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Image normalization of Wiener–Hopf operators and boundary-transmission value problems for a junction of two half-planes

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ABSTRACT

The present paper deals with an application of the image normalization technique for certain classes of Wiener–Hopf operators (WHOs) associated to ill-posed boundary-transmission value problems. We briefly describe the method of normalization and then apply it to boundary-transmission value problems issued from diffraction problems for a junction of two half-planes, which are relevant in mathematical physics applications. For each boundary-transmission value problem, we analyze the conditions under which the associated operator and the equivalent WHO are not normally solvable, and define the corresponding image normalized operators.

1. Introduction

We are interested in operators, namely Wiener–Hopf operators (WHOs), which arise in the context of diffraction problems of electromagnetic and acoustic waves and are strongly related to the operator description of the corresponding boundary-transmission value problems. In general, for many relevant physical situations of the boundary the corresponding boundary value problems are ill-posed, i.e., the associated operators are not normally solvable (see e.g. [5,11,7,8]). This was the main reason why one of the authors of the present paper developed in her PhD work a method of image normalization in order to convert not normally solvable WHOs into operators with closed image. This method was firstly applied to boundary value problems on the half-plane [8], but can be successfully used in case of other geometries of the boundary. In this paper we describe how to apply it to a junction of two half-planes. The case of a strip is treated in paper [1]. The method of image normalization is one of the possible ways of normalizing bounded linear operators acting between Banach spaces (see [4]), and works very effectively for the operators under consideration.

In Section 2 we describe the class of boundary-transmission value problems which arise from the diffraction of a plane wave by a junction of two half-planes. Diffraction by a two-part plane is relevant in many practical applications (see e.g. [9,10,12]). Starting from the standard operator procedure of the classical survey of Meister and Speck [5], we associate with the physical problem an operator and then prove the equivalence of this operator to a WHO. In Section 2 we also establish notation. Section 3 is dedicated to a summary of the method of image normalization of the WHOs under consideration. We present there the main results of [8] without proofs and use a more convenient notation for our present purpose. The next three sections describe and analyze chosen examples of image normalization of WHOs coming from different boundary-transmission conditions on the two half-planes. For instance, in Section 4 we first derive the WHO for boundary-transmission conditions of arbitrary orders on the two banks of the two half-planes, and then consider the image

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normalization when all four orders are even. Section 5 is devoted to consider boundary-transmission conditions with only normal derivatives of the same order on the upper and lower banks of the two half-planes, respectively. Finally, in Section 6 we consider a simpler boundary condition on the left half-plane and a boundary-transmission condition with oblique derivatives on the right half-plane.

2. Boundary-transmission problems and WHOs

In order to study the WHOs, we begin with the formulation of the following general boundary-transmission value problem, we call it Problem $\mathcal{P}$, for the diffraction of a plane wave by a junction of two half-planes in the natural setting of locally finite energy norm.

Problem $\mathcal{P}$. Find $\varphi \in L^2(\mathbb{R}^2)$, with $\varphi_{|\mathbb{R}\times\mathbb{R}_\pm} = \varphi^\pm \in H^1(\mathbb{R}\times\mathbb{R}_\pm)$, such that

$$\big(\Delta + k_0^2\big)\varphi^\pm = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}_\pm, \tag{1}$$

$$B_j^-\varphi(x) = \sum_{\sigma_1+\sigma_2\leqslant m_j} a_{\sigma,j}^+\big(D^\sigma\varphi^+\big)(x,0) + a_{\sigma,j}^-\big(D^\sigma\varphi^-\big)(x,0) = h_j(x) \quad \text{on } \mathbb{R}_-, \tag{2}$$

$$B_j^+\varphi(x) = \sum_{\sigma_1+\sigma_2\leqslant m'_j} b_{\sigma,j}^+\big(D^\sigma\varphi^+\big)(x,0) + b_{\sigma,j}^-\big(D^\sigma\varphi^-\big)(x,0) = g_j(x) \quad \text{on } \mathbb{R}_+, \tag{3}$$

