Lyndon factorization of the Prouhet words

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Abstract

Prouhet words are a natural generalization, over alphabets with more than two letters, of the well known binary Thue–Morse word.

We give a unique factorization of these words in a sequence of decreasing Lyndon words, then generalizing such a decomposition given by Ido and Melançon for the Thue–Morse word.

1. Introduction

In 1906, Thue was the first to publish a paper [18] in which the combinatorial properties of some strings of letters were explicitly studied. In 1912, he published another paper [19] in which a binary infinite overlap-free word, now called the Thue–Morse word was in particular introduced. Thue’s results were rediscovered independently years after (see [8] and, to know what Thue exactly did, [4]). In particular, Morse [13] proved again that the Thue–Morse word \( t \) is overlap-free. This word is generated by the Thue–Morse morphism \( \mu \) defined on the alphabet \( A = \{0, 1\} \) by \( \mu(0) = 01, \mu(1) = 10 \). Thus

\[
t = 01101100110110011010011001101001101001101001101101010011010010110011010011001101011001101010011010100101100110100110100110110101001101010010110011010011010011011010100101100110101100110...
\]

A construction made by Prouhet in 1851 [14] surprisingly produced an algorithm which, when applied with two letters, gives the Thue–Morse word. The Prouhet words are those obtained when the algorithm runs with \( n \) letters \((n \geq 2)\) (see, e.g., [1] or [2]). All of them are overlap-free and generated by morphisms (see [3,15,16]).

Lyndon words are words on a totally ordered alphabet that are lexicographically smaller than all their proper suffixes (see, e.g., [7,17], and for general properties of Lyndon words, [10]).

In 1997, Ido and Melançon [9] gave a (unique) decomposition of the Thue–Morse word as an infinite sequence of strictly decreasing Lyndon words. We generalize this...
result by giving such a decomposition for all the Prouhet words. Remark that, recently, Černý [6] studied another generalization of the Thue–Morse word: the infinite binary words $T_n$, obtained by counting, for a given word $u$ over the alphabet $\{0, 1\}$, the number of occurrences of $u$ in the binary expansion of the integers. For some values of $u$, he gave the decomposition of $T_n$ as an infinite sequence of strictly decreasing Lyndon words.

The paper is organized as follows. In Section 2, we give the main notations and definitions. Section 3 is dedicated to the Prouhet words and morphisms. In the main section, Section 4, we first recall results about the Lyndon factorization of finite or infinite words. Then we prove the existence of a unique factorization of each Prouhet word as an infinite product of decreasing finite Lyndon words and we give an algorithm to compute effectively such a factorization.

2. Notations and definitions

The terminology and notations are mainly those of Lothaire [11].

Let $A$ be a finite set called alphabet and $A^*$ the free monoid generated by $A$.

The elements of $A$ are called letters and those of $A^*$ are called words. The empty word $\varepsilon$ is the neutral element of $A^*$ for the concatenation of words (the concatenation of two words $u$ and $v$ is the word $uv$), and we denote by $A^+$ the semigroup $A^* \backslash \{\varepsilon\}$.

If $u \in A^*$, then $|u|$ is the length of $u$ (in particular $|\varepsilon| = 0$).

A word $w$ is called a factor (resp. a prefix, resp. a suffix) of $u$ if there exist words $x, y$ such that $u = xwy$ (resp. $u = wy$, resp. $u = xw$). The factor (resp. the prefix, resp. the suffix) is proper if $xy \neq \varepsilon$ (resp. $y \neq \varepsilon$, resp. $x \neq \varepsilon$).

An infinite word (or sequence) over $A$ is an application $a : \mathbb{N} \to A$. It is written as $a = a_0a_1 \cdots a_i \ldots, i \in \mathbb{N}, a_i \in A$.

The notion of factor is extended to infinite words as follows: a (finite) word $u$ is a factor (resp. prefix, resp. suffix) of an infinite word $a$ over $A$ if there exist $n \in \mathbb{N}$ (resp. $n = 0$) and $m \in \mathbb{N}$ ($m = |u|$) such that $u = a_n \cdots a_{n+m-1}$ (by convention $a_n \cdots a_{n-1} = \varepsilon$).

An overlap is a word $axax \ldots$ where $a \in A, x \in A^*$. A (finite or infinite) word $u$ over $A$ is overlap-free if none of its factors is an overlap.

In what follows, we will consider morphisms on $A$.

A morphism on $A$ (in short morphism) is an application $f : A^* \to A^*$ such that $f(uv) = f(u)f(v)$ for all $u, v \in A^*$. It is uniquely determined by its value on the alphabet $A$. A morphism $f$ is $k$-uniform, $k \in \mathbb{N}$, if $|f(a)| = k$ for all $a \in A$.

A morphism is nonerasing if $f(a) \neq \varepsilon$ for all $a \in A$. It is prolongable on $x_0$, $x_0 \in A$, if there exists $u \in A^+$ such that $f(x_0) = x_0u$. In what follows all morphisms will be nonerasing and prolongable on at least one letter $x_0$. In this case, for all $n \in \mathbb{N}$ the word $f^n(x_0)$ is a proper prefix of the word $f^{n+1}(x_0)$ and this defines a unique infinite word $x = x_0uf(u)f^2(u) \cdots f^n(u) \ldots$, which is the limit of the sequence $(f^n(x_0))_{n \geq 0}$. We write $x = f^0(x_0)$ and say that $x$ is generated by $f$. 
Convention. In the rest of this paper $n$ is a fixed integer, $n \geq 2$, and $A_n$ is the $n$-letter alphabet $A_n = \{0, \ldots, (n-1)\}$.

So $0, 1, \ldots, (n-1)$ will be sometimes integers and sometimes letters. Except if it is necessary for the comprehension we will use these symbols both as integers or as letters without noticing in which case we are. In particular computations will be done with the letters $0, 1, \ldots, (n-1)$ considered as integers and, in this case, all letters will be implicitly “computed” modulo $n$. Consequently, if $a$ is a letter of $A_n$ and $j \in \mathbb{Z}$, the letter $(a + j)$ is the letter of $A_n$ which corresponds to the integer $(a + j) \mod n$.

3. Prouhet words and morphisms

The $n$-uniform morphism $\pi_n$ is defined for all $a \in A_n$ by

$$\pi_n(a) = a(a + 1) \cdots (a + n - 1)$$

(all letters being of course “computed” modulo $n$).

Morphisms $\pi_n$ are called Prouhet morphisms and the $n$-letter Prouhet word is $P_n = \pi_n(0)$.

Prouhet morphisms and words appear as a natural generalization of the well known Thue–Morse morphism ($\pi_2 = \mu$) and word ($P_2 = t$) (see, e.g., [19, 13, 4]), but the construction made by Prouhet was older [14].

