty} \frac{dt}{\sqrt{\int_{0}^{t} f(e^{-h(s)})e^{G(e^{-h(s)})}\, ds}}$$
$$= \int_{0}^{\infty} \frac{dt}{\sqrt{\int_{e^{-h(t)}}^{1} f(\sigma)e^{2G(\sigma)}\, d\sigma}} = \int_{0}^{1} \frac{e^{G(\rho)}\, d\rho}{\sqrt{\int_{\rho}^{1} f(\sigma)e^{2G(\sigma)}\, d\sigma}}.$$

From now on the discussion and the conclusion are the same as in the previous case.

3. Existence via a direct method

To prove Theorem 1 we have used Lemma 1. Actually, the condition used in that lemma (that is $f'(t) \leqslant 0$) is not necessary to have existence. Therefore, we develop here a direct method which gives the existence result in some situation for which Theorem 1 does not apply.

We start to investigate the radial case. Let B be a ball of $\mathbb{R}^N$. We consider radial positive solutions of the BVP

$$\Delta u + g(u)|\nabla u|^2 + f(u) = 0 \quad \text{in } B, \qquad u = 0 \quad \text{on } \partial B, \tag{20}$$

where g and f are continuous functions in $(0, +\infty)$ with $f(t) > 0$. We do not assume $g(t)$ and $f(t)$ to be bounded neither as $t \to 0$ nor as $t \to \infty$.

We will need some qualitative theory of classical solutions of the Cauchy Problem

$$v'' + \frac{N-1}{r} v' + g(v)|v'|^2 + f(v) = 0, \qquad v(0) = v_0, \qquad v'(0) = 0, \tag{21}$$

for a positive constant v_0.

Lemma 3. *Suppose that $f > 0$ and both f and g are continuous non-increasing functions in $(0, +\infty)$. Then a classical solution v of Cauchy Problem (21) is a decreasing and concave function.*

Proof. See the proof of Lemma 2.1 of [10]. Note that the statement of Lemma 2.1 of [10] requires f to be decreasing, but, for $N > 1$, the proof works also with f non-increasing. □

Using Lemma 3 we show that Cauchy Problem (21) can also be used to solve the Boundary Value Problem for the ordinary differential equation

$$v'' + \frac{N-1}{r} v' + g(v)(v')^2 + f(v) = 0, \qquad v'(0) = 0, \qquad v(R) = 0. \tag{22}$$

Lemma 4. *Let $f > 0$ and both f and g be continuous non-increasing functions in $(0, +\infty)$. If $v \in C^2([0, R))$ is the maximal positive solution of (21) then*

$$\lim_{r \to R^-} v(r) = 0. \tag{23}$$

Proof. Define $g^+(t) = \max[g(t), 0]$. Multiplying Eq. (22) by

$$e^{-2\int_v^{v_0} g^+(t)\,dt} v'$$

we find

$$v'' e^{-2\int_v^{v_0} g^+(t)\,dt} v' + g^+(v)(v')^2 e^{-2\int_v^{v_0} g^+(t)\,dt} v' + f(v) e^{-2\int_v^{v_0} g^+(t)\,dt} v' \leqslant 0,$$

which implies

$$\left(\frac{(v')^2}{2} e^{-2\int_v^{v_0} g^+(t)\,dt} \right)' \leqslant f(v)(-v').$$

Integration over $(0, r)$ yields

$$\frac{(v')^2}{2} \leqslant e^{2\int_v^{v_0} g^+(t)\,dt} \int_v^{v_0} f(t)\,dt. \tag{24}$$

Since $v'(0) = 0$ and $v(r)$ is concave, R must be finite. By (24) we see that $v'(r)$ is finite when $v(r) > 0$. Hence $v(R) = 0$, as it was to be shown. $\square$

We will also need the following information about the dependence of the maximal R from the initial value v_0.

Lemma 5. *Suppose that $f > 0$ and g are continuous non-increasing in $(0, \infty)$. Let $R = R(v_0)$ be the length of the maximal interval $[0, R)$ for a positive solution of (21). Then $R(v_0)$ is a continuous increasing function such that*

$$\lim_{v_0 \to 0^+} R(v_0) = 0. \tag{25}$$

Proof. Monotonicity follows from comparison results for quasilinear equations, see [11, Theorem 10.1]. Indeed, suppose $v_0 < z_0$, and let v and z be the solutions of the boundary value problems

$$\Delta v + g(v)|\nabla v|^2 + f(v) = 0 \quad \text{in } B\big(R(v_0)\big), \qquad v = 0 \quad \text{on } \partial B\big(R(v_0)\big),$$

and

$$\Delta z + g(z)|\nabla z|^2 + f(z) = 0 \quad \text{in } B\big(R(z_0)\big), \qquad z = 0 \quad \text{on } \partial B\big(R(z_0)\big).$$

