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On p-saturable groups ☆

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Abstract

A pro-p group G is a PF-group if it has central series of closed subgroups $\{N_i\}_{i\in\mathbb{N}}$ with trivial intersection satisfying $N_1 = G$ and $[N_i, G, \overset{p-1}{\ldots}, G] \leqslant N_{i+1}^p$. In this paper, we prove that a finitely generated pro-p group G is a p-saturable group, in the sense of Lazard, if and only if it is a torsion free PF-group. Using this characterization, we study certain families of subgroups of p-saturable groups. For example, we prove that any normal subgroup of a p-saturable group contained in the Frattini is again p-saturable.

Keywords: Pro-p groups; p-Saturable; PF-groups; p-Adic analytic; Lazard; Powerful; Uniformly powerful; Potent

1. Introduction

Lazard, in his seminal paper *Groupes analytiques p-adiques* [7] from 1965, proved that a topological group is p-adic analytic if and only if it has an open p-saturable subgroup. In the case of pro-p groups this means that the open p-saturable subgroup must be of finite index. In 1980s, Lubotzky and Mann reinterpreted Lazard's work in terms of uniformly powerful groups replacing the condition of having an open p-saturable subgroup by the condition of having an open uniformly powerful subgroup (see [1]). These two concepts are strongly related but do not coincide (although, for some time, it was thought they did): Lazard proved that the pro-p Sylows of $GL_n(\mathbb{Z}_p)$ and $SL_n(\mathbb{Z}_p)$ with $n \leqslant p-2$ are p-saturable and Klopsch realized that they are not uniformly powerful (see [7] and [5]). We recall that a pro-p group G is powerful if $[G,G] \leqslant G^p$

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for $p \geqslant 3$ and $[G, G] \leqslant G^4$ for $p = 2$. A pro-p group is uniformly powerful if it is powerful and torsion free.

Since the moment powerful groups were introduced by Lubotzky and Mann they have gained a prominent role in the theory of finite p-groups and pro-p groups. In a way, the concept of uniformly powerful pro-p groups has successfully replaced the original definition of Lazard. However, as suggested by Klopsch, there might be a price to pay for using uniformly powerful pro-p groups: for example, Klopsch in [6] used p-saturable groups to give a positive solution to a problem posed by Shalev for which the use of uniformly powerful groups was not enough.

As we mentioned above, the concepts of powerful and p-saturable pro-p groups do not coincide. This does not mean that there are not links between both definitions. Uniformly powerful groups are p-saturable and conversely, if a group G is p-saturable, then G^p is powerful and therefore uniformly powerful.

More recently (see [2]), a new family of pro-p groups has been defined: a closed normal subgroup N of a pro-p group G is a PF-embedded in G if there exists a central series of subgroups $\{N_i\}_{i\in\mathbb{N}}$ starting at N with trivial intersection, and with the property that $[N_i, G, \overset{p-1}{\ldots}, G] \leqslant N_{i+1}^p$. If $G = N$ we say that G is a PF-group. This new family of subgroups has been successfully used to study the power structure of pro-p groups. For example, it has been proved in [2] that the torsion elements of a finitely generated pro-p group with the condition $\gamma_{h(p-1)}(G) \leqslant G^{p^{h+1}}$ form a finite subgroup (this result was already proven for $p > 2$ by Wilson in [10], but the bounds on the exponent of the torsion subgroup were not precise enough). One of the most remarkable facts about PF-embedded subgroups is that have they a very nice power-commutator structure inside the group. More precisely, if G is a pro-p group and N is a PF-embedded subgroup of G, then for all positive integer i the following holds

$$\left[N^{p^i}, G\right] = \left[N, G^{p^i}\right] = [N, G]^{p^i}.$$

The identity above will play a prominent role in the study of p-saturable groups in this paper (see Proposition 2.1).

The aim of this paper is to describe p-saturable groups in terms of the group structure. Once we have a nice description of them, we will try to conclude some facts on certain families of subgroups. We now present the first result of this paper.

Theorem A. *Let G be a torsion-free finitely generated pro-p group. Then the following conditions are equivalent*:

(1) *G is a p-saturable group.*
(2) *G is a PF-group.*
(3) *$G/\Phi(G)^p$ is a PF-group, where $\Phi(G)$ is the Frattini subgroup of G.*

One of the most remarkable properties about p-saturable groups (or uniformly powerful groups) is that they have a natural $\mathbb{Z}_p$-Lie lattice structure. This construction is, in fact, an isomorphism of categories between the category of p-saturable groups and the category of p-saturable Lie algebras. This isomorphism was already constructed by Lazard in [7, 3.2.6, Chapter 4]. Using the new characterization of p-saturable groups that we have introduced, it is possible to prove that this correspondence also works for certain families of subgroups.

Theorem B. *Let $G = \mathcal{G}$ be a p-saturable group or a p-saturable Lie algebra. Then*

(1) *$N \leqslant G$ is a PF-embedded subgroup of G if and only if $\mathcal{N}$ is a PF-embedded Lie ideal of $\mathcal{G}$. Even more, $[N, G]_G = [\mathcal{N}, \mathcal{G}]_L$. (The concept of a PF-embedded Lie subalgebra is analogous to that of PF-embedded subgroup and it will be introduced in Section 4.)*
(2) *Let $M = \mathcal{M} \leqslant N = \mathcal{N}$ be a PF-embedded subgroups of G. Then N/M is central in G/M if and only if $\mathcal{N}/\mathcal{M}$ is central in $\mathcal{G}/\mathcal{M}$.*
(3) *The lower central series of G and $\mathcal{G}$ coincide. In particular G is nilpotent if and only if $\mathcal{G}$ is nilpotent and the nilpotency class of G and $\mathcal{G}$ coincide.*
(4) *The derived series of G and $\mathcal{G}$ coincide. In particular G is solvable if and only if $\mathcal{G}$ is solvable and the derived length of G and $\mathcal{G}$ coincide.*

One can finally prove that most normal subgroups of a p-saturable group are again p-saturable.

Theorem C. *Let G be a p-saturable group and N a closed normal subgroup of G. Then*:

(1) *If N is contained in $\Phi(G)$, N is p-saturable.*
(2) *N is a Lie subalgebra of $\mathcal{G} = G$ if and only if $N^p = \{n^p \mid n \in \mathbb{N}\}$.*

We briefly sketch the structure of the rest of the paper. In Section 2, we recall some results about PF-groups and general properties about pro-p groups. Sections 3–5 will be devoted to prove Theorems A, B and C, respectively.

Notation. We use standard notation in group theory. If G is a pro-p group then all subgroups of G considered will be understood in a topological sense: i.e., when written a subgroup generated by a subset of G, a verbal subgroup of G, etc., we will always mean the topological closure of the corresponding abstract subgroup. Commutators will be written using left notation and we will write $[N,_k M]$ to denote the commutator $[N, M, \dots, M]$ with M appearing k times. $\Phi(G)$ will denote the Frattini subgroup.