For results on the Prouhet morphisms and words, see [3, 15, 16]. Here we only use the following

**Theorem 3.1** (Allouche and Shallit [3] and Séébold [15]). The Prouhet words $P_n$ are overlap-free.

Convention. In all the rest of this paper, since $n \geq 2$ is a fixed integer, we simplify the notation by omitting the subscript $n$ in $\pi_n$. So we write $\pi$ in place of $\pi_n$.

We end this section by proving a useful property of $\pi$.

**Lemma 3.2.** Let $j$ be an integer, $1 \leq j \leq n - 1$, and $u \in A_n^*$. The word $\pi^j(u)$ contains $00$ as a factor if and only if the word $u$ contains the factor $j0$. In particular, if $a$ is a letter of $A_n$,

(a) $\pi^{n-1}(a)$ does not contain $00$ as a factor;

(b) $\pi^{n-1}(a)$ ends with $0$ if and only if $a = (n - 1)$.

**Proof.** Let $x$ be a letter of $A_n$. By definition of $\pi$, for any letter $a$, $\pi(x)$ ends with the letter $(a - 1)$ if and only if $x = a$. Thus $\pi^j(x)$ ends with the letter $(a - j)$ if and only if $x = a$, which implies $\pi^j(x)$ ends with the letter $0$ if and only if $x = j$ (this gives also point (b) of the lemma). Consequently, since $\pi^j(v)$ starts with the letter $0$ if and only if the word $v$ starts with $0$, the condition is sufficient.

To prove that it is a necessary condition we first remark that, by definition of $\pi$, $\pi(x)$ never contains a factor $i0$ with $0 \leq i \leq n - 2$. If $\pi(u)$ contains $00$ as a factor for some $u \in A_n^*$, then $u$ contains the factor $10$, and the property is true for $j = 1$. Suppose now that the property is true for $j - 1$, $1 \leq j \leq n - 2$. If $\pi^j(u)$ contains $00$ then, since
\( \pi'(u) = \pi^{j-1}(\pi(u)) \), by hypothesis \( \pi(u) \) contains \((j-1)0\) as a factor. Thus from what precedes, since \( j \leq n - 2 \), \( \pi(u) \) contains a factor \((j-1)0\) at the border between \( \pi(x_i) \) and \( \pi(x_{i+1}) \) where \( x_i \) and \( x_{i+1} \) are two letters such that \( x_i x_{i+1} \) is a factor of \( u \). But in this case \( \pi(x_i) \) ends with \((j-1)\) which implies \( x_i = j \), and \( \pi(x_{i+1}) \) starts with 0 which implies \( x_{i+1} = 0 \). Thus \( u \) contains \( j0 \) as a factor. \( \square \)

4. Lyndon factorization of the Prouhet words

In this section, we suppose \( A_n \) totally ordered by \( 0 < 1 < \cdots < (n-1) \). The lexicographic order on \( A_n^+ \) is then defined as follows: for any \( u, v \in A_n^+ \), \( u < v \) if either \( u \) is a proper prefix of \( v \) or \( u = ras, v = rbt \) with \( a, b \in A_n \), \( a < b \), \( r, s, t \in A_n^* \).

Here we observe the following.

Lemma 4.1. For any \( u, v \in A_n^+ \), \( u0 < v0 \) if and only if \( u < v \).

Proof. If \( u \) is a proper prefix of \( v \), there exist a letter \( a \) and a word \( v' \) such that \( v = uav' \). Since 0 is the smallest letter in lexicographic order, one has \( 0 \leq a \), then either \( u0 \) is a proper prefix of \( v0 \) (if \( a = 0 \)) or \( u0 < v0 \) (if \( a > 0 \)).

If \( u = ras, v = rbt \) with \( a < b \), then \( u0 = ras0 \) and \( v0 = rbt0 \) which implies \( u0 < v0 \).

For the converse, two cases are possible:

- either \( u0 \) is a proper prefix of \( v0 \), which implies \( u \) is a proper prefix of \( v \),
- or \( u0 = u1u2, v0 = u1b v2 \) with \( a < b \). Since 0 is the smallest letter in lexicographic order, one has \( b \neq 0 \), thus \( u1b \) is a prefix of \( v \).

If \( u2 \neq v \) then \( u1a \) is a prefix of \( u \), and \( u1a < u1b \) implies \( u < v \).

If \( u2 = v \) then \( u = u1 \), and \( u1 < u1b \) implies again \( u < v \). \( \square \)

Lemma 4.2. The morphism \( \pi \) preserves the lexicographic order: for any \( u, v \in A_n^+ \), if \( u < v \) then \( \pi(u) < \pi(v) \).

Proof. This is a straightforward consequence of the following: for any two letters \( a, b \), one has \( |\pi(a)| = |\pi(b)| \) and if \( a < b \) then \( \pi(a) < \pi(b) \). \( \square \)

A word of \( A_n^+ \) is a Lyndon word if it is smaller than all its proper nonempty suffixes. The set of Lyndon words will be denoted as \( L \) (see, e.g., [10]).

Example. If \( n = 2 \), \( A_2 = \{0, 1\} \) with \( 0 < 1 \) and the first 11 words of \( L \) (ordered by length) are: 0, 1, 01, 001, 011, 0011, 0111, 00001, 00011, 00101.

The following fundamental result ([7], see also [10]) will be used in the sequel.

Theorem 4.3. Any word \( w \in A_n^+ \) may be written uniquely as a nonincreasing product of Lyndon words.

(This means \( w = w_1 \cdots w_p, w_i \in L \), and for \( 1 \leq j \leq p - 1 \), \( w_j \) is lexicographically greater than or equal to \( w_{j+1} \).)
This result was extended by Siromoney et al. [17] to (right) infinite words.

**Theorem 4.4.** Any infinite word $x$ may be written uniquely as

\[ x = \prod_{k \geq 0} l_k \text{ where } (l_k)_{k \geq 0} \text{ is an infinite nonincreasing sequence of finite Lyndon words} \]

\[ \text{or } x = l_0 \cdots l_{m-1} y \text{ where } l_0 \geq \cdots \geq l_{m-1} > y, m \geq 0, \text{ are finite Lyndon words, and } y \text{ is an infinite word with an infinite number of prefixes being finite Lyndon words.} \]

**Example.** Theorem 4.5 below gives an example of a factorization of type (1). Another important family of infinite words for which such a factorization is known is that of the characteristic Sturmian words ([12], see also [5]).

For a factorization of type (2), choose for example the word $x = 1^m01^o$, for some $m \in \mathbb{N}$: for any $n \in \mathbb{N}$, $01^n$ is a Lyndon word thus $1^o$ is an infinite word with an infinite number of prefixes being Lyndon words; on the other hand, $1^o$ is not a Lyndon word if $n \geq 2$ and, for any $n \in \mathbb{N}$, $01^n < 1$: consequently, $1^o$ cannot be obtained as the product of the elements of an infinite nonincreasing sequence of finite Lyndon words. Since $1 > 0$, the only factorization of $x$ following Theorem 4.4 is of type (2) with $x = l_0 \cdots l_{m-1} y$, $y = 01^o$ and $l_i = 1, 0 \leq i \leq m - 1$.

