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Inference on periodograms of infinite dimensional discrete time periodically correlated processes

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ABSTRACT

In this work we shall consider two classes of weakly second-order periodically correlated and strongly second-order periodically correlated processes with values in separable Hilbert spaces. The periodogram for these processes is introduced and its statistical properties are studied. In particular, it is proved that the periodogram is asymptotically unbiased for the spectral density of the processes, where the type of the convergence is fully specified.

1. Introduction

In this paper we let X be a separable Hilbert space with an inner product $(\cdot,\cdot)_X$, and consider a random sequence $\boldsymbol{\xi} = \{\xi^n,\ n \in \mathbb{Z}\}$ in X, where each $\xi^n : \Omega \to X$ is $\mathcal{F}/\mathcal{B}$ measurable, $(\Omega, \mathcal{F}, P)$ is a probability space, $\mathcal{B}$ is the Borel field in X, and $\mathbb{Z}$ is the set of integers. We refer to $\boldsymbol{\xi}$ as an X-valued discrete time stochastic process. This process is called second order (SO in abbreviation) if every $\xi^n \in L^2(\Omega, \mathcal{F}, P)$, where the latter is the Hilbert space of all mean zero complex random variables u defined on $(\Omega, \mathcal{F}, P)$ with $E|u|^2 < \infty$, equipped with the inner product $Eu\overline{v}$, where E stands for the expectation. The process $\boldsymbol{\xi}$ is called *weakly second order* (WSO in abbreviation) if $\xi_x^n = (\xi^n, x)_X \in L^2(\Omega, \mathcal{F}, P)$, and *strongly second order* (SSO in abbreviation) if $\|\xi^n\|_X \in L^2(\Omega, \mathcal{F}, P)$, for all $n \in \mathbb{Z},\ x \in X$.

An X-valued (WSO or SSO) stochastic process is said to be *periodically correlated* (PC in abbreviation) if there exists an integer $T > 0$ such that for every $x, y \in X$ and $m, n \in \mathbb{Z}$, $E\xi_x^n\overline{\xi_y^m} = E\xi_x^{n+T}\overline{\xi_y^{m+T}}$. The smallest such T is the period of the process. If $T = 1$, then the process is called stationary.

Basic spectral foundations of Hilbert space-valued WSO *stationary* processes are established by Rozanov [1], Salehi and Soltani [2], among others. Hilbert space-valued SSO stationary processes are also intensively studied by different authors: Gihman and Skorohod [3], Bosq [4], Chen and White [5], among others. Hilbert space-valued WSO PC processes are studied by Soltani and Shishebor [6], where basic spectral structures of such processes are provided.

In this work, we introduce the periodogram of periodically correlated (PC) processes of type WSO or SSO with values in separable Hilbert spaces. The periodogram is commonly defined on a segment of the process; see Section 4. We prove that the periodogram is asymptotically unbiased for the corresponding spectral density, as the length of the segment tends to infinity. The asymptotic unbiasedness of the periodogram appears to be in the weak sense for WSO PC processes, and in

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the strong sense for SSO PC processes, as given in Theorem 1. Our methodology is to derive the results for an auxiliary PC process, which is more convenient to work with, then extend the results to the process itself.

Periodograms are useful tools in time series for estimating the spectral densities and highlighting active frequencies. Although periodograms of multivariate stationary processes are intensively studied, to the best of our knowledge this is the first work on periodograms of infinite dimensional PC processes. This work is inspired by the work of Soltani and Azimmohseni [7], Hurd [8] on periodograms of univariate PC processes; and also Pourahmadi and Salehi [9]. The work of Makagon, Miamee, Salehi and Soltani [12] gives insights to the spectral domin of PC processes. This article is organized as follows.

Preliminaries are given in Section 2. In Section 3, basic properties of the finite Fourier transform for various segments corresponding to the PC processes are introduced and studied. Periodograms are defined in Section 4, and are proved to be asymptotically unbiased for the corresponding spectral densities.