During the course of the paper (for example in Section 4), we will deal with sets which have both group and Lie algebra structure. In order to distinguish both algebraic structures we will write $G, N, M, \dots$ to denote a group, subgroup, ... and $\mathcal{G}, \mathcal{N}, \mathcal{M}, \dots$ to denote a Lie algebra, Lie subalgebra, Lie ideal, ... The commutator in the group will be denoted by $[x, y]_G$ and the Lie product in the Lie algebra by $[x, y]_L$.

2. Some preliminary matters

In this section we recall some properties concerning pro-p groups. We start by giving a few definitions. Consider G a pro-p group and $\{N_i\}_{i\in\mathbb{N}}$ a decreasing central series of closed normal subgroups with trivial intersection. Suppose that for all i, $[N_i,_{p-1} G] \leqslant N_{i+1}^p$. Then we say that $\{N_i\}_{i\in\mathbb{N}}$ is a *potent filtration* of G. For ease of notation we will write $\{N_i\}$ instead of $\{N_i\}_{i\in\mathbb{N}}$.

If there is a potent filtration of G starting at a subgroup N, we say that N is *PF-embedded* on G. A pro-p group G is a *PF-group* if G is PF-embedded in G. We summarize the main properties concerning potent filtrations and PF-embedded subgroups in the following propositions.

Proposition 2.1. *Let G be a pro-p group, N a PF-embedded subgroup and $\{N_i\}_{i\in\mathbb{N}}$ a potent filtration starting at N. Then*:

(1) *N/K is PF-embedded in G/K for every closed normal subgroup K of G.*
(2) *If $g \in G$ and $x \in N_i$, then $(xg)^p \equiv x^p g^p \pmod{N_{i+1}^p}$.*
(3) *$\{N_i^p\}$ and $\{[N_i, G]\}$ are potent filtrations and N^p and $[N, G]$ are PF-embedded subgroups.*
(4) *$N^{p^k} = \{x^{p^k} \mid x \in N\}$.*
(5) *For all $i, j \geqslant 0$ $[N^{p^i}, G^{p^j}] = [N, G]^{p^{i+j}}$.*

Proof. See Propositions 3.1–3.3, Theorem 3.4 and Corollary 3.5 of [2]. □

In the next proposition we prove that roots can be taken in an unique way in a torsion free PF-group.

Proposition 2.2. *Let G be a pro-p group and N a torsion free PF-embedded subgroup of G. Then if $x^{p^k} \in N^{p^k}$, it follows that $x \in N$. Even more, if $x, y \in N$ and $x^{p^k} = y^{p^k}$, then $x = y$.*

Proof. It is enough to prove the proposition for the case $k = 1$. Consider a potent filtration $\{N_i\}$ starting at N. We put $x_1 = x$. By hypothesis $x_1^p = a_1^p$ for some $a_1 \in N_1$. If we put $x_2 = x_1 a_1^{-1}$, by part (2) of the previous proposition $x_2^p \in N_2^p$. In general, if $x_i^p \in N_i^p$, there exists $a_i \in N_i$ such that $x_i^p = a_i^p$, and then $x_{i+1} = x_i a_i^{-1}$ satisfies that $x_{i+1}^p \in N_{i+1}^p$ and $x_{i+1} \equiv x_i \pmod{N_i}$.

Therefore we have constructed a Cauchy sequence $\{x_i\}_{i \in \mathbb{N}}$ such that $x_i^p \in N_i^p$. Then $(\lim_{i\to\infty} x_i)^p = \lim_{i\to\infty} x_i^p = 1$. Since N is torsion-free, $\lim_{i\to\infty} x_i = 1$. Since multiplying by x and taking inverses are continuous operations, $h_i = x x_i^{-1}$ is also a Cauchy sequence. But h_i is contained in N ($h_1 = x x_1^{-1} = 1$ and $h_{i+1} = x x_{i+1}^{-1} \equiv x x_i^{-1} \bmod N$). Hence $\lim_{i\to\infty} h_i \in N$ and $x = \lim_{i\to\infty} h_i x_i = (\lim_{i\to\infty} h_i)(\lim_{i\to\infty} x_i) \in N$.

Consider now $x, y \in N$ such that $x^p = y^p$. It will be enough to prove that $xy^{-1} \in N_i$ for all i. We argue by induction on i being the case $i = 1$ obvious. Suppose that the assumption holds for all i and let us see it for $i + 1$. By part (2) of the previous proposition $(xy^{-1})^p \equiv x^p y^{-p} \pmod{N_{i+1}^p}$. Then $(xy^{-1})^p \in N_{i+1}^p$ and therefore $xy^{-1} \in N_{i+1}$. □

We finish this section giving some general results concerning pro-p groups.

Lemma 2.3. *Let G be a pro-p group and N a closed normal subgroup of G. Then*:

(1) *If M is another closed normal subgroup of G such that $N \leqslant M[N, G]$, then $N \leqslant M$.*
(2) *For every $\ell \geqslant 0$, $[N^p,_\ell G] \leqslant [N,_\ell G]^p [N,_{(p-1)+\ell} G]$.*

Proof. (1) Since a pro-p group is the inverse limit of finite p-groups we can reduce the proof to finite p-groups. Now applying the inclusion several times we get that $N \leqslant M[N,_k G]$. Now, the lemma follows from the fact that finite p-groups are nilpotent. The second part of the lemma is a particular case of Theorem 2.5 of [2]. □

3. The relation between PF-groups and Lazard's p-saturable groups

Let G be a finitely generated pro-p group. We say that G is *p-valued* if there exists a map $\omega : G \to \mathbb{R}_{>0} \cup \{\infty\}$, which we call *valuation*, such that the following properties hold for all $x, y \in G$:

(i) $\omega(x) > (p-1)^{-1}$,
(ii) $\omega(x) = \infty$ if and only if $x = 1$,
(iii) $\omega(xy^{-1}) \geqslant \min\{\omega(x), \omega(y)\}$,
(iv) $\omega([x, y]) \geqslant \omega(x) + \omega(y)$,
(v) $\omega(x^p) = \omega(x) + 1$.

Clearly, p-valued pro-p groups are torsion-free. If G is p-radical with respect to ω (i.e., if for every $x \in G$ with $\omega(x) > p(p-1)^{-1}$ there exists $y \in G$ such that $x = y^p$), then G is called a *p-saturable* pro-p group.

The definition we have given of p-saturable group is not literally the same as Lazard's definition, which requires two more conditions, but it is equivalent to it. Lazard uses the valuation ω to define a topology on G by choosing the subgroups $G_\nu = \{g \in G \mid \omega(g) \geqslant \nu\}$, with $\nu \in \mathbb{R}_{>0}$, as a fundamental system of neighborhoods of the identity. Then the two extra conditions that required are:

(L1) The group G is complete with respect to the topology coming from the valuation.
(L2) The group G has finite rank, i.e., there is a bound on the number of generators needed for the closed subgroups of G.