In the rest of the paper we will use the following useful notation.

Let $w \in A^*_n$. The inverse of $w$, say $\hat{w}$, is defined by $w\hat{w} = \hat{w}w = e$. Note that this is a mere notation, i.e., for $u_1, u_2, w \in A^*_n$, the words $\overline{u_1}w$ and $w\overline{u_2}$ are defined only if $u_1$ and $u_2$ are, respectively, a prefix and a suffix of $w$. If $f$ is a morphism and $u \in A^*_n$ then $f(\overline{u}) = f(u)$ (thus, in the above case, $f(\overline{u_1})f(w) = f(\overline{u_1}w)$ and $f(w)f(\overline{u_2}) = f(\overline{u_2})$).

In [9], the authors gave the following (unique) Lyndon factorization of type (1) of the Thue–Morse word.

**Theorem 4.5.** Let $w_1 = 011$, $w_2 = 01$, and for all $q \geq 2$, $w_{q+1} = 0 \mu(w_q) \overline{0}$. The words $(w_q)_{q \geq 1}$ form a strictly decreasing sequence of Lyndon words, and we have

\[ t = \prod_{q \geq 1} w_q. \]

Also, they asked for a possible generalization of this result in the case of alphabets with more than two letters.

4.1. Existence of a factorization of type (1)

Our aim in this subsection is to answer positively the question of Ido and Melançon by computing a (unique) Lyndon factorization of type (1) for each Prouhet word.

Because this decomposition is quite technical, this will be done in two steps of decomposition:

- first, we factorize $P_n$ in a sequence of finite words $(w_i)_{i \geq 1}$. When $n = 2$, these $w_i$’s are exactly those of Ido and Melançon (Theorem 4.5). But when $n \geq 3$ the words $w_i, i \geq 2$, are not Lyndon words;
then, we decompose the sequence \((w_i)_{i \geq 1}\) in a decreasing sequence of factors and prove that all these factors are Lyndon words. Unicity of such a decomposition is ensured by Theorem 4.4.

4.1.1. A first decomposition

Let \(t \in A_n^*\) be such that \(\pi(0) = 0 t (n - 1)\), i.e.,

\[
t = \begin{cases} 
  e & \text{if } n = 2, \\
  1 \cdots (n - 2) & \text{if } n \geq 3.
\end{cases}
\]

Let \((w_i)_{i \geq 1}\) be the sequence of words defined by

\[
w_1 = \pi^2(0)(\bar{0} t),
\]

\[
w_2 = 0 t \left( \prod_{i=2}^{n-1} \pi'[t(n-1)] \right) \pi^n(t) \pi^{n-1}[(n-1)] \bar{0},
\]

\[
w_{i+1} = 0 \pi^{n-1}(w_i) \bar{0}, \quad i \geq 2.
\]

First we remark that, for each value of \(i \geq 1\), \(w_i\) is well defined. Indeed, in any case the word \(w_1\) is defined because, for \(n \geq 2\), \(\pi^2(0)\) ends with \(\pi[(n-1)]\), and \(\pi[(n-1)] = (n-1)0 \bar{0}\) ends with \(0 t\).

Now, for any \(n \geq 2\), \(\pi^{n-1}[(n-1)]\) ends with \((n-1)0\). Thus \(w_2\) is defined because it ends with \(\pi^{n-1}[(n-1)] \bar{0}\); and for any \(i \geq 2\), \(\pi^{n-1}(w_i)\) ends with \((n-1)0\). Consequently, for any \(i \geq 2\), \(w_{i+1}\) ends with \((n-1)0 \bar{0}\), thus is well defined.

Now to prove that, for any \(n \geq 2\), \((w_i)_{i \geq 1}\) factorizes the Prouhet word \(P_n = \pi_n^a(0)\) (Proposition 4.7), we need a technical lemma.

**Lemma 4.6.** For any integer \(q \geq 1\), \(\prod_{i=1}^{q+1} w_i = \pi^q(n-1)(w_1) \prod_{i=q-1}^0 \pi^{q(n-1)}(\bar{0})\).

Remark that here \(\prod_{i=1}^{q+1} w_i\) is the classical concatenation product \(w_1 w_2 \cdots w_{q+1}\), when \(\prod_{i=q-1}^0 x_i\) represents the product \(x_{q-1} x_{q-2} \cdots x_0\).

**Proof of Lemma 4.6.** By definition, \(w_1 = \pi^2(0)(\bar{0} t)\) and \(w_2 = 0 t (\prod_{i=2}^{n-1} \pi'[t(n-1)] \pi^n(t) \pi^{n-1}[(n-1)] \bar{0})\).

Moreover, \(\pi(0) = 0 t (n - 1)\).

Thus,

\[
w_1 w_2 = \pi^2(0)(\bar{0} t) 0 t \left( \prod_{i=2}^{n-1} \pi'[t(n-1)] \right) \pi^n(t) \pi^{n-1}[(n-1)] \bar{0}
\]

\[
= \pi^2(0) \left( \prod_{i=2}^{n-1} \pi'[t(n-1)] \right) \pi^n(t) \pi^{n-1}[(n-1)] \bar{0}
\]

\[
= \pi^2(0 t (n - 1)) \left( \prod_{i=3}^{n-1} \pi'[t(n-1)] \right) \pi^n(t) \pi^{n-1}[(n-1)] \bar{0}
\]
\[
\begin{align*}
\pi^n[0 \ t(n-1)] \left( \prod_{i=4}^{n-1} \pi^k[0 \ t(n-1)] \right) \pi^n(t) \pi^n[(n-1) \ 0] \\
\vdots \\
= \pi^n[0 \ t(n-1)] \pi^n[(n-1) \ 0] \\
= \pi^n[0 \ t(n-1)] \pi^n(t) \pi^n[(n-1) \ 0] \\
= \pi^n(0 \ t) \pi^n[(n-1) \ 0] (\text{because } \pi[(n-1)] = (n-1) \ 0) \\
= \pi^n(0 \ t) \pi^n[(n-1) \ 0] \\
= \pi^n[\pi^2(0) \ (0) \ 0] \\
= \pi^n[(n-1) \ 0].
\end{align*}
\]

This implies the property is true if \( q = 1 \).

Now, we prove that if the property is true for \( q \geq 1 \) then it is true for \( q + 1 \).