where $\sigma = (\sigma_1,\sigma_2)$, $\sigma_j \in \mathbb{N}_0$, and $m = (m_1, m_2)$, $m' = (m'_1, m'_2)$, $m_j, m'_j \in \mathbb{N}_0$, represent the order of the boundary operators B_j^- and B_j^+, respectively, with $j = 1,2$ corresponding to the upper and lower banks of both left $\mathbb{R}_-$ and right $\mathbb{R}_+$ half-lines.[1] The coefficients $a_{\sigma,j}^\pm, b_{\sigma,j}^\pm \in \mathbb{C}$ simulate physical properties of the boundaries. For instance, for $\sigma = (0,1)$, $m_j = (0,1)$, $a_{\sigma,j}^+ = 1$, $a_{\sigma,j}^- = -1$ and $h = (h_1,h_2) = (0,0)$ in (2), B_j^- consists of the trivial jump of Dirichlet data and Neumann data on $\mathbb{R}_-$ and is usually known as the transmission condition for the Sommerfeld problem (see e.g. [5]). On the other hand for $\sigma_1 + \sigma_2 \leqslant 1$ in (3), B_j^+ consists of a linear combination of Dirichlet, Neumann, and oblique derivative data as considered in [7]. It is also physically meaningful to consider linear combinations of higher derivatives [11] both normal and tangential, with coefficients $a_{\sigma,j}^\pm, b_{\sigma,j}^\pm \in \mathbb{C}$ depending on the materials of the boundary. In the Helmholtz equation (1) k_0 stands for the complex wave number with positive real and imaginary part, i.e., $\operatorname{Re} k_0 > 0$ and $\operatorname{Im} k_0 > 0$.

Similarly to the method used by Meister and Speck in the classical survey [5], we describe Problem $\mathcal{P}$ by a single equation

$$\mathcal{P}\varphi = g, \tag{4}$$

where $\mathcal{P}: D(\mathcal{P}) \to H^{1/2-m'_1}(\mathbb{R}_+) \times H^{1/2-m'_2}(\mathbb{R}_+)$ is a linear operator *associated* to Problem $\mathcal{P}$. The domain $D(\mathcal{P})$ consists of the elements of $H^1(\mathbb{R}\times\mathbb{R}_\pm)$ satisfying the Helmholtz equation (1) and the boundary-transmission conditions in (2). The image of $\mathcal{P}$ consists of the data $g = (g_1, g_2) \in H^{1/2-m'_1}(\mathbb{R}_+) \times H^{1/2-m'_2}(\mathbb{R}_+)$ according to the trace theorem (see below) and the representation formula applied to (3) with $m' = (m'_1, m'_2)$.

We recall that the space H^s is a Bessel potential space of order s defined by

$$H^s = \big\{\phi \in \mathcal{S}'\colon\ \mathcal{F}^{-1}\big(\xi^2+1\big)^{s/2}\cdot\mathcal{F}\phi \in L^2\big\}$$

where $\mathcal{F}$ represents the Fourier transformation.[2] We also need to consider the space H_+^s, which represents a subspace of H^s distributions supported on $\overline{\mathbb{R}}_+$, and $H^s(\mathbb{R}_+)$, a subspace of restrictions of H^s distributions on $\mathbb{R}_+$ that already appeared in the last paragraph. These are well-known Hilbert spaces whose topologies are the usual subspace topology for H_+^s and the quotient space topology for $H^s(\mathbb{R}_+)$, respectively.

The next goal is to show that the equivalence relation $\mathcal{P} = EWF$ holds, where E and F are bounded invertible linear operators and W is a linear operator acting on $H_+^r = H_+^{r_1} \times H_+^{r_2}$ whose image is $H^s(\mathbb{R}_+) = H^{s_1}(\mathbb{R}_+) \times H^{s_2}(\mathbb{R}_+)$.

We start with the standard representation formula for the solutions of the Helmholtz equation (1) (see e.g. [13])

$$\varphi(x,y) = \mathcal{K}\varphi_0(x,y) = \mathcal{F}^{-1}_{\xi\to x}\big\{e^{-\beta(\xi)y}\hat{\varphi}_0^+(\xi)\chi_+(y) + e^{\beta(\xi)y}\hat{\varphi}_0^-(\xi)\chi_-(y)\big\}, \tag{5}$$

where $\varphi_0 = (\varphi_0^+, \varphi_0^-) \in H^{1/2}\times H^{1/2}$ is the trace vector of $\varphi^\pm$ corresponding to the banks of $\mathbb{R}_\pm$, $\hat{\varphi}_0^\pm$ represents the Fourier transform of the traces, $\chi_\pm$ denotes the characteristic function of the positive and negative half-line, respectively, and

[1] As a consequence of the physics of the wave diffraction the boundaries, i.e., the two half-planes, can be identified with these two subsets of the real line.

[2] In the context of Problem $\mathcal{P}$, it is also commonly used $(\xi^2 - k_0^2)^{s/2}$ in the definition of the Bessel potential spaces, in which case the branch cuts are defined along $\pm k_0 \pm i\epsilon$, $\epsilon \geqslant 0$.