In fact, the concept of rank used by Lazard is different, but as Klopsch indicates in [5], it is equivalent to have finite rank in one sense or the other. The following proposition shows that our definition implies property (L1).

Proposition 3.1. *Let G be a p-valued group. Then the topology of G coming from the valuation coincides with the topology of G as a pro-p group.*

Proof. We know from Theorem 1.17 of [1] that the open subgroups of G in the pro-p topology are exactly the subgroups of finite index. Now, according to (iv) and (v) of the definition of valuation, if $n = \lceil \nu \rceil$ then $\gamma_{n(p-1)}(G)G^{p^n}$ is contained in G_ν. Consequently G_ν has finite index in G and is open in the pro-p topology. It follows from Proposition 2.1.5(b) of [9] that the subgroups G_ν form a fundamental system of neighborhoods of the identity in the pro-p topology. Thus the topology coming from the valuation coincides with the pro-p topology. □

The fact that (L2) is also a consequence of our definition is immediate from Theorem 3.4 and Corollary 3.6 below: a p-saturable group in our sense is in particular a PF-group, and consequently p-adic analytic. As is well known, a p-adic analytic group has finite rank.

After this little digression, let us begin with the preliminaries needed for the proof of Theorem 3.4. Our next lemma shows that, in order to get a potent filtration beginning at a closed normal subgroup N, we do not have to worry so much that the subgroups have trivial intersection: it suffices if one of them already lies in $[N, G]N^p$. This result can also be applied to a subgroup N which is already known to be PF-embedded in order to obtain a particularly nice potent filtration whose first term is N.

Lemma 3.2. *Let G be a pro-p group and let N be a closed normal subgroup for which there exists a series of subgroups $N = N_1 \geqslant N_2 \geqslant \cdots \geqslant N_{t+1} = [N, G]N^p$ satisfying that $[N_i, G] \leqslant N_{i+1}$ and $[N_i,_{p-1} G] \leqslant N_{i+1}^p$ for $1 \leqslant i \leqslant t$. If we define*

$$M_i = \begin{cases} [N_{s,j}\, G]N^p, & \text{if } i = jt + s \leqslant t(p-1) \text{ with } 1 \leqslant s \leqslant t, \\ M^p_{i-t(p-1)}, & \text{if } i > t(p-1), \end{cases}$$

then $\{M_i\}$ *is a potent filtration of* G. *In particular* N *is a PF-embedded subgroup of* G.

Proof. It is clear that $\{M_i\}$ is a decreasing series of closed normal subgroups of G with trivial intersection. We will prove $[M_i, G] \leqslant M_{i+1}$ and $[M_{i,\,p-1}\, G] \leqslant M^p_{i+1}$ by induction on i.

Suppose first that $i \leqslant t(p-1)$. If we write $i = jt + s$ with $1 \leqslant s \leqslant t$, then $M_i = [N_{s,j}\, G]N^p$. Now we have $[M_i, G] = [[N_{s,j}\, G]N^p, G] \leqslant [N_{s+1,j}\, G]N^p = M_{i+1}$. On the other hand $[M_{i,\,p-1}\, G] = [[N_{s,j}\, G]N^p{}_{,\,p-1}\, G] \leqslant [N^p_{s+1},{}_j\, G][N^p{}_{,\,p-1}\, G]$. We first prove that $[N^p_k{}_{,j}\, G] \leqslant [N_{k,j}\, G]^p$ for $1 \leqslant k \leqslant t$. By part (2) of Lemma 2.3,

$$\big[N^p_k{}_{,j}\, G\big] \leqslant [N_{k,j}\, G]^p[N_{k,\,j+p-1}\, G] \leqslant [N_{k,j}\, G]^p\big[N^p_{k+1},{}_j\, G\big],$$

and by repeating this argument,

$$\big[N^p_k{}_{,j}\, G\big] \leqslant [N_{k,j}\, G]^p\big[N^p_{t+1},{}_j\, G\big]. \tag{1}$$

Applying again Lemma 2.3, in this case to $[N^p_{t+1},{}_j\, G]$, we get that

$$\begin{aligned} \big[N^p_{t+1},{}_j\, G\big] &\leqslant [N_{t+1,j}\, G]^p[N_{t+1,\,j+p-1}\, G] \\ &\leqslant [N_{t+1,j}\, G]^p\big[N^p{}_{,\,j+p-1}\, G\big][N_{,\,j+p}\, G] \leqslant [N_{t+1,j}\, G]^p\big[N^p{}_{,\,j+1}\, G\big]. \end{aligned} \tag{2}$$

Consequently

$$\begin{aligned} \big[N^p_k{}_{,j}\, G\big] &\leqslant [N_{k,j}\, G]^p\big[N^p{}_{,\,j+1}\, G\big] \leqslant [N_{k,j}\, G]^p[N_{,\,j+1}\, G]^p\big[N^p_{t+1},{}_{j+1}\, G\big] \\ &\leqslant [N_{k,j}\, G]^p\big[N^p_k{}_{,\,j+1}\, G\big], \end{aligned}$$

where the second inclusion follows from (1) with $k = 1$. Then we conclude from part 1 of Lemma 2.3 that $[N^p_k{}_{,j}\, G] \leqslant [N_{k,j}\, G]^p$. If we now apply this result to $[N^p_{s+1},{}_j\, G][N^p{}_{,\,p-1}\, G]$ (and also (2) if $s = t$), it follows that $[M_{i,\,p-1}\, G] \leqslant M^p_{i+1}$.

Suppose now that $i > t(p-1)$. Then $M_i = M^p_{i-t(p-1)}$, and by the induction hypothesis,

$$\begin{aligned} \big[M^p_{i-t(p-1)}, G\big] &\leqslant [M_{i-t(p-1)}, G]^p[M_{i-t(p-1),\,p}\, G] \\ &\leqslant [M_{i-t(p-1)}, G]^p\big[M^p_{i-t(p-1)+1}, G\big] \leqslant M_{i+1}. \end{aligned}$$

Using a similar argument we also obtain $[M_{i,\,p-1}\, G] \leqslant M^p_{i+1}$. □

As an easy corollary, we obtain the following criterion for a finitely generated closed normal subgroup to be PF-embedded.

Corollary 3.3. *Let* G *be a pro-*p *group and let* N *be a finitely generated closed normal subgroup of* G. *Then* N *is PF-embedded in* G *if and only if* $N/([N, G]N^p)^p$ *is PF-embedded in* $G/([N, G]N^p)^p$.