Since, for any \( i \geq 2, \ w_{i+1} = 0 \pi^{n-1}(w_i) \ 0 \), one has for \( q \geq 1 \)

\[
\begin{align*}
w_{q+2} &= 0 \pi^{n-1}(w_{q+1}) \ 0 \\
&= 0 \pi^{n-1}[0 \pi^{n-1}(w_q) \ 0] \ 0 \\
\vdots \\
&= \prod_{i=0}^{q-1} \pi^{(n-1)(0)} \pi^{(n-1)(w_2)} \prod_{i=q-1}^{0} \pi^{(n-1)(0)}. \\
\end{align*}
\]

Consequently, since \( \prod_{i=1}^{q+1} w_i = \pi^{q(n-1)}(w_1) \prod_{i=q-1}^{0} \pi^{(n-1)(0)} \) by induction hypothesis, one has

\[
\begin{align*}
\prod_{i=1}^{q+2} w_i &= \left( \prod_{i=1}^{q+1} w_i \right) w_{q+2} \\
&= \pi^{q(n-1)}(w_1) \prod_{i=q-1}^{0} \pi^{(n-1)(0)} \pi^{(n-1)(w_2)} \prod_{i=q-1}^{0} \pi^{(n-1)(0)} \\
&= \pi^{q(n-1)}(w_1) \pi^{q(n-1)}(w_2) \prod_{i=q-1}^{0} \pi^{(n-1)(0)}.
\end{align*}
\]
\[ P_n = \prod_{q \geq 1} w_q. \]

**Proof.** Since the \( w_i \)'s are all defined, \( \prod_{q \geq 1} w_q \) is also well defined.

Now, since \( n \geq 2 \), \( |w_1| \geq 3 \). Thus \( |w_1 w_2| = |\pi^{n-1} (w_1) 0| = (|w_1| \times n^{n-1}) - 1 > |w_1| \).

Moreover, for any \( a \in A_n \), \( |\pi(a)| \geq 2 \), so, for any \( q \geq 2 \), \( |w_{q+1}| > |w_q| > 0 \).

Consequently, for any \( q \geq 1 \), \( |\prod_{j=1}^{q+1} w_j| > |\prod_{j=1}^{q} w_j| \).

Hence, from Lemma 4.6, and because \( \pi \) is prolongable on 0 and \( w_1 \) is a prefix of \( \pi^2(0) \), \( \prod_{q \geq 1} w_q = \lim_{q \to \infty} [\pi^{0(n-1)}(w_1) \prod_{i=q-1}^{q} \pi^{i(n-1)}(0)] = \pi^{\omega}(w_1) = \pi^{\omega}(0) = P_n. \)

We have proved that, for any \( n \geq 2 \), the sequence \( (w_i)_{i \geq 1} \) factorizes the Prouhet word \( P_n \). However, when \( n \geq 3 \), this is not a Lyndon factorization. Indeed, for example, if \( n = 3 \) then

\[
\begin{align*}
w_2 &= 01 \pi^2(12) \pi^3(1) \pi^2(2) \hat{0} \\
&= 01120201012201012120120201012201012120120101212
\end{align*}
\]

So \( w_2 \) starts with 01 and contains 00, proving it is surely not a Lyndon word.

Thus to obtain the Lyndon factorization of type (1) of \( P_n \) when \( n \geq 3 \), we have to realize a new step of decomposition.

### 4.1.2. The Lyndon factorization

In the following, if \( u \) is a finite word then the Lyndon factorization of \( u \) will be a shortcut for the unique factorization of \( u \) as a nonincreasing product of Lyndon words.

We know, from Theorem 4.3, that \( w_2 \) has a Lyndon factorization.

Let \( v_{2,1}, v_{2,2}, \ldots, v_{2,k} \) be this decomposition. We will prove the following.

**Theorem 4.8.** \( P_n = w_1 \prod_{q \geq 2} (\prod_{j=1}^{k} v_{q,j}) \) where, for any \( q \geq 2 \), \( v_{q+1,j} = 0 \pi^{n-1} (v_{q,j}) \hat{0} \), \( 1 \leq j \leq k \), and \( w_1 \) and the \( v_{q,j} \)'s form a strictly decreasing sequence of Lyndon words.

**Proof.** Since for \( n = 2 \) this is the result of Ido and Melançon (Theorem 4.5), we suppose in the following that \( n \geq 3 \).

The proof will be in three steps: first, we prove that \( w_1 \) and all the \( v_{q,j} \)'s \( (q \geq 2, 1 \leq j \leq k) \) are well defined (Step 4.12); then we show that they form a strictly decreasing sequence of words (Step 4.13); to end, we prove that all these words are Lyndon words (Step 4.14).

The result follows from Proposition 4.7 because, for any \( q \geq 2 \), \( \prod_{j=1}^{k} v_{q,j} = w_q \).
Indeed, $\prod_{j=1}^{k} v_{2,j} = v_{2,1} v_{2,2} \cdots v_{2,k} = w_2$.

Now, if we assume $\prod_{j=1}^{k} v_{q,j} = w_q$ for some $q \geq 2$, then

$$\prod_{j=1}^{k} v_{q+1,j} = v_{q+1,1} v_{q+1,2} \cdots v_{q+1,k}$$

$$= 0 \pi^{n-1}(v_{q,1}) \tilde{0} 0 \pi^{n-1}(v_{q,2}) \tilde{0} \cdots 0 \pi^{n-1}(v_{q,k}) \tilde{0}$$

$$= 0 \pi^{n-1}(v_{q,1} v_{q,2} \cdots v_{q,k}) \tilde{0}$$

$$= 0 \pi^{n-1}(w_q) \tilde{0}$$

$$= w_{q+1}.$$ 

Before these three steps, we need some preliminaries.

Let us define the words $x_i$ (1 ≤ $i$ ≤ $n$ − 1) and $y_i$, $t_i$ (1 ≤ $i$ ≤ $n$ − 2) by:

• $x_i = i(i + 1) \cdots (n − 1)$;
• $y_i = 0 \cdots (i − 1)$;
• $t_i = 0 \cdots (i − 1)(i + 1) \cdots (n − 1)$.

For each letter $i \in A_n$, if 1 ≤ $i$ ≤ ($n$ − 2) then $\pi(i) = x_i y_i$ and $t_i = y_i x_{i+1}$. Moreover, since $t = 1 \cdots (n − 2)$, one has $\pi(t) = \pi[1 \cdots (n − 2)] = x_1 y_1 x_2 y_2 \cdots x_{n−2} y_{n−2} = x_1 t_1 t_2 \cdots t_{n−3} y_{n−2}$. Thus $\pi(t)(n − 1) = x_1 t_1 t_2 \cdots t_{n−3} y_{n−2}$ and we have:

**Lemma 4.9.** $x_1 t_1 t_2 \cdots t_{n−3} y_{n−2}$ is the Lyndon factorization of the word $\pi(t)(n − 1)$.

**Proof.** By definition $x_1 > t_1$ and for each integer $i$, 1 ≤ $i$ ≤ $n$ − 3, $t_i > t_{i+1}$.

Moreover, the words $x_1$, $t_i$ (1 ≤ $i$ ≤ $n$ − 2) are obviously all Lyndon words.

Thus $x_1 t_1 t_2 \cdots t_{n−3} y_{n−2}$ is a strictly decreasing product of Lyndon words. From Theorem 4.3 this decomposition is unique: this is the Lyndon factorization of $\pi(t)(n − 1)$. □

Now, we have to compute the values of $v_{2,1}$ and $v_{2,k}$.