2. Notation and preliminaries

Following the notions and terminology given in Section 1, let $\boldsymbol{\xi} = \{\xi^n,\ n \in \mathbb{Z}\}$ be an X-valued WSO or SSO PC process. We also write $\boldsymbol{\xi} = \{\xi^n_x,\ n \in \mathbb{Z},\ x \in X\}$, where every ξ^n_x, as a function in x, is in $L(X)$, where $L(X)$ stands for the bounded linear operators on X. In this case, every ξ^n is also specified by $\{\xi^n_{e_i},\ i = 1, 2, \dots\}$, where $\{e_1, e_2, \dots\}$ is an orthonormal basis for X. Evidently every SSO PC process is also WSO PC, and $\sum_{i=1}^{\infty} E|(\xi(\cdot), e_i)_X|^2 < \infty$. Under the latter condition, every WSO PC process is SSO PC.

Univariate second-order PC processes were introduced and studied by Gladyshev [10]. The X-valued WSO PC processes were studied by Soltani and Shishebor [6]. It is proved in this reference that such a process is harmonizable, i.e.,

$$\xi^n = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\lambda} \mathcal{Z}(\mathrm{d}\lambda), \quad n \in \mathbb{Z},$$

where the spectral random measure $\mathcal{Z}$ does not necessarily have orthogonal increments, but possesses the property that $E\mathcal{Z}(\mathrm{d}s)x\overline{\mathcal{Z}(\mathrm{d}t)y} = (x, F(\mathrm{d}s, \mathrm{d}t)y)$ defines a so called spectral distribution $F(\cdot,\cdot)$ on $[0, 2\pi) \times [0, 2\pi)$, which is supported by the parallel lines $d_k = \{(s, t) \in [0, 2\pi)^2, s - t = \frac{2\pi k}{T}\},\ k = 1 - T, \dots, T - 1$. The spectral distribution is indeed specified by T, $L(X)$-valued distributions $\{F_0, F_1, \dots, F_{T-1}\}$ on $[0, 2\pi)$ such that the matrix $\mathcal{F}$ defined by

$$\mathcal{F}(\mathrm{d}s) = \left[F_{p-l}\left(\mathrm{d}s + \frac{2\pi p}{T}\right)\right]_{p,l=0,\dots,T-1}, \quad s \in \left[0, \frac{2\pi}{T}\right),$$

is positive definite; see [6]. We also assume that the spectral density $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}(\mathrm{d}s)$ exists:

$$\mathbf{f}(s) = \left[f_{p-l}\left(\mathrm{d}s + \frac{2\pi p}{T}\right)\right]_{l,p=0,\dots,T-1}, \quad s \in \left[0, \frac{2\pi}{T}\right).$$

An alternative time dependent spectral representation is also given by Soltani and Shishebor [6,13], namely

$$\xi^n_x = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ns} \Phi(\mathrm{d}s) V_n(s)x, \quad n \in \mathbb{Z},$$

in the sense that

$$E\xi^n_x\overline{\xi^m_y} = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(n-m)s}(V_n(s)x, V_m(s)y)_X \mathrm{d}s, \quad n, m \in \mathbb{Z},$$

where Φ is an orthogonally scattered random measure, and $V_n(s) = \sum_{k=0}^{T-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi kn}{T}} a_k(s + \frac{2\pi k}{T})$ is a sequence of T-periodic, $L(X)$-valued functions for $s \in [0, 2\pi)$ and $n \in \mathbb{Z}$. Furthermore $\mathbf{f}(s) = \mathbf{A}^*(s)\mathbf{A}(s)$, $s \in [0, \frac{2\pi}{T})$, where $\mathbf{A}(s) = [\overline{a_{j-k}(s + \frac{2\pi j}{T})}]_{k\le j}$, $k, j = 0, \dots, T-1$. The operator-matrix $\mathbf{A}$ is the Cholesky factor of the spectral density $\mathbf{f}$.

The mapping $\mathcal{K}(\xi^n_x) = \mathrm{e}^{-\mathrm{i}ns}V_n(s)x$ establishes an isometry between the time domain and the spectral domain of the PC process $\boldsymbol{\xi}$. The spectral domain of $\boldsymbol{\xi}$ is a closed subset of $L^2(X)$, generated by $\{\mathrm{e}^{-\mathrm{i}n\cdot}V_n(\cdot)x,\ n \in \mathbb{Z}, x \in X\}$. We recall that $L^2(X)$ is the Hilbert space of all X-valued functions u on $[0, 2\pi)$ such that $\int_0^{2\pi} \|u(s)\|^2_X \mathrm{d}s < \infty$, with the inner product $(u, v)_{L^2(X)} = \int_0^{2\pi}(u(s), v(s))_X \mathrm{d}s$. Note that $E\xi^n_x\overline{\xi^m_y} = (\mathcal{K}(\xi^n_x), \mathcal{K}(\xi^m_y))_{L^2(X)}$.