Proof. If N is PF-embedded in G, then we know from part (1) of Proposition 2.1 that $N/([N,G]N^p)^p$ is PF-embedded in $G/([N,G]N^p)^p$. Conversely, consider a potent filtration $\{N_i/([N,G]N^p)^p\}$ beginning at $N/([N,G]N^p)^p$. Since N is finitely generated, the subgroup $[N,G]N^p$ is open and consequently $N_t \leqslant [N,G]N^p$ for some t. Then it suffices to consider the series of subgroups $N = N_1[N,G]N^p \geqslant N_2[N,G]N^p \geqslant \cdots \geqslant N_t[N,G]N^p = [N,G]N^p$ and to apply the previous lemma. □

We can now prove the main theorem of this section.

Theorem 3.4. *Let G be a torsion-free finitely generated pro-p group. Then the following conditions are equivalent*:

(1) *G is a p-saturable group.*
(2) *G is a PF-group.*
(3) *$G/\Phi(G)^p$ is a PF-group, where $\Phi(G)$ is the Frattini subgroup of G.*

Proof. The equivalence between (2) and (3) follows from Corollary 3.3. Let us prove the equivalence between (1) and (2).

Suppose first that G is a PF-group. Consider a family of subgroups $G = N_1 \geqslant N_2 \geqslant \cdots \geqslant N_{t+1} = \Phi(G)$ such that $[N_i, G] \leqslant N_{i+1}$ and $[N_i,_{p-1} G] \leqslant N_{i+1}^p$ for all $i \leqslant t$. For every positive integer i, write $i = rt(p-1) + jt + s$ with $0 \leqslant j \leqslant p-2$ and $1 \leqslant s \leqslant t$, and define $M_i = ([N_s,_j G]G^p)^{p^r}$. By Lemma 3.2, $\{M_i\}$ is a potent filtration of G.

We define a valuation ω on G as follows. For every $x \in M_i \backslash M_{i+1}$, let

$$\omega(x) = \frac{1}{p-1}\left(\frac{2^{sp+j+1}}{2^{(t+1)p}} + j + 1\right) + r,$$

where r, j and s are as above and $\omega(1) = \infty$. Conditions (i)–(iii) of the definition of a valuation are obviously satisfied. On the other hand, if $x^p \in ([N_s,_j G]G^p)^{p^r}$ then, by Proposition 2.2, $x \in ([N_s,_j G]G^p)^{p^{r-1}}$. Therefore (v) also holds. In order to check that (iv) is fulfilled, consider $x_1, x_2 \in G$ and suppose that $x_k \in M_{i_k} \backslash M_{i_k+1}$. Write $M_{i_k} = ([N_{s_k},_{j_k} G]G^p)^{p^{r_k}}$. Then, by Proposition 2.1,

$$\begin{aligned}[x_1, x_2] &\in \left[\left([N_{s_1},_{j_1} G]G^p\right)^{p^{r_1}}, \left([N_{s_2},_{j_2} G]G^p\right)^{p^{r_2}}\right] \\ &= \left[[N_{s_1},_{j_1} G]G^p, [N_{s_2},_{j_2} G]G^p\right]^{p^{r_1+r_2}} \leqslant \left([N_{s_1},_{j_1+j_2+1} G]G^p\right)^{p^{r_1+r_2}}.\end{aligned}$$

We may assume without loss of generality that $s_1p + j_1 \geqslant s_2p + j_2$. If $j_1 + j_2 < p-2$ then

$$\omega([x_1, x_2]) \geqslant \frac{1}{p-1}\left(\frac{2^{s_1p+j_1+j_2+2}}{2^{(t+1)p}} + j_1 + j_2 + 2\right) + r_1 + r_2.$$

Since

$$\omega(x_1) + \omega(x_2) = \frac{1}{p-1}\left(\frac{2^{s_1p+j_1+1} + 2^{s_2p+j_2+1}}{2^{(t+1)p}} + j_1 + j_2 + 2\right) + r_1 + r_2$$

and

$$2^{s_1 p+j_1+j_2+2} \geqslant 2^{s_1 p+j_1+2} \geqslant 2^{s_1 p+j_1+1} + 2^{s_2 p+j_2+1},$$

we obtain that $\omega([x_1, x_2]) \geqslant \omega(x_1) + \omega(x_2)$. On the other hand, if $j_1 + j_2 \geqslant p - 2$ then

$$\begin{aligned}
[x_1, x_2] &\in \big[[N_{s_1, j_1} G]G^p, [N_{s_2, j_2} G]G^p\big]^{p^{r_1+r_2}} \\
&\leqslant \big([N_{s_1, j_1+j_2+1} G][N_{s_1, j_1+1} G]^p[N_{s_2, j_2+1} G]^p G^{p^2}\big)^{p^{r_1+r_2}} \\
&\leqslant \big([N_{s_1+1, j_1+j_2-p+2} G]^p[N_{s_1, j_1+1} G]^p[N_{s_2, j_2+1} G]^p G^{p^2}\big)^{p^{r_1+r_2}} \\
&\leqslant \big([N_{s_1+1, j_1+j_2-p+2} G]G^p\big)^{p^{r_1+r_2+1}},
\end{aligned}$$

and the result follows similarly.

Finally, we need to prove that G is p-radical with respect to ω, i.e. that for every $x \in G$ with $\omega(x) > p(p-1)^{-1}$ there exists $y \in G$ such that $x = y^p$. Suppose $x \in ([N_{s,j} G]G^p)^{p^r}$. Since $\omega(x) > p(p-1)^{-1}$, it is clear from the definition of ω that $r \geqslant 1$. Thus $x \in G^p$ and, by Proposition 2.1, $x = y^p$ for some $y \in G$.

Conversely, let us suppose that G has a p-valuation ω such that G is a p-saturable group. Define on $((p-1)^{-1}, +\infty]$ the topology whose open sets are those of the form $(\nu, +\infty]$ with $\nu \geqslant (p-1)^{-1}$, together with the empty set. This makes ω a continuous map, since G_ν is open in G. Now, since G is compact, $\omega(G)$ is also compact. Thus there exists $\delta > 0$ such that $\omega(G) \subseteq [(p-1)^{-1} + \delta, +\infty]$. Let us see that $N_i = \{x \in G \mid \omega(x) \geqslant (p-1)^{-1} + \delta i\}$ defines a potent filtration of G. It is obvious that $\{N_i\}_{i\in\mathbb{N}}$ form a decreasing series of closed subgroups with trivial intersection. That they form a central filtration of closed normal subgroup follows from the fact that if $x \in N_i$ and $y \in G$,

$$\omega\big([x, y]\big) \geqslant \omega(x) + \omega(y) \geqslant (p-1)^{-1} + i\delta + (p-1)^{-1} + \delta \geqslant (p-1)^{-1} + (i+1)\delta.$$