**Fact 4.10.** $v_{2,1} = 0 t \pi(t)(n − 1)$ and $v_{2,2}$ starts with $0 t 0 t(n − 1)$.

**Proof.** Since $n \geq 3$, $w_2$ starts with $0 t \pi^2(1) = 0 t \pi(t)(n − 1) 0 t 0 t(n − 1)$.

From Lemma 4.9, the smallest Lyndon word in the Lyndon factorization of $\pi(t)(n − 1)$ is its suffix $t_{n−2} = 0 \cdots (n − 3)(n − 1)$. Since $0 t < 0 \cdots (n−3)(n−1)$, this implies that $0 t \pi(t)(n − 1)$ is undecomposable for Theorem 4.3, thus is a prefix of $v_{2,1}$. But, for each $u$ prefix of 0 $t$, if $u \neq e$, then $u < 0 t \pi(t)(n − 1)$, so $v_{2,1} = 0 t \pi(t)(n − 1)$.

Moreover, 0 $t 0 t(n − 1)$ is a Lyndon word thus $v_{2,2}$ starts with $0 t 0 t(n − 1)$. □

**Fact 4.11.** $v_{2,k} = 0 \pi^{n−1}[0 \cdots (n − 3)(n − 1)] \tilde{0}$. 

Proof. One has
\[
\begin{align*}
  w_2 &= 0t \left( \prod_{i=2}^{n-1} \pi'[t(n-1)] \right) \pi^n(t) \pi^{n-1}[(n-1)] \tilde{0} \\
  &= \left( n - 1 \right)(n - 1) 0 t \pi^2(t) \left( \prod_{i=2}^{n-1} \pi'[t(n-1)] \pi^{i+1}(t) \right) \pi^{n-1}[(n-1)] \tilde{0} \\
  &= \left( n - 1 \right) \pi[(n-1) \pi(t)] \left( \prod_{i=2}^{n-1} \pi'[t(n-1) \pi(t)] \right) \pi^{n-1}[(n-1)] \tilde{0} \\
  &= \left( n - 1 \right) \left( \prod_{i=1}^{n-2} \pi'[t(n-1)] \right) \pi^{n-1}[(n-1) \pi(t)(n-1)] \tilde{0}.
\end{align*}
\]

Now the proof is in two steps (note that \( \pi(t) \) starts with \( x_1 \)):
(a) first, we prove that \( \left( \prod_{i=1}^{n-2} \pi'[t(n-1) \pi(t)] \right) \pi^{n-1}[(n-1)] \pi^{n-1}(x_1) \tilde{0} \) does not contain \( 00 \) as a factor;
(b) second, we show that \( \pi^{n-1}(x_1) \tilde{0} \) factorizes as \( \pi^{n-1}(x_1) \tilde{0} \left( \prod_{i=1}^{n-2} 0 \pi^{n-1}(t_i) \tilde{0} \right) \) where \( 0 \pi^{n-1}(t_i) \tilde{0} \) is a strictly decreasing sequence of Lyndon words.

From (a) we deduce that the Lyndon factorization of \( w_2 \) ends with the Lyndon factorization of \( \left( \prod_{i=1}^{n-2} 0 \pi^{n-1}(t_i) \tilde{0} \right) \) (because this starts with the first occurrence of \( 00 \) in \( w_2 \)), and from (b) we conclude that this Lyndon factorization ends with \( 0 \pi^{n-1}(t_{n-2}) \tilde{0} = 0 \pi^{n-1}[0 \cdots (n-3)(n-1)] \tilde{0} \).

(a) One has
\[
\begin{align*}
  (n-1) \pi(t) \\
  &= (n-1) \pi[1 \cdots (n-2)] \\
  &= (n-1) 1 \cdots (n-1)0 2 \cdots (n-2)01 \cdots (n-2)(n-1)01 \cdots (n-3).
\end{align*}
\]

Thus, for any \( i, 1 \leq i \leq (n-2), \) the word \( (n-1) \pi(t) \) does not contain \( 0 \tilde{0} \) as a factor which implies, from Lemma 3.2 and since \( \pi'[t(n-1)] \) starts with \( (n-1) \) for any \( j \) such that \( 1 \leq j \leq (n-1), \) that \( \left( \prod_{i=1}^{n-2} \pi'[t(n-1) \pi(t)] \right) \) does not contain \( 00 \) as a factor.

Also, again from Lemma 3.2, \( \pi^{n-1}[(n-1)] \) and \( \pi^{n-1}(x_1) \) do not contain \( 00 \) as a factor.

Thus, since \( \pi^{n-1}(x_1) \) starts with \( 1, \left( \prod_{i=1}^{n-2} \pi'[t(n-1) \pi(t)] \right) \pi^{n-1}[(n-1)] \pi^{n-1}(x_1) \tilde{0} \) does not contain \( 00 \) as a factor.

(b) We know (Lemma 4.9) that the Lyndon factorization of \( \pi(t)(n-1) \) is \( x_1 t_1 \cdots t_{n-2}. \)

Since \( x_1 \) and each \( t_i \) \( (1 \leq i \leq n-2) \) end with the letter \( (n-1), \) from Lemma 3.2 the words \( \pi^{n-1}(x_1) \) and \( \pi^{n-1}(t_i) \) end with \( 0, \) which implies that the word \( \pi^{n-1}(x_1) \tilde{0} \) and all the words \( 0 \pi^{n-1}(t_i) \tilde{0} \) are well defined.

Now, from Lemma 4.2, \( \pi \) preserves the lexicographic order thus from \( t_i > t_{i+1} \) \( (1 \leq i \leq n-3) \) we deduce \( 0 \pi^{n-1}(t_i) > 0 \pi^{n-1}(t_{i+1}) \) from which we have \( 0 \pi^{n-1}(t_i) \tilde{0} > 0 \pi^{n-1}(t_{i+1}) \tilde{0} \) (Lemma 4.1).

Thus we just have to prove that each \( 0 \pi^{n-1}(t_i) \tilde{0} \) \( (1 \leq i \leq n-2) \) is a Lyndon word.
Because the only occurrence of the letter 0 in \( t_i \) is always at the beginning, from Lemma 3.2, \( \pi^{n-1}(t_i) \) does not contain 00 as a factor for \( 1 \leq i \leq n-2 \). Thus \( 0 \pi^{n-1}(t_i) \) (which starts with 00) is always smaller than all its proper suffixes, which implies \( 0 \pi^{n-1}(t_i) \tilde{\in} L \). \( \square \)

Now we prove the three steps.

First of all, since \( w_2 \) starts with 0 and \( v_{2,1} \geq v_{2,2} \geq \cdots \geq v_{2,k} \), all the \( v_{2,j} \)'s start with 0. But, from Theorem 3.1, \( P_n \) is overlap-free thus all the \( v_{2,j} \)'s are different.