We note that a WSO PC process is SSO if and only if $\sum_{i=1}^{\infty}\int_0^{2\pi} \|V_n(s)e_i\|^2_X \mathrm{d}s < \infty$, for $n = 0, \dots, T-1$. It is straightforward to verify that this is equivalent to $\sum_{i=1}^{\infty} \|a_n(s)e_i\|^2_X \in L^1([0, 2\pi), \mathrm{d}s)$ for $n = 0, \dots, T-1$. Also $E(\sum_{i=1}^{\infty} |(\xi^n(\cdot), e_i)_X|^2) = \sum_{i=1}^{\infty}\int_0^{2\pi} \|V_n(s)e_i\|^2_X \mathrm{d}s < \infty$. It readily follows that the matrix-operator $\mathbf{f}$ is the spectral density of a SSO PC process if and only if it is nuclear, i.e. each of its entries is nuclear. Equivalently, the matrix-operator $\mathbf{A}$ is Hilbert–Schmidt.

Let us record the following properties that will be used in subsequent sections. (i) Φ defines a WSO X-valued random measure; (ii) $\Phi(\mathrm{d}s, \omega)V_n(s)$ defines a SSO X-valued random measure. It follows from (i) that $\eta^n = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ns}\Phi(\mathrm{d}s),\ n \in \mathbb{Z}$, defines an X-valued WSO stationary process.

3. Asymptotic equivalences

Let $d_\zeta^N(\lambda) = N^{-1/2}\sum_{t=0}^{N-1}\zeta^t e^{it\lambda}$, $\lambda \in [0, 2\pi)$, denote the finite Fourier transform (FFT) of a segment $\zeta^1, \zeta^2, \ldots, \zeta^N$ in X. As is customary, we define the FFT terms to be step functions with jumps at Fourier frequencies $\frac{2\pi k}{N}$, $k = 0, \ldots, N-1$, and drop the superscript N whenever there is no ambiguity.

Let $\boldsymbol{\xi}$ be an X-valued (WSO or SSO) process, and let $\{\xi^1, \ldots, \xi^N\}$ be a segment of $\boldsymbol{\xi}$. We assume $N = mT$. Also let

$$\tilde{\xi}_N^n = N^{-1/2}\sum_{p=0}^{N-1} e^{-in\lambda_p} d_\eta(\lambda_p)V_n(\lambda_p), \quad n \in \mathbb{Z}.$$

Clearly, if $\boldsymbol{\xi}$ is SSO, then $d_\xi^N(\lambda)$ and $d_{\tilde{\xi}}^N(\lambda)$, $\lambda \in [0, 2\pi)$ are SSO. It is straightforward to verify that

$$E|d_\eta(\lambda_p)V_n(\lambda_p)e_i|^2 = 2\pi\ \|V_n(\lambda_p)e_i\|_X^2, \quad \text{and} \quad E(d_\eta(\lambda_p)V_n(\lambda_p)x)\overline{(d_\eta(\lambda_q)V_n(\lambda_q)y)} = 0,$$

$p, q = 0, \ldots, N-1$, $p \neq q$. Therefore $d_\eta(\lambda_p)V_n(\lambda_p)$, $p = 0, \ldots, N-1$, possesses the covariance operator $[B_{\eta,n}(p, q)]_{p,q=0,\ldots,N-1}$, where $B_{\eta,n}(p, q) = 0$ for $p \neq q$ and $B_{\eta,n}(p, p) = V_n^*(\lambda_p)V_n(\lambda_p)$, which is nuclear if the process is SSO. This also implies that the X-valued random variables $d_\eta(\lambda_p)V_n(\lambda_p)$, $p = 0, \ldots, N-1$, are pairwise uncorrelated. It will be more convenient to work with $\tilde{\boldsymbol{\xi}}$ and $d_{\tilde{\xi}}$ rather than $\boldsymbol{\xi}$ and d_ξ. We will show in this section that the corresponding terms share the same asymptotic properties.