Finally, if $x \in N_i$ and $y_1, \dots, y_{p-1} \in G$ then

$$\begin{aligned}
\omega\big([x, y_1, \dots, y_{p-1}]\big) &\geqslant (p-1)^{-1} + i\delta + 1 + \delta(p-1) \\
&\geqslant p(p-1)^{-1} + (i+1)\delta > p(p-1)^{-1}.
\end{aligned}$$

Therefore $[x, y_1, \dots, y_{p-1}] = a^p$ with $\omega(a) = \omega([x, y_1, \dots, y_{p-1}]) - 1 \geqslant (p-1)^{-1} + (i+1)\delta$. Thus $a \in N_{i+1}$ and $[N_i, {}_{p-1} G] \leqslant N_{i+1}^p$, which yields the last condition of being a potent filtration. □

Corollary 3.5. *Let G be a pro-p group such that $\gamma_p(G) \leqslant \Phi(G)^p$. Then G is a PF-group.*

Let K be a finite extension of $\mathbb{Q}_p$ with ramification index e, and let $\mathcal{O}$ be its valuation ring. This last corollary can be used to provide an alternative proof of Lazard's result that the Sylow pro-p subgroups of $GL_n(\mathcal{O})$ and $SL_n(\mathcal{O})$ are p-saturable if $en < p - 1$ (see (3.2.7.5) in Chapter III of [7]). To see this, let G be one of these Sylow subgroups and note that G is torsion-free if $en < p - 1$. Since $\gamma_{en+2}(G) \leqslant (G')^p$ for every matrix size n, in particular we have $\gamma_p(G) \leqslant \Phi(G)^p$ for $en < p - 1$, and the result follows.

As we could expect from Theorem 3.4, PF-groups can be used in the same way as powerful groups, uniformly powerful groups or p-saturable groups in order to characterize p-adic analytic groups.

Corollary 3.6. *Let G be a finitely generated pro-p group. Then G is p-adic analytic if and only if G has an open PF-subgroup.*

Proof. The "only if" part is immediate from Theorem 3.4. Conversely, if N is an open PF-subgroup of G then N^p is an open powerful subgroup of G: by part (5) of Proposition 2.1 $[N^p, N^p] \leqslant [N, N]^{p^2}$ and $[N, N]^{p^2}$ is contained in $(N^p)^p$ if $p \geqslant 3$ and in $(N^2)^4$ if $p = 2$. Consequently G is p-adic analytic. □

We finish this section by giving an example of a p-saturable group G that does not satisfy the condition $\gamma_p(G) \leqslant \Phi(G)^p$. In particular G will not be a uniformly powerful group.

Example. Consider $A = \langle x_1, \ldots, x_p\rangle$ the direct product of p copies of $\mathbb{Z}_p$ and $\langle\alpha\rangle \cong \mathbb{Z}_p$ acting on A by $x_i^\alpha = x_i x_{i+1}$ for $1 \leqslant i \leqslant p-2$, $x_{p-1}^\alpha = x_{p-1}x_p^p$ and $x_p^\alpha = x_p$. Let G be the semidirect product between A and $\langle\alpha\rangle$. Consider $N_1 = G$ and $N_i = \langle x_i, x_{i+1}, \ldots, x_p\rangle$. One has that $\{N_i\}$ is a potent filtration of G. Therefore, since G is torsion free, G is p-saturable. But G does not satisfy the condition $\gamma_p(G) \leqslant \Phi(G)^p$.

4. The correspondence between PF-embedded normal subgroups and PF-embedded ideals

As in the case of pro-p groups, we can also introduce the concept of a p-saturable Lie algebras. Let L be a finitely generated $\mathbb{Z}_p$-Lie lattice. We say that L is *p-valued* if there exists a map $\omega : L \to \mathbb{R}_{>0} \cup \{\infty\}$, which we call *valuation*, such that the following properties hold for all $x, y \in L$:

(i) $\omega(x) > (p-1)^{-1}$,
(ii) $\omega(x) = \infty$ if and only if $x = 0$,
(iii) $\omega(x - y) \geqslant \min\{\omega(x), \omega(y)\}$,
(iv) $\omega([x, y]_L) \geqslant \omega(x) + \omega(y)$,
(v) $\omega(px) = \omega(x) + 1$.

If L is p-radical with respect to ω, i.e. if for every $x \in L$ with $\omega(x) > p(p-1)^{-1}$ there exists $y \in L$ such that $x = py$, then L is called a *p-saturable* Lie algebra.

In the same way we did for pro-p groups, we can introduce PF-Lie algebras. Consider $\mathcal{L}$ a finitely generated $\mathbb{Z}_p$-Lie algebra and $\{\mathcal{I}_i\}_{i\in\mathbb{N}}$ a central series of Lie ideals with trivial intersection. We say that $\{\mathcal{I}_i\}_{i\in\mathbb{N}}$ is a *potent filtration* of $\mathcal{L}$ if $[\mathcal{I}_i, {}_{p-1}\mathcal{L}]_L \leqslant p\mathcal{I}_{i+1}$. Again we will write $\{\mathcal{I}_i\}$ instead of $\{\mathcal{I}_i\}_{i\in\mathbb{N}}$. In this case, if $\mathcal{I} = \mathcal{I}_1$, we say that $\mathcal{I}$ is PF-embedded in $\mathcal{L}$. A $\mathbb{Z}_p$-Lie algebra $\mathcal{L}$ is a *PF-Lie algebra* if $\mathcal{L}$ is PF-embedded in $\mathcal{L}$. It is easy to check that $\{[\mathcal{I}_i, \mathcal{L}]_L\}$ and $\{p\mathcal{I}_i\}$ are again potent filtrations of $\mathcal{L}$. Therefore, as in the case of pro-p groups, $[\mathcal{I}, \mathcal{L}]_L$ and $\mathcal{I}$ will be PF-embedded in $\mathcal{L}$. One can also characterize p-saturable Lie algebras in terms of p-Lie algebras.

Theorem 4.1. *Let $\mathcal{L}$ be a finitely generated $\mathbb{Z}_p$-Lie lattice. Then the following conditions are equivalent*:

(1) *$\mathcal{L}$ is p-saturable.*
(2) *$\mathcal{L}$ is a PF-Lie algebra.*
(3) *$\mathcal{L}/(p[\mathcal{L},\mathcal{L}]_L + p^2\mathcal{L})$ is a PF-Lie algebra.*

Proof. The proof is very similar to that one of Theorem 3.4. □

For a p-saturable Lie algebra we can write the Baker–Campbell–Hausdorff formula as

$$\mathcal{H}(x,y) = x + y + \sum_{i\in\mathbb{N}} u_i(x,y),$$

where $u_i(x,y)$ is a Lie polynomial in x and y with coefficients in $\mathbb{Q}$ and $p^{\lfloor \frac{n-1}{p-1} \rfloor} u_i(x,y)$ has coefficients in $\mathbb{Z}_p$ (see Theorem 3.2.2, Chapter 4 of [7]). The Baker–Campbell–Hausdorff formula transforms any p-saturable Lie algebra into a p-saturable group. One can also write the conjugation of an element x by an element y in terms of Lie products in the following way:

$$\mathcal{H}\big(\mathcal{H}(-x,y),x\big) = y + \sum_{i=1}^{n} \frac{1}{n!}[y,_n x].$$

Conversely, if G is a p-saturable group we can define the following operations:

- $x + y = \lim_{n\to\infty}(x^{p^n} y^{p^n})^{p^{-n}}$,
- $[x,y]_L = \lim_{n\to\infty}[x^{p^n}, y^{p^n}]_G^{p^{-2n}}$.