So \( w_2 = v_{2,1} v_{2,2} \cdots v_{2,k} \) where the \( v_{2,j} \)'s form a strictly decreasing sequence of Lyndon words.

**Step 4.12.** \( w_1 \) and all the \( v_{q,j} \)'s \( (q \geq 2, 1 \leq j \leq k) \) are well defined.

**Proof.** We already saw (at the beginning of this subsection) that \( w_1 \) is well defined.

The words \( v_{2,1}, \ldots, v_{2,k} \) are well defined by construction.

We have to show that, for any two integers \( q, j \) with \( q \geq 2 \) and \( 1 \leq j \leq k \), \( 0 \pi^{n-1}(v_{q,j}) \tilde{\in} \) is well defined, i.e., the word \( \pi^{n-1}(v_{q,j}) \) ends with 0.

From Lemma 3.2(b), if \( i \) is a letter of \( A_n \) then \( \pi^{n-1}(i) \) ends with 0 if and only if \( i = (n-1) \).

Thus we have to prove that \( v_{q,j} \) ends with the letter \((n-1)\).

First, we remark that it is enough to prove it for \( q = 2 \). Indeed, if the word \( v_{q,j} \) ends with the letter \((n-1)\) for some \( q \geq 2 \) (and \( 1 \leq j \leq k \)) then \( \pi^{n-1}(v_{q,j}) \) ends with \( \pi^{n-1}((n-1)) \) which ends with \((n-1)0\). Thus \( v_{q+1,j} \) ends with the letter \((n-1)\).

Since the word \( v_{2,2} \) starts with \( 0ti0 \) (Fact 4.10) and \( v_{2,2} > v_{2,3} > \cdots > v_{2,k} \), for any \( j, 2 \leq j \leq k \), the word \( v_{2,j} \) starts with a factor \( 0ux \) where \( u \) is a prefix (maybe empty) of \( t \) and \( x \) is a letter whose value is smaller than or equal to the value of the last letter of \( u \) \( (x = 0 \text{ if } u = e \text{ or } u = t) \). By definition of \( \pi \), this can happen only if \( 0u \) is a suffix of \( \pi(i) \) for some letter \( i \). In this case, since \( |0u| \leq n-1 \), the word \( \pi(i) \) ends with \((n-1)0u \). Thus \( v_{2,j} \) ends with the letter \((n-1)\).

To end, since \( w_2 \) ends with \( \pi^{n-1}((n-1)) \), it ends with the letter \((n-1)\). Consequently, \( v_{2,k} \) also ends with the letter \((n-1)\). \( \square \)

**Step 4.13.** \( w_1 > v_{2,1} \) and, for any \( q \geq 2 \) and \( 1 \leq j \leq k \), \( v_{q,j} > v_{q,j+1} \) and \( v_{q,k} > v_{q+1,1} \).

**Proof.** Since \( w_1 \) starts with \( \pi(0) = 0t(n-1) \) and, from Fact 4.10, \( v_{2,1} \) starts with \( 0t \pi(t) \), i.e., with \( 0t1 \), one has \( w_1 > v_{2,1} \).

Now, since \( \pi \) preserves the lexicographic order (Lemma 4.2), if \( v_{q,j} > v_{q,j+1} \), then \( \pi^{n-1}(v_{q,j}) > \pi^{n-1}(v_{q,j+1}) \).

From this and Lemma 4.1 we have

\[
v_{q+1,j} = 0 \pi^{n-1}(v_{q,j}) \tilde{\in} > 0 \pi^{n-1}(v_{q,j+1}) \tilde{\in} = v_{q+1,j+1}.
\]

Also if \( v_{q,k} > v_{q+1,1} \), then \( v_{q+1,k} = 0 \pi^{n-1}(v_{q,k}) \tilde{\in} > 0 \pi^{n-1}(v_{q+1,1}) \tilde{\in} = v_{q+2,1} \).

Thus it is enough to prove the properties for \( q = 2 \).

First we already know that \( v_{2,1} > \cdots > v_{2,k} \).
Second, from Fact 4.10, \( v_{3,1} = 0 \pi^{n-1}(v_{2,1}) \tilde{0} = 0 \pi^{n-1}[0 t \pi(t)(n - 1)] \tilde{0} \) which starts with \( 0 \pi^{n-1}[0 \cdots (n-3)](n-2) \) and, from Fact 4.11, \( v_{2,k} = 0 \pi^{n-1}[0 \cdots (n-3)(n-1)] \tilde{0} \) which starts with \( 0 \pi^{n-1}[0 \cdots (n-3)](n-1) \). Thus \( v_{2,k} > v_{3,1} \). □

**Step 4.14.** \( w_1 \in L \) and, for any \( q \geq 2 \) and \( 1 \leq j \leq k \), \( v_{q,j} \in L \).

**Proof.** First,

\[
\begin{align*}
    w_1 &= \pi^2(0)(\overline{0t}) \\
    &= \pi[0 t(n-1)](\overline{0t}) \\
    &= \pi(0)\pi(t)(n-1) \\
    &= 0 \cdots (n-3)(n-2)(n-1)\pi(t)(n-1).
\end{align*}
\]

We know from Lemma 4.9 that \( t_{n-2} \) is the smallest Lyndon word in the Lyndon factorization of \( \pi(t)(n-1) \). But \( t_{n-2} = 0 \cdots (n-3)(n-1) \) and \( 0 \cdots (n-3)(n-2) \) which starts with \( 0 \pi^{n-1}[0 \cdots (n-3)](n-1) \).

Moreover, \( 0 \cdots (n-3)(n-2)(n-1) \in L \).

Thus \( w_1 \) is smaller than all its proper nonempty suffixes: it is a Lyndon word.

Now we prove that all the \( v_{q,j} \)'s are Lyndon words.

By construction \( v_{2,j} \in L \), \( 1 \leq j \leq k \).

Now suppose that for some \( q \geq 2 \) and \( j \), \( 1 \leq j \leq k \), \( v_{q,j} \in L \) but \( v_{q+1,j} \notin L \). Since \( v_{q+1,j} = 0 \pi^{n-1}(v_{q,j}) \tilde{0} \), this means that there exists a proper suffix of \( 0 \pi^{n-1}(v_{q,j}) \tilde{0} \), say \( X \), such that \( X < 0 \pi^{n-1}(v_{q,j}) \tilde{0} \).

Let \( u \in A^*_n \) be such that \( v_{q,j} = 0u \). Then \( 0 \pi^{n-1}(v_{q,j}) \tilde{0} = 0 \pi^{n-1}(0u) \tilde{0} \), thus it starts with a factor 00.

But \( X < 0 \pi^{n-1}(v_{q,j}) \tilde{0} \), thus \( X \) must also start with 00. Since \( X \) is a suffix of \( \pi^{n-1}(0u) \tilde{0} \), this implies from Lemma 3.2 that there exist \( u_1, u_2 \in A^*_n \) such that \( 0u = u_1 (n-1)0u_2 \) and \( X = 0 \pi^{n-1}(0u_2) \tilde{0} \).