First we observe that $R(v_0) \neq R(z_0)$, otherwise by the comparison principle $v = z$ in $B(R(v_0)) = B(R(z_0))$. Suppose now that $B(R(v_0)) > B(R(z_0))$. Then it would be $v \geqslant z$ on $\partial B(R(z_0))$ and hence, again by the comparison principle, $v \geqslant z$ in $B(R(z_0))$, in particular $v_0 \geqslant z_0$, a contradiction. Therefore, $R(v_0) < R(z_0)$, and monotonicity is proved.

We prove the continuity from the left. Let $v_{0,k}$ be an increasing sequence converging to v_0. Let $v(r)$ and $v_k(r)$ be, respectively, the maximal solutions corresponding to the initial values v_0 and $v_{0,k}$. If $R_0 = R(v_0)$ and $R_k = R(v_{0,k})$, by monotonicity we have

$$\lim_{k \to \infty} R_k = \sup_{k \in \mathbb{N}} R_k = \overline{R} \leqslant R_0. \tag{26}$$

We must prove that equality holds in (26). Inequality (24) for the solution $v_k(r)$ yields

$$\frac{(v_k')^2}{2} \leqslant e^{2\int_{v_k}^{v_0} g^+(t)\,dt} \int_{v_k}^{v_0} f(t)\,dt. \tag{27}$$

Fix $\epsilon > 0$ and $\delta > 0$ small. By monotonicity we have $\delta \leqslant v_k(r)$ on $[0, \overline{R} - \epsilon]$ for $k > k_{\epsilon,\delta}$. Hence, for these values of k we have

$$\frac{(v_k')^2}{2} \leqslant e^{2\int_{\delta}^{v_0} g^+(t)\,dt} \int_{\delta}^{v_0} f(t)\,dt \quad \forall r \in [0, \overline{R} - \epsilon].$$

Moreover, since $v_k(r)$ and $v_k'(r)$ are decreasing, by Lemma 3, we have

$$0 \leqslant -v_k'' = \frac{N-1}{r} v_k' + g(v_k)\big(v_k'\big)^2 + f(v_k) \leqslant g^+(\delta)\big(v_k'\big)^2 + f(\delta).$$

The last inequalities imply that $v_k(r)$ and $v_k'(r)$ are uniformly bounded and equicontinuous in $[0, \overline{R} - \epsilon]$. We can take a subsequence (denoted again) $v_k(r)$ such that $v_k(r) \to \phi(r)$ and $v_k'(r) \to \phi'(r)$ in $[0, \overline{R})$. Let us write the equation for v_k as

$$\big(r^{N-1} v_k'\big)' + r^{N-1}\big(g(v_k)\big(v_k'\big)^2 + f(v_k)\big) = 0.$$

Integrating over $(0, r)$ we find

$$r^{N-1}v_k' + \int_0^r \rho^{N-1}\big(g(v_k)\big(v_k'\big)^2 + f(v_k)\big)\,d\rho = 0.$$

Passing to the limit as $k \to \infty$ for $r < \overline{R}$ we find

$$r^{N-1}\phi' + \int_0^r \rho^{N-1}\big(g(\phi)(\phi')^2 + f(\phi)\big)\,d\rho = 0, \qquad \phi(0) = v_0, \qquad \phi'(0) = 0.$$

By the uniqueness of the Cauchy problem we must have $\phi(r) = v(r)$ in $[0, \overline{R})$. To prove that equality holds in (26) we must prove that $v(\overline{R}) = 0$. By contradiction, let $v(\overline{R}) > 0$. Let r_k be a sequence such that

$$v_k(r_k) = \frac{v(\overline{R})}{2}.$$

The sequence r_k is increasing and converging to $\overline{R}$. Also, $v(r_k) \to v(\overline{R})$ because we are assuming $\overline{R} < R_0$. We have

$$v_{0,k} - v_k(r_k) = \int_{r_k}^0 v_k'(r)\,dr. \tag{28}$$

Note that $v(\overline{R})/2 \leqslant v_k(r) \leqslant v_0$ in $[0, r_k)$. Hence, by (27) we find

$$\frac{(v_k')^2}{2} \leqslant e^{2\int_{v(\overline{R}/2)}^{v_0} g^+(t)\,dt} \int_{v(\overline{R}/2)}^{v_0} f(t)\,dt.$$

Therefore, we can pass to the limit for $k \to \infty$ in (28) and find

$$v_0 - \frac{v(\overline{R})}{2} = \int_{\overline{R}}^0 v'(r)\,dr = v_0 - v(\overline{R}).$$

Hence, $v(\overline{R}) = 0$, contradicting the assumption $v(\overline{R}) > 0$. The continuity from the left follows.