The Dirichlet kernel $D_N(\theta) = \sum_{t=0}^{N-1} e^{it\theta}$ and the Fejer kernel $K_N(\theta) \equiv |D_N(\theta)|^2/(2\pi N) = \sin^2(N\theta/2)/\{(2\pi N)\sin^2(\theta/2)\}$, $\theta \in [0, 2\pi)$, are useful tools in the theory of Fourier transformations. These kernels have potential applications in the spectral estimation of stationary processes. The following kernel which appears to be useful in the spectral estimation of PC processes was introduced by Soltani and Azimmohseni [7]:

$$S_N(\theta; \eta, \eta') = \frac{D_N(\theta-\eta)D_N(\theta-\eta')}{N}, \quad \theta \in [0, 2\pi),\ \eta, \eta' \in [0, 2\pi).$$

It has the following properties: (i) $S_N(\theta; \eta, \eta) = 2\pi K_N(\theta - \eta)$, where K_N is the Fejer kernel; (ii) $S_N(\theta; \eta, \eta') \to 0$, $N \to \infty$, for $\eta \neq \eta'$, $\theta \in [0, 2\pi)$, $\theta \neq \eta$ and η'; (iii) for any $0 < \delta < \frac{1}{2}|\eta - \eta'|$, $|S_N(\theta; \eta, \eta')| < 1/\sin^2(\frac{\delta}{2})$, $N \geq 1$, $\theta \in [0, 2\pi)$, $\eta, \eta' \in [0, 2\pi)$, $\eta \neq \eta'$.

Our first asymptotic result below shows that the PC process $\tilde{\xi}$ converges to ξ. For $x \in X$, $n = 0, \ldots, T-1$, let us set $u_{n,x}(\theta) = \|V_n(\theta)x\|_X^2$, $v_n(\theta) = \sum_{i=0}^\infty \|V_n(\theta)e_i\|_X^2$, and also set $u_{n,x}(\theta, \theta') = (V_n(\theta)x, V_n(\theta')x)_X$, $v_n(\theta, \theta') = \sum_{i=0}^\infty (V_n(\theta)e_i, V_n(\theta')e_i)_X$.

Lemma 1. (i) *Let ξ be a WSO PC process for which $u_{n,x}(\cdot)$, $n = 0, \ldots, T-1$, are continuous and of bounded variation on $[0, 2\pi)$. Then for each $t \in \mathbb{Z}$, $x \in X$,*

$$E|\tilde{\xi}_x^t - \xi_x^t|^2 \to 0, \quad N \to \infty. \tag{3.1}$$

(ii) *Let ξ be a SSO PC process for which $v_n(\cdot)$, $n = 0, \ldots, T-1$, are continuous and of bounded variation on $[0, 2\pi)$. Then for each $t \in \mathbb{Z}$,*

$$E\|\tilde{\xi}^t - \xi^t\|_X^2 \to 0, \quad N \to \infty. \tag{3.2}$$

Proof. It follows from the Kolmogorov isomorphism that the terms in (3.1) and (3.2) are respectively equal to

$$\int_0^{2\pi}\left\|\frac{1}{N}\sum_{p=0}^{N-1} e^{-i\lambda_p t}D_N(\lambda_p - \theta)V_t(\lambda_p)x - e^{-it\theta}V_t(\theta)x\right\|_X^2 d\theta,$$

and

$$\int_0^{2\pi}\sum_{i=0}^\infty\left\|\frac{1}{N}\sum_{p=0}^{N-1} e^{-i\lambda_p t}D_N(\lambda_p - \theta)V_t(\lambda_p)e_i - e^{-it\theta}V_t(\theta)e_i\right\|_X^2 d\theta,$$

which can be written as $a_t^N + b_t^N - e_t^N - \overline{e}_t^N$, where for the WSO process

$$a_t^N = \int_0^{2\pi}\frac{1}{N^2}\sum_{p,p'=0}^{N-1} e^{-i\lambda_p t + i\lambda_{p'}t}D_N(\lambda_p - \theta)D_N(\lambda_{p'} - \theta)u_{t,x}(\lambda_p, \lambda_{p'})d\theta,$$

$$b_t^N = \int_0^{2\pi} u_{t,x}(\theta)d\theta,$$

$$e_t^N = \frac{1}{N}\sum_{p=0}^{N-1} e^{-i\lambda_p t}\int_0^{2\pi} D_N(\lambda_p - \theta)e^{it\theta}u_{t,x}(\lambda_p, \theta)d\theta;$$