Now, since \( v_{q,j} \in L \), one has \( 0u_2 > 0u \), thus, from Lemma 4.2, \( \pi^{n-1}(0u_2) \tilde{0} > \pi^{n-1}(0u) \tilde{0} \). This implies \( X > 0 \pi^{n-1}(v_{q,j}) \tilde{0} \), a contradiction. □

**4.2. An effective computation of the factorization**

We saw in the previous subsection that, for any \( n \geq 2 \), there exists a factorization of \( P_n \) as a strictly decreasing sequence of Lyndon words. This factorization is obtained from the one of the word \( w_2 \), and we know from Theorem 4.3 that this last one always exists and is unique. However, due to the length of \( w_2 \), such a factorization can be hard to compute effectively: indeed, from the proof of Lemma 4.6, one has \( |w_1w_2| = |\pi^{n-1}(w_1)| - 1 = n^n - |w_1| - 1 \) which implies \( |w_2| = |w_1|(n^n - 1) - 1 \); if, for example, \( n = 6 \) this gives \( |w_1| = 31 \) and \( |w_2| = 31(6^6 - 1) - 1 = 241024 \).

So, in what follows, we give for any \( n \geq 2 \) the value of each \( v_{2,i} \), \( 1 \leq i \leq k \), only depending on \( n \). From this we deduce for any \( n \geq 2 \) the exact length of each \( v_{2,i} \),
3. For each integer \( j \), \( 1 \leq i \leq k \). Thus, to obtain the Lyndon factorization of \( w_2 \) it is enough either to construct each \( v_{2,i} \), \( 1 \leq i \leq k \), or to compute the prefix of length \( |\pi^n-1(w_1)|-1 \) of \( P_n \), and then, after removing a prefix of length \( |w_1| \), to cut the resulting word at the given lengths.

4.2.1. The computation

From (a) and (b) of the proof of Fact 4.11, we know that the Lyndon factorization of \( w_2 \) is made of the Lyndon factorization of

\[
\frac{(n-1)}{(n-1)} \left( \prod_{i=1}^{n-2} \pi'[[n-1] \pi(t)] \right) \pi^{n-1}(x_1) \tilde{0}
\]

Thus, to obtain the Lyndon factorization of

\[
\prod_{i=1}^{n-2} \pi'[[n-1] \pi(t)] \pi^{n-1}(x_1) \tilde{0}
\]

We will prove that \( w'_2 \) factorizes in \( w'_2 = v_{2,1} \cdots v_{2,n-2} 0 t + 1 \), and compute the value of each of these \( v_{2,i} \).

We start with some observations.

1. Since \( 0 t < t_{n-2} \), and from Lemma 4.9, \( x_1 t_1 \cdots t_{n-2} 0 t \) is the Lyndon factorization of \( \pi(t)(n-1) 0 t \).
2. For each integer \( i \), \( 1 \leq i \leq n-2 \), \( \pi(x_i) \) and \( \pi(t) \) \((1 \leq j \leq n-2) \) end with \( 01 \cdots (n-i-1) \) and \( \pi'(0 t) \) ends with \( 01 \cdots (n-i-2) \).
3. For each integer \( j \), \( 2 \leq j \leq n-1 \), \( 01 \cdots (n-j) \pi^{j-1}(x_1) \pi 01 \cdots (n-j) \in L \) (because it starts with \( 01 \cdots (n-j) 1 \), and any other occurrence of \( 01 \cdots (n-j) \) is followed by a letter greater than 1).
4. For each integer \( j \), \( 2 \leq j \leq n-1 \), \( 01 \cdots (n-j) \pi^{j-1}(y) \pi 01 \cdots (n-j) \in L \) for \( y \in \{t_1, \ldots, t_{n-2}\} \) (because it starts with \( 01 \cdots (n-j) 0 \).
5. For each integer \( j \), \( 2 \leq j \leq n-2 \), \( 01 \cdots (n-j) \pi^{j-1}(0 t) \pi 01 \cdots (n-j-1) \in L \) (because it starts with \( 01 \cdots (n-j) 0 \).

From this we deduce that, for \( 0 \leq p \leq n-3 \),

\[
v_{2,pn+1} = 01 \cdots (n-p-2) \pi^{p+1}(x_1) \pi 01 \cdots (n-p-2),
\]

\[
v_{2,pn+i+1} = 01 \cdots (n-p-2) \pi^{p+1}(t_i) \pi 01 \cdots (n-p-2) \quad (1 \leq i \leq n-2),
\]
and for $0 \leq p \leq n - 4$,
\[ v_{2,(p+1)n} = 0 \cdots (n - p - 2) \pi^{p+1}(0t) 0 \cdots (n - p - 3). \]

Hence we have $w'_2 = (v_{2,1} \cdots v_{2,(n-3)n+n-1}) 0 \pi^{n-2}(0t) \pi^{n-1}(x_1) \tilde{0}$.
But one has $\pi^{n-2}(t_{n-2}) \tilde{0} = 0 \pi^{n-2}(0 \cdots (n - 3)(n - 2))$, thus the Lyndon factorization of $w'_2$ ends with the Lyndon factorization of $0 \pi^{n-2}(0t) \pi^{n-1}(x_1) \tilde{0}$.

Also, $\pi^{n-1}(x_1) \tilde{0}$ ends with $0 \pi^{n-2}(0t) \tilde{0}$ and $0 \pi^{n-2}(0t) \pi^{n-1}(x_1) \tilde{0}$ is lexicographically smaller than all its proper nonempty suffixes, except $0 \pi^{n-2}(0t) \tilde{0}$.

Consequently, the Lyndon factorization of $0 \pi^{n-2}(0t) \pi^{n-1}(x_1) \tilde{0}$ contains two Lyndon words: $v_{2,(n-2)n} = 0 \pi^{n-2}(0t) \pi^{n-1}(12 \cdots (n - 2)) \pi^{n-2}(n - 1) \tilde{0}$ and $v_{2,(n-2)(n+1)} = 0 \pi^{n-2}(0t) \tilde{0}$.

So the Lyndon factorization of $w'_2$ is $w'_2 = v_{2,1} \cdots v_{2,(n-2)n+1}$.

In $w_2$, this factorization is followed by the $n-2$ Lyndon words $v_{2,(n-2)n+1+i} = 0 \pi^{n-1}(i, \tilde{0}) \tilde{0}$ ($1 \leq i \leq n - 2$).

From this we deduce that $k = (n - 2)n + 1 + n - 2 = (n - 1)n - 1$.

4.2.2. Length of the $v_{2,i}$’s

From what precedes we deduce the following.

For any $n \geq 2$, $w_1$ is the prefix of $\pi^i(0)$ of length $1 + (n - 1)n$. The word $w_2$ is such that $|w_1w_2| = |\pi^{n-1}(w_1)| - 1$ (this implies that $w_1w_2$ is a prefix of $\pi^n(0)$).