With this new operations $(G, +, [\,,\,]_L)$ is a p-saturable Lie algebra.

Even more, the two operations defined above are compatible one with each other. The facts given above are summarized in the following theorem due to Lazard.

Theorem 4.2.

(1) *Let $\mathcal{L}$ be a p-saturable Lie algebra. Then $(\mathcal{L},\mathcal{H})$ is a p-saturable group.*
(2) *Let G be a p-saturable group. Then $(G,+,[\,,\,]_L)$ is a p-saturable Lie algebra.*

Even more, if L is a p-saturable Lie algebra and $G = (L,\mathcal{H})$, then $L = (G,+,[\,,\,]_L)$. Conversely, if G is a p-saturable group and $L = (G,+,[\,,\,]_L)$, then $G = (L,\mathcal{H})$.

Proof. See (3.2.6) in Chapter 4 of [7]. □

It seems natural to ask what happens with PF-embedded subgroups and PF-embedded ideals. Consider G a PF-embedded subgroup in a p-saturable group. We know that G is also a p-saturable group. This means that G becomes a p-saturable sub Lie algebra. Our aim in the following sections will be to prove that, indeed, G is a PF-embedded ideal. Conversely, one can prove that PF-embedded ideals are PF-embedded normal subgroups. We start by proving the following two technical lemmas.

Lemma 4.3. *Let G be a pro-p group and N a PF-embedded subgroup of G. Then, for all $x \in G$ and $y \in N$ there exists $a \in N$ such that $x^{p^n} y^{p^n} = (xa)^{p^n}$.*

Proof. Consider a potent filtration $\{N_i\}$ starting at N. We first proceed with the case when N is finite, and prove it of all N_i. We argue by induction on n. For $n = 1$ we argue by reverse induction on i. For i big enough one has that $N_i = 1$ and the result follows. Suppose that the result is true for $i + 1$. Then, by part (2) of Proposition 2.1, $x^p y^p = (xy)^p \pmod{N_{i+1}^p}$. Hence, applying part (4) of Proposition 2.1, $x^p y^p = (xy)^p z^p$ with $z \in N_{i+1}$. Now, by induction hypothesis $x^p y^p = (xy)^p z^p = (xyb)^p$ for some $b \in N_{i+1}$. In particular if we take $a = yb$ we are done.

For the general n we apply the case $n = 1$ to $(x^{p^{n-1}})^p (y^{p^{n-1}})^p$ and to the PF-group $N^{p^{n-1}}$. Then $x^{p^n} y^{p^n} = (x^{p^{n-1}} b)^p$ with $b \in N^{p^{n-1}}$. Again, by part (4) of Proposition 2.1 there exists $c \in N$ such that $b = c^{p^{n-1}}$. Applying the induction hypothesis on $x^{p^{n-1}} c^{p^{n-1}}$ we are done.

Suppose now that N is not necessarily finite. From previous argument one has that for any open normal subgroup H of G there exists $a_H \in N$ such that, $x^{p^n} y^{p^n} \equiv (xa_H)^{p^n} \pmod H$ (Note that NH/H is also a PF-group.) $\{a_H \mid H$ open normal subgroup of $G\}$ is a net of G. Therefore, since G is compact, it has a cluster point. Let a be such a point. It is clear that $a \in N$ and $x^{p^n} y^{p^n} = (xa)^{p^n}$. □

Lemma 4.4. *Let $\mathcal{L}$ be a p-saturable Lie algebra and $\mathcal{I}$ a PF-embedded ideal of $\mathcal{L}$. Consider $\{\mathcal{I}_i\}$ a potent filtration starting at $\mathcal{I}$. Then for all $x \in \mathcal{L}$ and $y \in \mathcal{I}_i$ the following holds*:

(1) $\mathcal{H}(x, y) \equiv x + y \pmod{\mathcal{I}_{i+1}}$.
(2) $\mathcal{H}(\mathcal{H}(-x, y), x) \equiv y \pmod{[\mathcal{I}_i, \mathcal{L}]_L}$.

Proof. Let us prove (1). We have that $\mathcal{H}(x, y) = x + y + \sum_{n\in\mathbb{N}} u_n(x, y)$ where the $u_n(x, y)$ is a Lie polynomial in x and y and $p^{\lfloor \frac{n-1}{p-1} \rfloor} u_n(x, y)$ has coefficients in $\mathbb{Z}_p$. Since any Lie polynomial in x and y of length n and coefficients in $\mathbb{Z}_p$ is contained in $p^{\lfloor \frac{n-1}{p-1} \rfloor} \mathcal{I}_{i+1}$, one has that $u_n(x, y)$ is contained in $\mathcal{I}_{i+1}$ and the assertion follows.

In order to prove (2) we recall that $\mathcal{H}(\mathcal{H}(-x, y), x) = y + \sum_{i=1}^{n} \frac{1}{n!}[y,_n x]_L$. Since the p-valuation of $n!$ is bounded above by $\lfloor \frac{n-1}{p-1} \rfloor$, it follows that $[y,_n x]_L \in p^{\lfloor \frac{n-1}{p-1} \rfloor}[\mathcal{I}_i, \mathcal{L}]_L$. Therefore $\frac{1}{n!}[y,_n x]_L \in [\mathcal{I}_i, \mathcal{L}]_L$. □

Now we are ready to prove the correspondence between PF-embedded subgroups and PF-embedded Lie ideals.

Theorem 4.5. *Let $G = \mathcal{G}$ be a p-saturable group. Then $N \subseteq G$ is a PF-embedded subgroup of G if and only if $\mathcal{N} = N$ is a PF-embedded Lie ideal of $\mathcal{G}$. Even more, in this circumstances $\mathcal{G}/\mathcal{N} = G/N$ and $[\mathcal{G}, \mathcal{N}]_L = [G, N]_G$.*

Proof. Consider N a PF-embedded subgroup of G and $\{N_i\}$ a potent filtration starting at N. Since N_i is p-saturable, then $\mathcal{N}_i$ is a sub Lie algebra of $\mathcal{G}$. Even more, by part (5) of Proposition 2.1, if $x \in N_i$ and $y \in G$, it follows that $[x^{p^n}, y^{p^n}]_G \in [N_i^{p^n}, G^{p^n}]_G = [N_i, G]_G^{p^{2n}}$. Hence, by Proposition 2.2, we have that $[x^{p^n}, y^{p^n}]_G^{p^{-2n}} \in [N_i, G]_G$. Then $[\mathcal{N}_i, \mathcal{G}]_L \subseteq [N_i, G]_G$, and in particular $[\mathcal{N}_i, \mathcal{G}]_L \leqslant \mathcal{N}_{i+1}$ and $[\mathcal{N}_i, {}_{p-1}\,\mathcal{G}]_L \subseteq [N_i, {}_{p-1}\,G]_G \leqslant N_{i+1}^p = p\mathcal{N}_{i+1}$. Therefore $\{\mathcal{N}_i\}$ is a potent filtration and $\mathcal{N}$ is a PF-embedded Lie ideal of $\mathcal{G}$.