$\beta(\xi) = \sqrt{\xi^2 - k_0^2}$. Moreover, define $\tilde{B}_- = \operatorname{Rst} B_- : [H^{1/2}]^2 \to H_+^{1/2-m_1} \times H_+^{1/2-m_2}$ as the restricted operator with the same domain and Fourier symbol as B_-, but whose image is defined to be an H_+^s space.

For operator $\mathcal{K}$ in (5) the following result holds.

Theorem 2.1. *Let $B_- = \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F}$ be the bounded linear operator*

$$B_- : \left[H^{1/2}\right]^2 \to H^{1/2-m_1} \times H^{1/2-m_2}$$

with Fourier symbol

$$\Phi_- = \begin{bmatrix} \sum_{|\sigma|\leqslant m_1} a_{\sigma,1}^+(-i\xi)^{\sigma_1}(-\beta(\xi))^{\sigma_2} & \sum_{|\sigma|\leqslant m_1} a_{\sigma,1}^-(-i\xi)^{\sigma_1}\beta(\xi)^{\sigma_2} \\ \sum_{|\sigma|\leqslant m_2} a_{\sigma,2}^+(-i\xi)^{\sigma_1}(-\beta(\xi))^{\sigma_2} & \sum_{|\sigma|\leqslant m_2} a_{\sigma,2}^-(-i\xi)^{\sigma_1}\beta(\xi)^{\sigma_2} \end{bmatrix} \tag{6}$$

such that $\det \Phi_- \neq 0$, *and consider the restricted operator* $\tilde{B}_- = \operatorname{Rst} B_-$.

Then the operator $\mathcal{K}$ in (5) *is invertible and its inverse is the trace operator $T_0 : D(\mathcal{P}) \to Y_0$, where $D(\mathcal{P})$ is a closed subspace of the direct sum $H^1(\mathbb{R} \times \mathbb{R}_+) \oplus H^1(\mathbb{R} \times \mathbb{R}_-)$. The image space of T_0 is given by*

$$Y_0 = \left\{\varphi_0 = \left(\varphi_0^+, \varphi_0^-\right) \in \left[H^{1/2}\right]^2 \colon \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F}\varphi_0 - \ell^{(c)}h \in H_+^{1/2-m_1} \times H_+^{1/2-m_2}\right\}, \tag{7}$$

where $h = (h_1, h_2)$ is the data from (2), *$\ell^{(c)}$ represents a continuous extension operator of even type if m_j even, and odd type if m_j odd, and its left inverse is the restriction operator r_+.*

Moreover, for $\varphi_0 = (\varphi_0^+, \varphi_0^-) = \tilde{B}_-^{-1}(v^+ + \ell^{(c)}h)$ with $v^+ \in H_+^{1/2-m_1} \times H_+^{1/2-m_2}$ in (5), *the operator $\tilde{B}_- T_0$ is continuously invertible and its inverse is the operator $\mathcal{K}\tilde{B}_-^{-1}$.*

Proof. The trace operator $T_0 : D(\mathcal{P}) \to Y_0$ is here defined as an operator that acts between spaces of order greater than or equal to 1/2. For these space orders, we automatically have surjectivity and right invertibility. The left invertibility is obtained by choosing the space Y_0 in (7) as a subspace of order 1/2 and such that it contains zero extensions of the corresponding trace values that appear in our problem (see e.g. [1] for a discussion when this fails). Therefore, we have invertibility for T_0 and its inverse is the operator $\mathcal{K}$ given by (5).

Consider now $\varphi_0 = \tilde{B}_-^{-1}(v^+ + \ell^{(c)}h)$ in the representation formula (5). Then $\tilde{B}_-\varphi_0 = v^+ + \ell^{(c)}h$ and we have

$$\tilde{B}_- T_0 \mathcal{K}\tilde{B}_-^{-1}\left(v^+ + \ell^{(c)}h\right) = \tilde{B}_- T_0 \mathcal{K}\varphi_0 = \tilde{B}_-\varphi_0 = v^+ + \ell^{(c)}h,$$

and also

$$\mathcal{K}\tilde{B}_-^{-1}\tilde{B}_- T_0\varphi = \mathcal{K}\tilde{B}_-^{-1}\tilde{B}_-\tilde{B}_-^{-1}\left(v^+ + \ell^{(c)}h\right) = \mathcal{K}\left(\tilde{B}_-^{-1}\left(v^+ + \ell^{(c)}h\right)\right) = \mathcal{K}\varphi_0 = \varphi,$$

i.e. $\tilde{B}_- T_0$ is continuously invertible and its inverse is given by $\mathcal{K}\tilde{B}_-^{-1}$. □

We now prove the following equivalence result.