We prove now the continuity from the right. Let $v_{0,k}$ be a decreasing sequence converging to v_0. Let $v(r)$ and $v_k(r)$ be, respectively, the maximal solutions corresponding to the initial values v_0 and $v_{0,k}$. If $R_0 = R(v_0)$ and $R_k = R(v_{0,k})$, by monotonicity we have

$$\lim_{k\to\infty} R_k = \inf_{k\in\mathbb{N}} R_k = \underline{R} \geqslant R_0. \tag{29}$$

We must prove that equality holds in (29). By contradiction, suppose $\underline{R} > R_0$. Let us show that

$$\inf_{k\in\mathbb{N}} v_k(R_0) = \underline{v} > 0. \tag{30}$$

Indeed, let us write Eq. (21) as

$$v'' + \frac{N-1}{r}v' + \big(g^+(v) - g^-(v)\big)(v')^2 + f(v) = 0, \tag{31}$$

where $g^-(t) = g^+(t) - g(t)$. By (31) we deduce that

$$\left(r^{N-1}v'\right)' + r^{N-1}\left(g^+(v)(v')^2 + f(v)\right) \geqslant 0.$$

Integrating over $(0, r)$ and using the monotonicity of v, v', g^+ and f we find,

$$\frac{v'(r)}{r} \geqslant -\frac{1}{N}\left(g^+(v)(v')^2 + f(v)\right),$$

which, inserted in (31), yields

$$v'' - g^-(v_0)(v')^2 + \frac{1}{N}f(v) \leqslant 0. \tag{32}$$

Here we have used, as we will make again just below, the fact that $g^-(t)$ is a non-decreasing function. Multiplying the last inequality by v' and integrating over $(0, r)$, we obtain

$$\frac{(v'(r))^2}{2} + \left(v'(r)\right)^2 g^-(v_0)v_0 \geqslant \frac{1}{N}f(v_0)\left(v_0 - v(r)\right).$$

If we write the last inequality for the solution v_k instead of v we find

$$\left(v_k'(r)\right)^2\left(1 + 2g^-(v_{0,k})v_{0,k}\right) \geqslant \frac{2}{N}f(v_{0,k})\left(v_{0,k} - v_k(r)\right).$$

Since $g^-(v_{0,k})v_{0,k} \leqslant g^-(v_{0,1})v_{0,1}$ and $f(v_{0,k}) \geqslant f(v_{0,1})$, there is a constant $c > 0$ such that

$$\frac{-v_k'(r)}{\sqrt{v_{0,k} - v_k(r)}} \geqslant c.$$

Integrating on (R_0, R_k) we get

$$2\left[\sqrt{v_{0,k}} - \sqrt{v_{0,k} - v_k(R_0)}\right] \geqslant c(R_k - R_0).$$

Multiplying by $\sqrt{v_{0,k}} + \sqrt{v_{0,k} - v_k(R_0)}$ we find

$$2v_k(R_0) \geqslant c\sqrt{v_{0,k}}(R_k - R_0) \geqslant c\sqrt{v_0}(\underline{R} - R_0),$$

which yields (30). Using (30) and (27) we find

$$\frac{(v_k')^2}{2} \leqslant e^{2\int_{\underline{v}}^{v_0} g^+(t)\,dt} \int\limits_{\underline{v}}^{v_0} f(t)\,dt.$$

Moreover,

$$0 \leqslant -v_k'' \leqslant g^+(\underline{v})\left(v_k'\right)^2 + f(\underline{v}).$$

As in the previous case, using the last inequalities we see that a subsequence of v_k converges to $v(r)$ in $[0, \underline{R})$. This implies $\underline{R} = R_0$, contradicting the assumption $\underline{R} > R_0$. The continuity of $R(v_0)$ is now proved.