Conversely, consider $\mathcal{N}$ a PF-embedded Lie ideal of $\mathcal{G}$ and $\{\mathcal{N}_i\}$ a potent filtration starting at $\mathcal{N}$. Since $\mathcal{N}_i$ is a p-saturable Lie algebra, N_i is a p-saturable subgroup of G. Even more, if $x \in N_i$ and $y \in G$, then, by previous lemma, $x^y = x + z$ with $z \in [\mathcal{N}_i, \mathcal{G}]$.

On the other hand, any Lie polynomial of length $k(p-1)+1$ in x and $x+z$ it will be contained in $p^k[\mathcal{N}, \mathcal{G}]_L$. Therefore $[x, y]_G = \mathcal{H}(\mathcal{H}(\mathcal{H}(-x, -y), x), y) = \mathcal{H}(-x, x^y)$ will be contained in $[\mathcal{N}_i, \mathcal{G}]_L$. Thus $[N_i, G]_G \leqslant [\mathcal{N}_i, \mathcal{G}]_L$. Arguing in a similar way as above, one concludes that N is a PF-embedded subgroup of G and that $\{N_i\}$ is a potent filtration of G.

The equality $[\mathcal{G}, \mathcal{N}]_L = [G, N]_G$ follows from the above construction. Finally, the fact that $\mathcal{G}/\mathcal{N} = G/N$ follows from the previous two lemmas. □

A direct consequence of the previous theorem is that lower central series and derived series of the group and of the Lie algebra coincide.

Corollary 4.6. *Let G be a p-saturable group and $\mathcal{G}$ its corresponding Lie algebra. Then the lower central series of G and $\mathcal{G}$ coincide. In particular G is nilpotent if and only if $\mathcal{G}$ is nilpotent and the nilpotency class of G and $\mathcal{G}$ coincide.*

Proof. Let $\gamma_i(G)$ and $\gamma_i(\mathcal{G})$ be the lower central series of G and $\mathcal{G}$, respectively. The corollary now follows from the previous theorem and from the fact that $\gamma_i(G)$ and $\gamma_i(\mathcal{G})$ are PF-embedded in G and $\mathcal{G}$, respectively. □

Corollary 4.7. *Let G be a p-saturable group and $\mathcal{G}$ its corresponding Lie algebra. Then the derived series of G and $\mathcal{G}$ coincide. In particular G is solvable if and only if $\mathcal{G}$ is solvable and the derived length of G and $\mathcal{G}$ coincide.*

Proof. Consider $G^{(i)}$ and $\mathcal{G}^{(i)}$ the derived series of G and $\mathcal{G}$, respectively. The corollary now follows from the previous theorem and from the fact that $G^{(i)}$ and $\mathcal{G}^{(i)}$ are p-saturable and $G^{(i+1)}$ and $\mathcal{G}^{(i+1)}$ are PF-embedded in $G^{(i)}$ and $\mathcal{G}^{(i)}$, respectively. □

5. Normal subgroups of p-saturable group

In this section we prove that a normal subgroup of a p-saturable group contained in $\Phi(G)$ is again p-saturable. First we recall the next theorem which is a key tool to understand normal subgroups of PF-groups.

Theorem 5.1. *Let G be a PF-group and N a closed normal subgroup of G. Then there exists a closed PF-subgroup T of G containing N and such that for every PF-embedded subgroup M of T, we have that $[M^{p^i}, T^{p^j}]^{p^k} = [M^{p^r}, N^{p^s}]^{p^t}$ whenever $i + j + k = r + s + t \geqslant 1$.*

Proof. This is a particular case of Theorem 3.7 of [2]. □

We start proving that normal subgroups contained in G^p are p-saturable.

Proposition 5.2. *Let G be a PF-group and N a closed normal subgroup of G contained in G^p. Then there exists a filtration*

$$N = N_1 \geqslant N_2 \geqslant N_3 \geqslant \cdots$$

such that $\bigcap_{i=1}^{\infty} N_i = 1$ and $[N_i, N] \leqslant N_{i+1}^p$. In particular N is a PF-group.

Proof. Consider $M = \langle x \in G \mid x^p \in N \rangle$. Since $N \leqslant G^p$ we have that $N \leqslant M^p$. On the other hand $M/N\gamma_p(M)$ is a regular group generated by elements of order p, therefore $(M/N\gamma_p(M))^p = 1$. Then $M^p \leqslant N\gamma_p(M)$.

By previous theorem there exists a normal subgroup T of G that contains M and a potent filtration $\{T_i\}$ of T starting at T such that $[T_i^p, T] = [M^p, T_i]$. We start proving that $[M^p, T_i] \leqslant [N, T_i]$. By Lemma 4.9 in Chapter 4 of [4]

$$\begin{aligned} \left[M^p, T_i\right] &\leqslant [N, T_i]\left[\gamma_p(M), T_i\right] \leqslant [N, T_i][T_{i,p} M] \\ &\leqslant [N, T_i]\left[T_{i+1}^p, T\right] \leqslant [N, T_i]\left[M^p, T_{i+1}\right]. \end{aligned}$$

Applying this argument several times we have that for all k, $[M^p, T_i] \leqslant [N, T_i][M^p, T_{i+k}]$. Therefore $[M^p, T_i] \leqslant [N, T_i]$. Now we conclude that

$$[N, N] \leqslant \left[M^p, M^p\right] \leqslant \left[T^p, T^p\right] \leqslant \left[M^p, T\right]^p \leqslant [N, T]^p$$

and

$$\begin{aligned} \left[[N, T_k], N\right] &\leqslant \left[M^p, T_k, M^p\right] \leqslant \left[\left[T^p, T_k\right], T^p\right] \\ &\leqslant \left[T_{k+1}^p, T\right]^p \leqslant \left[M^p, T_{k+1}\right]^p \leqslant [N, T_{k+1}]^p. \end{aligned}$$

That is, $N \geqslant [N, T_1] \geqslant [N, T_2] \geqslant [N, T_3] \geqslant \cdots$ satisfies the conclusions of the proposition. □

Corollary 5.3. *Let G be a p-saturable group and N a closed normal subgroup of G. Then N is a Lie subalgebra of $G = \mathcal{G}$ if and only if $N^p = \{n^p \mid n \in N\}$.*

Proof. Consider $x, y \in N$, then $x + y = \lim_{n\to\infty}(x^{p^n} y^{p^n})^{p^{-n}}$. On the other hand N^p is p-saturable. Then, since $N^p = \{n^p \mid n \in N\}$ and by Proposition 2.1, $N^{p^n} = \{n^{p^n} \mid n \in N\}$. Therefore $x^{p^n} y^{p^n} = z^{p^n}$ for some $z \in N$. Then by Proposition 2.2, $(x^{p^n} y^{p^n})^{p^{-n}} = z$. In particular, $x + y = \lim_{n\to\infty}(x^{p^n} y^{p^n})^{p^{-n}} \in N$.