the same expression for the SSO process will be true if the u functions are replaced by the v functions. We will show that the limits of a_t^N and e_t^N are the same as that of b_t^N. For a_t^N, by using the facts that $\int_0^{2\pi} D_N(\lambda-\theta)D_N(\lambda^*-\theta)\mathrm{d}\theta = 2\pi D_N(\lambda-\lambda^*)$, and $D_N(\lambda_p-\lambda_{p'}) = N\mathbf{1}(p=p')$ for the Fourier frequencies, we deduce that $a_t^N = \frac{2\pi}{N}\sum_{p=0}^{N-1} u_{t,x}(\lambda_p)$, which converges to $\int_0^{2\pi} u_{t,x}(\theta)\mathrm{d}\theta < \infty$. For e_t^N, let $C_{N,M} = \frac{1}{N}\sum_{p=0}^{N-1} \mathrm{e}^{-\mathrm{i}\lambda_p t}\int_0^{2\pi} D_M(\lambda_p-\theta)\mathrm{e}^{it\theta}u_{t,x}(\lambda_p,\theta)\mathrm{d}\theta$, and $C_N = \frac{2\pi}{N}\sum_{p=0}^{N-1} u_{t,x}(\lambda_p)$. Under the assumption that the u functions are continuous and are of bounded variation on $[0,2\pi)$, the integral term in $C_{N,M}$ converges to $2\pi \mathrm{e}^{\mathrm{i}\lambda_p t}u_{t,x}(\lambda_p)$ uniformly in λ_p, as $M\to\infty$; consequently $C_{N,M}$ converges to C_N uniformly in N as $M\to\infty$. Therefore $\lim_{N\to\infty} e_t^N = \lim_{N\to\infty} C_{N,N} = \lim_{N\to\infty}\lim_{M\to\infty} C_{N,M} = \lim_{N\to\infty} C_N = \int_0^{2\pi} u_{t,x}(\theta)\mathrm{d}\theta$. The proof is complete. □

Let us highlight some applications of Lemma 1. It can be used to test whether a segment $\xi^1,\ldots,\xi^n$ is a segment of a PC process with given $\{V_0(\cdot),\ldots,V_n(\cdot)\}$. More precisely, by replacing $\tilde{\xi}$ by ξ in $d_{\tilde{\xi}}^N(\lambda)$, one can solve the resulting linear equations for $\{d_\eta(\lambda_p), p=0,\ldots,N-1\}$ and then test whether the solutions are the FFT of a white noise process.

Also, since $E|\tilde{\xi}_N^n x|_X^2 = (2\pi)/N\sum_{p=0}^{N-1} u_{n,x}(\lambda_p)$, and $E\|\tilde{\xi}_N^n\|_X^2 = (2\pi)/N\sum_{p=0}^{N-1} v_n(\lambda_p)$, $n\in\mathbb{Z}, N=1,2,\ldots$, it readily follows that for large N the contribution of every Fourier frequency to the variances of a WSO or SSO PC process can be measured by solving a number of simultaneous equations. We note that the variances are periodic too.

In what follows, we prove that the mean square deviation between $d_{\tilde{\xi}}^N(\lambda)$ and $d_\xi^N(\lambda)$ goes to zero as N tends to infinity, under the mild assumption of continuity of the Cholesky factor. Let us introduce some functions that will be useful in the following lemma. Let

$$h_{k,x}(\theta) = \left\| a_k\left(\theta+\frac{2\pi k}{T}\right)x\right\|_X^2, \qquad g_k(\theta) = \sum_{i=0}^{\infty}\left\| a_k\left(\theta+\frac{2\pi k}{T}\right)e_i\right\|_X^2,$$

$$h_{k,k',x}(\theta,\theta') = \left(a_k\left(\theta+\frac{2\pi k}{T}\right)x, a_{k'}\left(\theta'+\frac{2\pi k'}{T}\right)x\right)_X,$$

$$g_{k,k'}(\theta,\theta') = \sum_{i=0}^{\infty}\left(a_k\left(\theta+\frac{2\pi k}{T}\right)e_i, a_{k'}\left(\theta'+\frac{2\pi k'}{T}\right)e_i\right)_X,$$