To complete the proof of the lemma, let $v_0 > 0$ be sufficiently small, say $g^-(v_0)v_0 < 1/2$. By (32) we deduce

$$v''v' \geqslant \frac{2}{N}f(v_0)(-v').$$

Integrating over $(0, r)$ we find

$$(v')^2 \geqslant \frac{4}{N}f(v_0)\left(v_0 - v(r)\right), \qquad \frac{-v'(r)}{\sqrt{v_0 - v(r)}} \geqslant 2\sqrt{\frac{f(v_0)}{N}}.$$

Finally, integrating over $(0, R(v_0))$, we have

$$R(v_0) \leqslant 2\sqrt{\frac{Nv_0}{f(v_0)}},$$

and therefore (25). □

From Lemma 5, using $v_0(R)$, the inverse function of $R(v_0)$, we obtain the following existence and uniqueness result in the balls.

Corollary 4. *Suppose that f and g are continuous non-increasing in $(0, \infty)$, with $f(t) > 0$. Let*

$$R_\infty = \sup_{v_0 > 0} R(v_0).$$

Then for any $R < R_\infty$ the BVP (20) in the ball $B = B_R(x_0)$ has a unique positive solution, which is radially symmetric and equals $v_0(R)$ in x_0.

Proof. For the existence, take $v_0 = v_0(R)$ (see Lemma 5) and solve the Cauchy Problem (21). Denoting by $v(r)$ the solution, then $u(x) = v(|x - x_0|)$ is a solution of the BVP (20) in $B_R(x_0)$. The uniqueness follows at once from comparison theorems for quasilinear equations [11, Theorem 10.1]. □

With the aid of the radial case, we construct solutions of BVP (1) in a bounded smooth domain D. Following [10] and using the monotone method of [12], we first solve the following approximation of the original BVP (1) with $\varepsilon > 0$:

$$\Delta u + g(u)|\nabla u|^2 + f(u) = 0 \quad \text{in } D, \qquad u = \varepsilon \quad \text{on } \partial D. \tag{33}$$

Lemma 6. *Suppose that f and g are continuous non-increasing in $(0, \infty)$, with $f(t) > 0$. Let D be a bounded smooth domain of $\mathbb{R}^N$ such that $D \subset B_d(x_0)$ for some $d > 0$ and some $x_0 \in \mathbb{R}^N$. Suppose also that $f(t) \leqslant M$ and $g(t) \leqslant L$ as $t \geqslant a$ for some real number $a \geqslant 1$, and positive constants L, M such that*

$$MLd^2 < \frac{\pi^2}{4} N. \tag{34}$$

Then for each $0 < \varepsilon < 1$ there exists a classical solution $u = u_\varepsilon$ of the approximating BVP (33) such that

$$\varepsilon \leqslant u_\varepsilon(x) \leqslant a + \frac{\log \sec(\alpha d)}{L}, \quad x \in D, \tag{35}$$

with $\alpha = \sqrt{ML/N}$.

Proof. Since $f > 0$, we easily check that $\underline{u} = \varepsilon$ is a subsolution.

To find a supersolution, we consider the radial function

$$v(r) = a + \frac{\log \sec(\alpha d)}{L} - \frac{\log \sec(\alpha r)}{L}.$$

Since $\alpha = \sqrt{ML/N} < \pi/2d$, we have

$$
\begin{aligned}
&v'' + \frac{N-1}{r}v' + g(v)(v')^2 + f(v) \\
&\quad \leqslant -\frac{\alpha^2}{L}\frac{1}{\cos^2(\alpha r)} - \frac{\alpha^2(N-1)}{L}\frac{\tan(\alpha r)}{\alpha r} + \frac{\alpha^2}{L}\tan^2(\alpha r) + M \\
&\quad \leqslant -\frac{\alpha^2 N}{L} + M = 0.
\end{aligned}
$$

Hence $\bar{u}(x) = v(|x - x_0|)$ is a supersolution such that $\bar{u} \geqslant \varepsilon$ on ∂D.

Thanks to the monotone method of [12], we can infer that for each $\varepsilon > 0$ there exists a solution $u = u_\varepsilon \in W^{1,p}(D)$, with $p > N$, of the approximating boundary value problem (33), such that (35) holds. By virtue of the classical regularity theory, we conclude that $u_\varepsilon \in C^{2,\beta}(\overline{D})$ for some $0 < \beta < 1$. □

As a consequence, we shall prove an existence result for BVP (1), which turns out to be valid in the case g is bounded from below.

Defining

$$\underline{H}(t) = \inf_{0<s\leqslant t} H(s),$$

for a continuous function H we get a continuous non-increasing function $\underline{H}$ such that

$$\underline{H}(t) \leqslant H(t).$$

We observe that, if H is bounded from below as $t \to 0^+$, so it will be $\underline{H}$.