The Lyndon factorization of $w_2$ is $w_2 = v_{2,1} \cdots v_{2,k}$ where:
- $k = |w_1| - 2 = (n - 1)n - 1$,
- $|v_{2,1}| = \cdots = |v_{2,n-1}| = (n - 1)n$,
- For $1 \leq p \leq n - 3$,
  \[ |v_{2,pn}| = 1 + (n - 1)n^p, \]
  \[ |v_{2,pn+1}| = \cdots = |v_{2,(p+1)n-1}| = (n - 1)n^{p+1}, \]
- $|v_{2,(n-2)n}| = (n - 1)n^{n-1}$,
- $|v_{2,(n-2)n+1}| = 1 + (n - 1)n^{n-2}$,
- $\cdots = |v_{2,(n-1)n-1}| = (n - 1)n^{n-1}$.

4.2.3. Examples

When $n = 2$ we find the result of Ido and Melançon (Theorem 4.5).

When $n = 3$, $w_1 = 0121202$ and $w_2 = v_{2,1} v_{2,2} v_{2,3} v_{2,4} v_{2,5}$ with

\[ v_{2,1} = 011202, \]
\[ v_{2,2} = 010122, \]
\[ v_{2,3} = 010121201202010122, \]
\[ v_{2,4} = 0101212, \]
\[ v_{2,5} = 001212020120101212. \]
Moreover, $\pi^{n-1} = \pi^2$ is given by

\[
0 \mapsto 012120201, \\
1 \mapsto 120201012, \\
2 \mapsto 201012120.
\]

Thus,

\[
v_{3,1} = 0 \pi^{n-1}(v_{2,1}) 0 \\
= 0012120201120201012120201012201012012012010121201012120,
\]

\[
v_{3,2} = 0 \pi^{n-1}(v_{2,2}) 0 \\
= 0012120201120201012012120201120201012201012120201012120,
\]

and so on.

When $n = 5$, $w_1 = 012341234023401340124$.

One has $|w_1| = 21$, thus $w_2$ will be cut in $k = 19$ parts with

\[
|v_{2,1}| = |v_{2,2}| = |v_{2,3}| = |v_{2,4}| = 20,
\]

\[
p = 1 : |v_{2,5}| = 21,
\]

\[
|v_{2,6}| = |v_{2,7}| = |v_{2,8}| = |v_{2,9}| = 100,
\]

\[
p = 2 : |v_{2,10}| = 101,
\]

\[
|v_{2,11}| = |v_{2,12}| = |v_{2,13}| = |v_{2,14}| = 500,
\]

\[
|v_{2,15}| = 2500,
\]

\[
|v_{2,16}| = 501,
\]

\[
|v_{2,17}| = |v_{2,18}| = |v_{2,19}| = 2500.
\]

In what follows, we will give both the values of $v_{2,1}$ to $v_{2,14}$ obtained from the computation realized in 4.2.1, and those obtained by cutting the prefix of $w_1 \pi^{n-1}(w_1)$ at the given lengths. After that, we also give (but not explicitly because the total length is more than 10000!) the values of the words $v_{2,15}$ to $v_{2,19}$.

Since $n = 5$, one has $n - 4 = 1$, $n - 3 = 2$, $n - 2 = 3$ and $t = 123$, $x_1 = 1234$, $t_1 = 0234$, $t_2 = 0134$, $t_3(= t_{n-2}) = 0124$.

\[
v_{2,1} = 0123\pi(1234)0123 = 01231234023401340124,
\]

\[
v_{2,2} = 0123\pi(0234)0123 = 01230123423401340124,
\]

\[
v_{2,3} = 0123\pi(0134)0123 = 01230123412340340124,
\]

\[
v_{2,4} = 0123\pi(0124)0123 = 01230123412340234014,
\]

\[
v_{2,5} = 0123\pi(0123)012 = 012301234123402340134,
\]
\[ v_{2.6} = 012\pi^2(1234)012 \]
\[ = 012123402340134012401230123423401340124012301234123403401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]

\[ v_{2.7} = 012\pi^2(0234)012 \]
\[ = 012012341234023401340124012323401340124012301234123403401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]

\[ v_{2.8} = 012\pi^2(0134)012 \]
\[ = 012012341234023401340124012312340234013401240123012343401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]
\[ = 012341234023401401230123412340234013401240123 \]

\[ v_{2.9} = 012\pi^2(0124)012 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]

\[ v_{2.10} = 012\pi^2(0123)012 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]
\[ = 01201234123402340134012401231234023401340124012301234234013401240123 \]

\[ v_{2.11} = 01\pi^3(1234)012 \]
\[ = 0110123402340134012401230123423401340124012301234340124012401230 \]
\[ = 12341234023401401230123412340234013401240123412340340124012401230 \]
\[ = 12341234023401401230123412340234013401240123412340340124012401230 \]
\[ = 23401340124012301234123403401240123012341234023401401230123412340340124012401230 \]
\[ = 23401340124012301234123403401240123012341234023401401230123412340340124012401230 \]
\[ = 34012401230123412342340134012401230123412340340124012401230123412340340124012401230 \]
\[ = 34012401230123412342340134012401230123412340340124012401230123412340340124012401230 \]
\[ = 34012401230123412342340134012401230123412340340124012401230123412340340124012401230 \]
\[ = 34012401230123412342340134012401230123412340340124012401230123412340340124012401230 \]
\[ v_{2,12} = 01\pi^3(0234)\delta \]
\[ = 010123412340234013401240123123402340134012401230123423401340124 \]
\[ + 012301234123403401240123012341234023401401230123423401340124 \]
\[ + 0123401340124012301234123403401240123012342340140123012342340134 \]
\[ + 340234013401201234123402340134012401231234023401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
\[ + 21234013401240123012341234034012401230123423401340124012301234 \]
\[ + 340124012301234123402340140123012342340134012401230123423401340 \]
\[ + 0134012401231234023401340124012301234234013401240123012342340 \]
\[ + 01230123412340234013401240123012342340134012401230123423401340 \]
\[ + 24012301234234013401240123012342340134012401230123423401340 \]
Since $v_{2,15} \; v_{2,16} = 0 \; 1 \; \pi^{n-2}(0 \; t) \; \pi^{n-1}(x_1) \; \bar{0}$, we have

$$v_{2,15} = 01\pi^3(0123)\pi^4(123)\pi^3(4)\bar{0}1,$$

and

$$v_{2,16} = 01\pi^3(0123)\bar{0}.$$

To end, we have

$$v_{2,17} = 0\pi^4(0234)\bar{0},$$

$$v_{2,18} = 0\pi^4(0134)\bar{0},$$

$$v_{2,19} = 0\pi^4(0124)\bar{0}.$$

Acknowledgements

I thank Gwénaël Richomme and the two referees whose remarks and recommendations helped me to improve the quality of this paper.

References