$x\in X$ and $k=0,\ldots,T-1$. We also note that

$$\mathcal{K}(d_\xi(\lambda)x) = N^{\frac{-1}{2}}\sum_{t=0}^{N-1}\mathrm{e}^{\mathrm{i}t(\lambda-\theta)}V_t(\theta)x = N^{\frac{-1}{2}}\sum_{k=0}^{T-1} D_N\left(\lambda-\theta-\frac{2\pi k}{T}\right)a_k\left(\theta+\frac{2\pi k}{T}\right)x,$$

and $\mathcal{K}(d_{\tilde{\xi}}(\lambda)x) = N^{\frac{-1}{2}}\sum_{t=0}^{N-1}\mathrm{e}^{\mathrm{i}t\lambda}\mathcal{K}(\tilde{\xi}^t)$ is equal to

$$N^{\frac{-3}{2}}\sum_{k=0}^{T-1}\sum_{p=0}^{N-1} D_N\left(\lambda-\lambda_p-\frac{2\pi k}{T}\right)D_N(\lambda_p-\theta)a_k\left(\lambda_p+\frac{2\pi k}{T}\right)x.$$

Lemma 2. (i) *Let ξ be a WSO PC process for which $h_{n,x}(\cdot)$, $n=0,\ldots,T-1$, are continuous on* $[0,2\pi)$. *Then at every* $\lambda\in[0,2\pi)$,

$$E|d_{\tilde{\xi}}(\lambda)x - d_\xi(\lambda)x|^2 \to 0, \quad x\in X,\ N\to\infty. \tag{3.3}$$

(ii) *Let ξ be a SSO PC process for which $g_n(\cdot)$, $n=0,\ldots,T-1$, are continuous on* $[0,2\pi)$. *Then at every* $\lambda\in[0,2\pi)$,

$$E\|d_{\tilde{\xi}}(\lambda) - d_\xi(\lambda)\|^2 \to 0, \quad N\to\infty. \tag{3.4}$$

Proof. Let us fix $\lambda\in[0,2\pi)$, then for each $N>1$ choose Fourier frequencies $\lambda_{q(N)}$ such that $\lambda_{q(N)}\le\lambda<\lambda_{q(N)+1}$, and $\lambda_{q(N)}\to\lambda$, as $N\to\infty$. Note that the terms in (3.3) and (3.4) are respectively equal to

$$E\left|d_{\tilde{\xi}}(\lambda)x - d_\xi(\lambda)x\right|^2 = \int_0^{2\pi}\|\mathcal{K}(d_{\tilde{\xi}}(\lambda)x) - \mathcal{K}(d_\xi(\lambda)x)\|_X^2\mathrm{d}\theta,$$

and

$$E\|d_{\tilde{\xi}}(\lambda) - d_\xi(\lambda)\|^2 = \int_0^{2\pi}\sum_{i=0}^{\infty}\|\mathcal{K}(d_{\tilde{\xi}}(\lambda)e_i) - \mathcal{K}(d_\xi(\lambda)e_i)\|_X^2\mathrm{d}\theta,$$

which can be written as $A_\lambda^N + B_\lambda^N - E_\lambda^N - \overline{E_\lambda^N}$, where, for each WSO process,

$$A_\lambda^N = \sum_{k=0}^{T-1}\int_0^{2\pi} K_N\left(\lambda_{q(N)}-\theta-\frac{2\pi k}{T}\right)h_{k,x}(\theta)\mathrm{d}\theta + \sum_{k\neq k'}\int_0^{2\pi} S_N\left(\theta, \lambda_{q(N)}-\frac{2\pi k}{T}, \lambda_{q(N)}-\frac{2\pi k'}{T}\right)h_{k,k',x}(\theta,\theta)\mathrm{d}\theta,$$

$$B_\lambda^N = \frac{1}{N}\sum_{k,k'=0}^{T-1}\sum_{p=0}^{N-1} S_N\left(\lambda_p, \lambda_{q(N)} - \frac{2\pi k}{T}, \lambda_{q(N)} - \frac{2\pi k'}{T}\right) h_{k,k',x}(\lambda_p, \lambda_p),$$

$$E_\lambda^N = \frac{1}{N^2}\int_0^{2\pi}\sum_{k,k'=0}^{T-1}\sum_{p=0}^{N-1} D_N\left(\theta + \frac{2\pi k}{T} - \lambda_{q(N)}\right) D_N\left(\lambda_p + \frac{2\pi k'}{T} - \lambda_{q(N)}\right) D_N(\lambda_p - \theta) h_{k,k',x}(\theta, \lambda_p)\mathrm{d}\theta.$$