Theorem 5. *Let $f(t)$ and $g(t)$ be continuous functions in $(0, +\infty)$ such that $\underline{f}(t)$ is positive, and let $g(t)$ be bounded from below as $t \to 0^+$. Let also*

$$M = \limsup_{t\to+\infty} f(t) < \infty, \qquad L = \limsup_{t\to+\infty} g(t) < \infty,$$

and d be the diameter of D. There exists a positive constant $C = C(N)$ such that, if

$$MLd^2 < C(N), \tag{36}$$

then BVP (1) has a positive classical solution. If we also suppose f and g to be decreasing, such a solution is unique.

Proof. Let K be a compact subdomain of D. Using Lemma 6 we find solutions u_ε of BVP (33). Inequalities (35) show that the functions u_ε are equibounded from above. We claim that they are also positively equibounded from below in K. In fact, if $0 < R < \min\{R_\infty, \operatorname{dist}(K, \partial D)\}$, choosing $v_0 = v_0(R)$ we can use Corollary 4 to construct for each point $x_0 \in K$ a positive solution $w(x) = v(|x - x_0|)$ of the BVP

$$\Delta w + \underline{g}(w)|\nabla w|^2 + \underline{f}(w) = 0 \quad \text{in } B_R(x_0), \qquad w = 0 \quad \text{on } \partial B_R(x_0),$$

such that $w(x_0) = v_0(R)$. Since

$$\Delta u_\varepsilon + \underline{g}(u_\varepsilon)|\nabla u_\varepsilon|^2 + \underline{f}(u_\varepsilon) \leqslant 0 \quad \text{in } B_R(x_0), \qquad u_\varepsilon \geqslant \varepsilon \quad \text{on } \partial B_R(x_0),$$

again by the above cited comparison principle [11, Theorem 10.1] we obtain $u_\epsilon(x) \geqslant w(x)$ in $B_R(x_0)$. In particular we have

$$u_\varepsilon(x_0) \geqslant w(x_0) = v_0(R), \quad x_0 \in K,$$

which yields the desired uniform bound from below for the u_ε in K.

Finally, using interior estimates for the gradient [14, Theorem 3.1] and standard Schauder estimates (see [11]), we deduce that the norms $\|u_\varepsilon\|_{C^{2,\alpha}(K)}$ are in turn equibounded. Since K is arbitrary, we can extract a sequence which converges to a positive solution of BVP (1).

If f and g are decreasing, uniqueness follows from the already mentioned comparison principle. $\square$

Remark 1. We notice that condition (36) is satisfied in all bounded domains D, no matter how large is the diameter, if

$$\text{either}\quad \limsup_{t\to+\infty} f(t)=0,\quad \text{or}\quad \limsup_{t\to+\infty} g(t)\leqslant 0.$$

Remark 2. Theorem 5 solves problems which cannot be solved by the method described in Section 2. For example, let

$$\Delta u+\frac{\beta}{u}|\nabla u|^2+u^{-\alpha}=0\quad \text{in } D,\qquad u=0\quad \text{on } \partial D,$$

with $\beta>0$ and $\alpha\geqslant 0$. By Theorem 5 we have existence of a positive solution. Now we find $G(t)=\log(t^\beta)$, therefore, condition (6) holds. We could apply Theorem 1 only when $t^{-\alpha+\beta}$ is non-increasing, that is when $\alpha\geqslant\beta$.

4. Boundary behaviour

Theorem 6. *Consider the problem*

$$\Delta u+f(u)=0\quad \textit{in } D,\qquad u=0\quad \textit{on } \partial D, \tag{37}$$

where $f(t)$ is continuous, positive, non-increasing and satisfies

$$\int_0^1 f(\tau)\,d\tau=\infty. \tag{38}$$

Define

$$F(t)=\int_t^1 f(\tau)\,d\tau,\qquad \psi(t)=\int_0^t \frac{d\tau}{\sqrt{2F(\tau)}}. \tag{39}$$

Then

$$\psi(bt)<b\psi(t)\quad \forall b\in(0,1), \tag{40}$$

and the solution $u(x)$ to problem (37) *satisfies*

$$\lim_{x\to\partial D}\frac{u(x)}{\phi(\delta(x))}=1, \tag{41}$$

where $\phi(\delta)$ is the inverse function of ψ.