For the Lie product we have that $[x, y]_L = \lim_{n\to\infty}[x^{p^n}, y^{p^n}]_G^{p^{-2n}}$. But, if we take a subgroup T as in Theorem 5.1, we have that $[x^{p^n}, y^{p^n}]_G \leqslant [N^{p^n}, N^{p^n}]_G = [N, T]_G^{p^{2n}}$, and arguing as above we have that $[x, y]_L \in N$. Therefore N is a Lie subalgebra of $\mathcal{G}$. □

Corollary 5.4. *Let $p \geqslant 3$ and let G be a finitely generated torsion free pro-p group such that $\gamma_{p-1}(G)^{\leqslant} G^p$. Then G is p-saturable and all closed normal subgroups are Lie subalgebras of $\mathcal{G} = G$.*

Proof. Since G is a potent group, $\gamma_p(G) \leqslant [G^p, G] = [G, G]^p$ (see Theorems 3.1 and 3.2 of [3]). Therefore, by Theorem 3.4, G is p-saturable and applying Theorem 6.1 of [3] we have that all normal subgroups of G satisfy the condition of the previous corollary. □

In order to extend the previous proposition to more normal subgroups we need to generalize the definition of potent filtrations and PF-groups. Consider G a pro-p group and let $1 \leqslant k \leqslant p-1$ and $m \geqslant 1$. We say that a normal subgroup N is *PF-embedded in G of type (k, m)* if there exists a decreasing central series of closed normal subgroups with trivial intersection $\{N_i\}$ such that $[N_i,_k G] \leqslant N_{i+1}^{p^m}$. In this circumstances, we say that $\{N_i\}$ is a *potent filtration of type (k, m)*. We say that a pro-p group G is a *PF-group of type (k, m)* if G is PF-embedded in G of type (k, m). PF-groups of type (k, m) generalize the concept of k-powerful groups (see [8]). The next theorem constitutes a key tool to understand PF-groups of type (k, m).

Theorem 5.5. *Let G be a PF-group of type (k, m) and N a closed normal subgroup of G. Then there exists T a closed PF-subgroup of type (k, m) of G containing N and such that for every PF-embedded subgroup of type (k, m) M of T, we have that $[M^{p^i}, T^{p^j}]^{p^k} = [M^{p^r}, N^{p^s}]^{p^t}$ whenever $i + j + k = r + s + t \geqslant k$.*

Proof. The proof is the same as that of Theorem 5.1. □

Lemma 5.6. *Let G be a PF-group of type (k, m) with $k + 1 \leqslant p - 1$ and N a closed normal subgroup of G. Then N is a PF-group of type $(k + 1, m)$.*

Proof. Let N be a normal subgroup of G. By previous theorem there exists a subgroup T of G that contains N and a potent filtration $\{T_i\}$ of type (k, m) starting at T such that $[T_i^{p^k}, T] = [N, T_i]^{p^k}$. Then $[N,_{k+1} N] \leqslant [T_2^{p^k}, T] = [N, T_2]^{p^k}$ and $[[N, T_i],_{k+1} N] \leqslant [T_{i+1}^{p^k}, T] = [T_{i+1}, N]^{p^k}$. Therefore

$$N \geqslant [N, T_2] \geqslant [N, T_3] \geqslant [N, T_4] \geqslant \cdots$$

satisfies the conditions of a potent filtration of type (k, m). □

Corollary 5.7. *Let G be a PF-group and N a closed normal subgroup of G contained in $\Phi(G)$. Then N is a PF-group.*

Proof. The case $p = 2$ follows from Proposition 5.2. For the case $p \geqslant 3$, consider T_i a potent filtration starting at G and $H_i = [T_i, G]T_i^p$. Then $[H_i,_{p-2} \Phi(G)] = [[T_i, G]T_i^p,_{p-2} \Phi(G)] \leqslant [[T_i, G],_{p-2} \Phi(G)][T_i^p,_{p-2} \Phi(G)]$. Now, since

$$\big[[T_i, G],_{p-2} \Phi(G)\big] \leqslant [T_i,_{p-1} G]^p [T_i,_p G] \leqslant \big(T_{i+1}^p\big)^p [T_{i+1}, G]^p \leqslant H_{i+1}^p$$

and

$$\big[T_i^p,_{p-2} \Phi(G)\big] \leqslant \big(T_{i+1}^p\big)^p [T_{i+1}, G]^p \leqslant H_{i+1}^p,$$

it follows that $[H_i,_{p-2}\Phi(G)] \leqslant H_{i+1}^p$. Then $\Phi(G)$ is a PF-group of type $(p-2,1)$ and by previous lemma, N is a PF-group. □

Now we are prepared to prove the main result of this section concerning normal subgroups of p-saturable groups.

Corollary 5.8. *Let G be a p-saturable group and N a closed normal subgroup of G contained in $\Phi(G)$. Then N is p-saturable.*

Proof. By previous corollary, N is a PF-group. Since G is torsion free and of finite rank, then N is also torsion free and finitely generated. Then, by Theorem 3.4, we have that N is a p-saturable group. □

We finish this section by giving an example of a normal subgroup of a p-saturable group that is not p-saturable.

Example. Let $A = \langle x_1, \dots, x_p\rangle$ the direct product of p copies of $\mathbb{Z}_p$ and $B = \langle\alpha\rangle \cong \mathbb{Z}_p$. Consider the semidirect product between A and B given by the action $x_i^\alpha = x_i x_{i+1}$ if $1 \leqslant i \leqslant x_{p-1}$, $x_{p-1}^\alpha = x_{p-1}x_p^{p^2}$ and $x_p^\alpha = x_p$. It is clear that $\gamma_p(G) \leqslant G^{p^2}$. Therefore G is a PF-group. Take $N = \langle \alpha, x_1, x_2, \dots, x_{p-1}, x_p^{p^2}\rangle$. N is a normal subgroup of G but $\alpha^p x_1^p$ is not a p-power in N. Therefore N cannot be a PF-group.

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