The same expressions for the SSO processes will hold if the h functions are replaced by the g functions. Let us treat these terms one by one. Under the continuity assumption, the integral involved in the first term of A_λ^N equals $\|a_k(\lambda_q)x\|_X^2$, which converges to $\|a_k(\lambda)x\|_X^2$ for each k as $N \to \infty$. The second term in A_λ^N converges to zero. Indeed, we note that for $k \neq k'$, $S_N(\theta, \lambda_{q(N)} - \frac{2\pi k}{T}, \lambda_{q(N)} - \frac{2\pi k'}{T}) \to 0$ for every $\theta \in [0, 2\pi)$ except for $\theta = \lambda_{q(N)} - \frac{2\pi k}{T}$ and $\theta = \lambda_{q(N)} - \frac{2\pi k'}{T}$. Moreover since a given θ cannot be equal to these exception points simultaneously, it follows from the property (iii) of the $S_N(\cdot,\cdot,\cdot)$ that for a.e. θ, w.r.t. the Lebesgue measure,

$$\left|S_N\left(\theta, \lambda_{q(N)} - \frac{2\pi k}{T}, \lambda_{q(N)} - \frac{2\pi k'}{T}\right)\right| \leq \frac{1}{\sin^2\left(\frac{\delta}{2}\right)},$$

where $\delta < \frac{\pi|k-k'|}{T}$. Since $\int_0^{2\pi} h_{k,k',x}(\theta,\theta)\mathrm{d}\theta < \infty$, we conclude, using the Dominated Convergence Theorem, that the integral in the second term in A_λ^N converges to zero. Therefore $A_\lambda^N \to \sum_{k=0}^{T-1}\|a_k(\lambda)x\|_X^2$ as $N \to \infty$. Similarly, it can be shown that B_λ^N and E_λ^N have the same limiting values as A_λ^N, and then we conclude the lemma. □

Corollary 1. *Under the assumption of Lemma 2, for $\lambda, \lambda' \in [0, 2\pi)$,*

$$E|d_{\tilde{\xi}}(\lambda)x\overline{d_{\tilde{\xi}}(\lambda')x} - d_\xi(\lambda)x\overline{d_\xi(\lambda')x}| \to 0, \quad N \to \infty,\ x \in X,$$

for WSO PC processes; and for SSO PC processes,

$$E\left|\sum_{i=0}^{\infty}\{d_{\tilde{\xi}}(\lambda)e_i\overline{d_{\tilde{\xi}}(\lambda')e_i} - d_\xi(\lambda)e_i\overline{d_\xi(\lambda')e_i}\}\right| \to 0, \quad N \to \infty.$$

4. Periodograms

For a segment $\zeta^1, \ldots, \zeta^N$ in the X,

$$\mathbf{d}_\zeta^T(\lambda) = \left(d_\zeta(\lambda), \ldots, d_\zeta\left(\lambda + \frac{2\pi(T-1)}{T}\right)\right)', \quad \lambda \in \left[0, \frac{2\pi}{T}\right). \tag{4.1}$$

If $\boldsymbol{\xi}$ is an X-valued PC (WSO or SSO) process with period T, then we call $\{\mathbf{d}_\xi^T(\lambda),\ \lambda : \text{Fourier frequency}\}$ the sample finite Fourier transform (SFFT). The periodogram for a WSO as well as a SSO PC process is defined to be

$$\mathbf{I}_\xi^T(\lambda) = [I_{k,\ell}(\lambda)]_{k,\ell=0,\ldots,T-1}, \quad \lambda \in \left[0, \frac{2\pi}{T}\right), \tag{4.2}$$

$$(I_{k,\ell}(\lambda)x, y) = d_\xi\left(\lambda + \frac{2\pi k}{T}\right)x\overline{d_\xi\left(\lambda + \frac{2\pi\ell}{T}\right)y}, \quad x, y \in X. \tag{4.3}$$

By using Theorem 12.8 of Rudin [11], we observe that each $I_{k,\ell}(\lambda)$ is a random $L(X)$-valued function on $[0, \frac{2\pi}{T})$. Moreover, for WSO or SSO processes,

$$E|(I_{k,\ell}(\lambda)x, y)| < \infty, \qquad E\|I_{k,\ell}(\lambda)x\|^2 < \infty, \quad k, \ell = 0, \ldots, T-1,$$

respectively, Let us give the main result of this article.