Proof. Inequality (40) holds because ψ is convex and $\psi(0)=0$. Let B_R be a ball of radius R, and let $v(r)=u(x)$, $r=|x|$, where $u(x)$ is the solution to problem (37) with $D=B_R$. We have

$$v''+\frac{N-1}{r}v'+f(v)=0,\qquad v(0)=v_0,\qquad v'(0)=0,\qquad v(R)=0.$$

Multiplying by v' and integrating on $(0, r)$ we find

$$\frac{(v')^2}{2} + (N-1)\int_0^r \frac{(v'(t))^2}{t}\,dt - F(v) + F(v_0) = 0.$$

Assumption (38) implies that $(v'(r))^2 \to \infty$ as $r \to R$. As a consequence, by a result of Lazer–McKenna [16, Lemma 2.1] we have

$$\lim_{r\to R} \frac{\int_0^r \frac{(v'(t))^2}{t}\,dt}{(v'(r))^2} = 0.$$

Hence, given $\epsilon > 0$ there is $r_\epsilon < R$ such that

$$(v')^2 > 2F(v)(1-\epsilon)^2 \quad \forall r \in (r_\epsilon, R),$$
$$\frac{-v'}{\sqrt{2F(v)}} > 1 - \epsilon.$$

Integration over (r, R) yields

$$\psi(v) > (1-\epsilon)(R-r),$$

and

$$v(r) > \phi\big((1-\epsilon)\delta\big), \quad \delta = R - r. \tag{42}$$

By (40) we get

$$(1-\epsilon)\phi(\delta) < \phi\big((1-\epsilon)\delta\big).$$

Hence, by (42) we find

$$v(r) > \phi(\delta)(1-\epsilon). \tag{43}$$

Let $A(R, \overline{R})$ be the annulus with radii R and $\overline{R}$, and let $w(r) = u(x)$, $r = |x|$, where $u(x)$ is the solution to problem (37) with $D = A(R, \overline{R})$. We have, for some r_0, $R < r_0 < \overline{R}$,

$$w'' + \frac{N-1}{r}w' + f(w) = 0, \qquad w(R) = 0, \qquad w'(r_0) = 0, \qquad w(r_0) = w_0.$$

Multiplying by w' and integrating on (r, r_0) we find

$$-\frac{(w')^2}{2} + (N-1)\int_r^{r_0} \frac{(w'(t))^2}{t}\,dt - F(w_0) + F(w) = 0.$$

Assumption (38) implies that $w'(r) \to \infty$ as $r \to R$. As a consequence, by the mentioned result of Lazer–McKenna we have

$$\lim_{r\to R} \frac{\int_0^r \frac{(w'(t))^2}{t}\,dt}{(w'(r))^2} = 0.$$

Hence, given $\epsilon > 0$ there is r_ϵ such that

$$(w')^2 < 2F(w)(1+\epsilon)^2 \quad \forall r \in (R, r_\epsilon),$$
$$\frac{w'}{\sqrt{2F(w)}} < 1 + \epsilon.$$

Integration over (R, r) yields

$$\psi(w) < (1+\epsilon)(r-R),$$

and

$$w(r) < \phi\big((1+\epsilon)\delta\big), \quad \delta = r - R. \tag{44}$$

By (40) we find

$$(1+\epsilon)\psi(t) < \psi\big((1+\epsilon)t\big),$$

whence,

$$\phi\big((1+\epsilon)\delta\big) < (1+\epsilon)\phi(\delta).$$

Hence, by (44) we find

$$w(r) < \phi(\delta)(1+\epsilon). \tag{45}$$

Now let D be a bounded domain with a smooth boundary, and let $P \in \partial D$. We can consider a small ball $B = B_R$ contained in D and tangent to ∂D in P. Furthermore, we can consider a suitable annulus $A = A(R, \overline{R})$ containing D and such that the inner boundary is tangent to ∂D in P. We may assume that the radius R of the ball B_R is equal to the inner radius of the annulus $A(R, \overline{R})$. If v, u, w are the solutions to problem (37) respectively in B, D and A then we have $v(x) \leqslant u(x) \leqslant w(x)$ in B. Using these inequalities together with (43) and (45) we get

$$\phi\big(\delta(x)\big)(1-\epsilon) \leqslant u(x) \leqslant \phi\big(\delta(x)\big)(1+\epsilon).$$