Theorem 1. *Let ξ be an X-valued PC process with the spectral density $\mathbf{f}(\lambda),\ \lambda \in [0, 2\pi)$. Let $\mathbf{A}(\lambda),\ \lambda \in [0, 2\pi)$ be the Cholesky factor of $\mathbf{f}$. Assume, for every $x \in X$, $\mathbf{A}(\lambda)x$ is continuous in λ w.r.t. the norm in X. Also let $\mathbf{d}_\xi^T(\lambda)$ and $\mathbf{I}_\xi^T(\lambda)$ be the corresponding SFFT and periodogram, respectively. Then:*

(i) *If ξ is WSO then for $k, \ell = 0, \ldots, T-1$,*

$$E(I_{k,\ell}(\lambda)x, y)_X \longrightarrow (2\pi f_{k,\ell}(\lambda)x, y)_X, \qquad N \to \infty, x, y \in X. \tag{4.4}$$

(ii) *If ξ is SSO then for $k, \ell = 0, \ldots, T-1$,*

$$E\|I_{k,\ell}(\lambda)x - 2\pi f_{k,\ell}(\lambda)x\|_X^2 \longrightarrow 0, \qquad N \to \infty, x \in X. \tag{4.5}$$

(iii) *For arbitrary frequencies $\lambda_1, \ldots, \lambda_J$ in $[0, \frac{2\pi}{T})$, SFFT $\mathbf{d}_\xi^T(\lambda_1), \ldots, \mathbf{d}_\xi^T(\lambda_J)$ are asymptotically uncorrelated with mean zero and covariance operators $\mathbf{f}(\lambda_1), \ldots, \mathbf{f}(\lambda_J)$.*

Proof. Indeed, we provide (4.4) and (4.5) for the periodogram of the auxiliary process $\tilde{\boldsymbol{\xi}}$, and then use Lemma 2 and Corollary 1 to deduce them for the periodogram of the process $\boldsymbol{\xi}$. It is straightforward to verify that for Fourier frequencies $\lambda_j = \frac{2\pi j}{T}$ that are in $[0, \frac{2\pi}{T})$,

$$\mathbf{d}^T_{\tilde{\xi}}(\lambda_j) = \mathbf{A}(\lambda_j)\mathbf{d}^T_{\eta}(\lambda_j),$$

where ad_η is understood as $d_\eta a$, for a and d_η entries of $\mathbf{A}(\lambda_j)$ and $\mathbf{d}^T_\eta(\lambda_j)$, respectively. This gives

$$E\mathbf{d}^T_{\tilde{\xi}}(\lambda_p)\mathbf{d}^T_{\tilde{\xi}}(\lambda_p)^* = \mathbf{A}(\lambda_p)E\left(\mathbf{d}^T_\eta(\lambda_p)\mathbf{d}^T_\eta(\lambda_p)^*\right)\mathbf{A}(\lambda_p)^* = 2\pi\mathbf{f}(\lambda_p), \tag{4.6}$$

in the sense that the inner products between the corresponding entries applied to x and y are the same, for all $x, y \in X$. Thus, in this sense, at Fourier frequencies in $[0, \frac{2\pi}{T})$,

$$E\mathbf{I}^T_{\tilde{\xi}}(\lambda_p) = 2\pi\mathbf{f}(\lambda_p). \tag{4.7}$$

For an arbitrary frequency in $\lambda \in [0, \frac{2\pi}{T})$, choose a sequence of Fourier frequencies $\{\lambda_{p(N)}\}$ that converge to λ, then apply (4.7) together with Lemma 2 to conclude the result. For (iii) note that $\mathbf{d}^T_{\tilde{\xi}}(\lambda_p)$ and $\mathbf{d}^T_{\tilde{\xi}}(\lambda_q)$ are uncorrelated for $p \neq q$. Then use Lemma 2 and (4.6). □

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