Since ϵ is arbitrary, the theorem follows. □

Remark 3. Theorem 6 continues to hold if we replace $F(t)$ by $F(t)+c$, c constant. Indeed, let

$$\int_0^{\phi_c(\delta)} \frac{d\tau}{\sqrt{2F(\tau)+c}} = \delta. \tag{46}$$

If $c > 0$ we have $\phi(\delta) < \phi_c(\delta)$. Moreover, given $\epsilon > 0$ we find $t_\epsilon > 0$ such that

$$2F(t) + c < 2(1+\epsilon)F(t)$$

for $0 < t < t_\epsilon$. Hence, for δ small we have

$$\begin{aligned}\int_0^{\phi(\delta)} \frac{dt}{\sqrt{2F(t)}} &= \int_0^{\phi_c(\delta)} \frac{dt}{\sqrt{2F(t)+c}} > \int_0^{\phi_c(\delta)} \frac{dt}{\sqrt{2(1+\epsilon)F(t)}} \\ &= \int_0^{\frac{\phi_c(\delta)}{1+\epsilon}} \frac{d\sigma}{\sqrt{2F((1+\epsilon)\sigma)/(1+\epsilon)}} > \int_0^{\frac{\phi_c(\delta)}{1+\epsilon}} \frac{d\sigma}{\sqrt{2F(\sigma)}}.\end{aligned}$$

In the last step we have used the inequality $F((1+\epsilon)\sigma)/(1+\epsilon) < F(\sigma)$, true because $F(\sigma)/\sigma$ is decreasing. It follows that

$$\lim_{\delta\to 0} \frac{\phi_c(\delta)}{\phi(\delta)} = 1.$$

The proof when $c < 0$ is similar.

Corollary 7. *Consider the problem* (1), *and define $G(t)$ as in* (5). *Suppose condition* (6) *holds, and define $h(s)$ as in* (7). *Suppose the assumptions of Theorem* 6 *hold with $f(h(t))e^{G(h(t))}$ instead of $f(t)$. Then we have*

$$\lim_{x\to\partial D}\frac{h^{-1}(u(x))}{\varphi(\delta(x))}=1, \tag{47}$$

with

$$\int_0^{\varphi(s)}\frac{dt}{\sqrt{2F(t)}}=s,\qquad F(t)=\int_t^1 f\big(h(s)\big)e^{G(h(s))}\,ds.$$

Proof. With the change $u=h(w)$ we are reduced to the statement of Theorem 6. □

In the particular case $g(t)=-\beta/t$ with $0<\beta<1$ we have $G(t)=\log(t^{-\beta})$ and

$$h(t)=\big((1-\beta)t\big)^{\frac{1}{1-\beta}}.$$

In this situation, setting $\Phi(s)=h(\varphi(s))$, we can rewrite (47) as

$$\lim_{x\to\partial D}\frac{u(x)}{\Phi(\delta(x))}=1, \tag{48}$$

with

$$\int_0^{\Phi(s)}\frac{\xi^{-\beta}\,d\xi}{\sqrt{2\int_\xi^{h(1)}f(\tau)\tau^{-2\beta}\,d\tau}}=s.$$

For example, when $f(t)=t^{-\alpha}$, all the assumptions of Corollary 7 hold when $\alpha>1-2\beta$. Recalling Remark 3 we can take

$$\Phi(s)=\left(\frac{\alpha+1}{\sqrt{2(\alpha+2\beta-1)}}s\right)^{\frac{2}{\alpha+1}}.$$

To treat the next case we recall a result on blow-up solutions.

Theorem 8. *Consider the problem*

$$\Delta u=f(u)\quad \text{in } D,\qquad u\to\infty\quad \text{as } x\to\partial D, \tag{49}$$

where $f(t)$ satisfies the conditions (i) *and* (8) *of Lemma* 2. *Moreover, if*

$$\psi(t)=\int_t^\infty\frac{d\tau}{\sqrt{2F(\tau)}},\qquad F(t)=\int_0^t f(\tau)\,d\tau, \tag{50}$$

suppose

$$\liminf_{t\to\infty}\frac{\psi(bt)}{\psi(t)}>1\quad \forall b\in(0,1). \tag{51}$$

If D is sufficiently smooth then a solution $u(x)$ to the boundary blow-up problem (49) *satisfies*

$$\lim_{x\to\partial D}\frac{u(x)}{\phi(\delta(x))}=1, \tag{52}$$

where $\phi(\delta)$ is the inverse function of ψ.

Proof. We refer to [2]. □

Corollary 9. *Consider the BVP* (1). *With $G(t)$ defined as in* (5), *suppose conditions* (10) *hold. Define $h(s)$ as in* (11), *and suppose the assumptions of Theorem* 8 *hold with $f(h(t))e^{G(h(t))}$ instead of $f(t)$. Then we have*

$$\lim_{x\to\partial D}\frac{h^{-1}(u(x))}{\varphi(\delta(x))}=1, \tag{53}$$

with

$$\int_{\varphi(s)}^{\infty}\frac{dt}{\sqrt{2F(t)}}=s, \quad F(t)=\int_0^t f\big(h(s)\big)e^{G(h(s))}\,ds.$$

Proof. With the change $u=h(w)$ we are reduced to the statement of Theorem 8. □

If $g(t)=-\beta/t$ with $\beta>1$ then $G(t)=\log(t^{-\beta})$ and

$$h(t)=\big((\beta-1)t\big)^{\frac{1}{1-\beta}}.$$

In this situation, with $\Phi(s)=h(\varphi(s))$, we can rewrite (53) as

$$\lim_{x\to\partial D}\frac{u(x)}{\Phi(\delta(x))}=1, \tag{54}$$

with

$$\int_0^{\Phi(s)}\frac{\xi^{-\beta}\,d\xi}{\sqrt{2\int_\xi^\infty f(\tau)\tau^{-2\beta}\,d\tau}}=s.$$

For example, if $f(t)=t^{-\alpha}$, all the assumptions of Corollary 9 hold when $\alpha>-1$. We find

$$\Phi(s)=\left(\frac{\alpha+1}{\sqrt{2(\alpha+2\beta-1)}}s\right)^{\frac{2}{\alpha+1}}.$$

To discuss the last case we recall a further result of Bandle–Marcus on blow-up solutions.

Theorem 10. *Consider the problem*

$$\Delta u=f(u)\quad \text{in } D,\qquad u\to\infty\quad \text{as } x\to\partial D, \tag{55}$$

where $f(t)$ satisfies the conditions (ii) *and* (8) *of Lemma* 2. *In addition, with $F(t)=\int_{-\infty}^t f(\tau)\,d\tau$, suppose that $F(t)t^{-2}$ is increasing for large t and that $F(t)t^{-4}\to\infty$ as $t\to\infty$. Moreover, suppose there exist numbers a,b with $1<a<b$, such that*

$$a\frac{F(t)}{f(t)}\leqslant\frac{\int_0^t\sqrt{F(s)}\,ds}{\sqrt{F(t)}}\leqslant b\frac{F(t)}{f(t)}\quad \text{for large } t.$$

Then a solution $u(x)$ to this boundary blow-up problem in a smooth domain D satisfies

$$\lim_{x\to\partial D}\big[u(x)-\phi\big(\delta(x)\big)\big]=0, \tag{56}$$

where $\phi(s)$ is defined as

$$\int_{\phi(s)}^{\infty} \frac{dt}{\sqrt{2F(t)}} = \delta. \tag{57}$$

Proof. We refer to [3, Theorem 4(iii)]. $\square$

Corollary 11. *Consider the BVP* (1). *With $G(t)$ defined as in* (5)*, suppose conditions* (14) *hold. Define $h(s)$ as in* (15)*, and suppose the assumptions of Theorem* 10 *hold with $f(e^{-h(t)})e^{G(e^{-h(t)})}$ instead of $f(t)$. Then we have*

$$\lim_{x\to\partial D}\left[h^{-1}\bigl(u(x)\bigr) - \varphi\bigl(\delta(x)\bigr)\right] = 0, \tag{58}$$

with

$$\int_{\varphi(s)}^{\infty} \frac{dt}{\sqrt{2F(t)}} = s, \quad F(t) = \int_{-\infty}^{t} f\bigl(e^{-h(\tau)}\bigr)e^{G(e^{-h(\tau)})}\,d\tau.$$

Proof. With the change $u = e^{-h(w)}$ we are reduced to the statement of Theorem 10 with $f(e^{-h(t)})e^{G(e^{-h(t)})}$ instead of $f(t)$. $\square$

If $g(t) = -1/t$ then $G(t) = \log(t^{-1})$ and $h(s) = s$. In this situation we can rewrite (58) as

$$\lim_{x\to\partial D}\left[\log\bigl(u(x)\bigr)^{-1} - \varphi\bigl(\delta(x)\bigr)\right] = 0. \tag{59}$$

If $f(t) = t^{-\alpha}$, with $\alpha > -1$ we find

$$F(t) = \frac{e^{(\alpha+1)t}}{\alpha+1},$$

and

$$\varphi(s) = -\frac{2}{\alpha+1}\log\left(\sqrt{\frac{\alpha+1}{2}}s\right).$$

Therefore, by (59) we get

$$\lim_{x\to\partial D}\left[\log\bigl(u(x)\bigr)^{-1} + \frac{2}{\alpha+1}\log\left(\sqrt{\frac{\alpha+1}{2}}\delta(x)\right)\right] = 0,$$

whence

$$\lim_{x\to\partial D} \frac{u(x)}{\left(\sqrt{\frac{\alpha+1}{2}}\delta(x)\right)^{\frac{2}{\alpha+1}}} = 1.$$

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