Theorem 2.2. *Let $B_+ = \mathcal{F}^{-1}\Phi_+ \cdot \mathcal{F}$ be the linear bounded operator*

$$B_+ : \left[H^{1/2}\right]^2 \to H^{1/2-m_1'} \times H^{1/2-m_2'}$$

with Fourier symbol

$$\Phi_+ = \begin{bmatrix} \sum_{|\sigma|\leqslant m_1'} b_{\sigma,1}^+(-i\xi)^{\sigma_1}(-\beta(\xi))^{\sigma_2} & \sum_{|\sigma|\leqslant m_1'} b_{\sigma,1}^-(-i\xi)^{\sigma_1}\beta(\xi)^{\sigma_2} \\ \sum_{|\sigma|\leqslant m_2'} b_{\sigma,2}^+(-i\xi)^{\sigma_1}(-\beta(\xi))^{\sigma_2} & \sum_{|\sigma|\leqslant m_2'} b_{\sigma,2}^-(-i\xi)^{\sigma_1}\beta(\xi)^{\sigma_2} \end{bmatrix} \tag{8}$$

such that $\det \Phi_+ \neq 0$, *and let the conditions of Theorem* 2.1 *hold. Then, the operator $\mathcal{P}$ in* (4) *is equivalent to the WHO*

$$W = r_+ B_+ \tilde{B}_-^{-1} : H_+^{1/2-m_1} \times H_+^{1/2-m_2} \to H^{1/2-m_1'}(\mathbb{R}_+) \times H^{1/2-m_2'}(\mathbb{R}_+) \tag{9}$$

with Fourier symbol $\Phi = \Phi_+\Phi_-^{-1}$. The equivalence relation is given by

$$\mathcal{P} = W\tilde{B}_- T_0, \tag{10}$$

i.e., the operators $\mathcal{P}$ and W coincide up to bijective factors.

Proof. Since by Theorem 2.1, we have $\varphi_0 = \tilde{B}_-^{-1}(v^+ + \ell^{(c)}h)$, it follows that $v^+ = \tilde{B}_-\varphi_0 - \ell^{(c)}h$ and

$$Wv^+ = W\left(\tilde{B}_-\varphi_0 - \ell^{(c)}h\right) = r_+ B_+ \tilde{B}_-^{-1}\left(\tilde{B}_-\varphi_0 - \ell^{(c)}h\right) = r_+ B_+\varphi_0 - r_+ B_+ \tilde{B}_-^{-1}\ell^{(c)}h = g - r_+ B_+ \tilde{B}_-^{-1}\ell^{(c)}h.$$

On the other hand, assuming that (10) holds, we can write

$$\mathcal{P}\varphi = W\tilde{B}_- T_0\varphi = W\tilde{B}_-\varphi_0 = W\tilde{B}_-\tilde{B}_-^{-1}\big(v^+ + \ell^{(c)}h\big) = Wv^+ + W\ell^{(c)}h,$$

which after substituting Wv^+ by the expression obtained before, yields $\mathcal{P}\varphi = g$. This proves the equivalence between the two operators with the equivalence relation given by (10). □

Next we study in more detail the general structure of the operators obtained above. Firstly, we formally rewrite the WHO as an operator

$$W = r_+ A|_{H_+^r} : H_+^r \to H^s(\mathbb{R}_+) \tag{11}$$

where $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ is a translation invariant homeomorphism with a matrix *Fourier symbol* $\Phi = \Phi_+\Phi_-^{-1} \in L^\infty_{\mathrm{loc}}$. Note that the elements of the Fourier symbol Φ of W in (9), due to (6) and (8), for any arbitrary orders m_j, m_j' and coefficients $a^\pm_{\sigma,j}, b^\pm_{\sigma,j}$, are rational functions of ξ and $\beta(\xi) = \sqrt{\xi^2 - k_0^2}$, see the next sections for details. Then, lifting the WHO W into L^2 (see e.g. [3]) we obtain the *lifted* WHO

$$W_0 = r_+ A_0|_{[L_+^2]^n} : \big[L_+^2\big]^2 \to \big[L^2(\mathbb{R}_+)\big]^2, \tag{12}$$

where $A_0 = \mathcal{F}^{-1}\Phi_0 \cdot \mathcal{F}$, $\Phi_0 \in L^\infty(\mathbb{R})^{2\times 2}$. In this paper we assume in the first place that $\Phi_0 \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})^{2\times 2}$, i.e., that the lifted Fourier symbol lies in the invertible algebra of Hölder continuous 2×2 matrix functions defined on $\ddot{\mathbb{R}} = [-\infty, +\infty]$. In fact, the elements of the lifted Fourier symbol Φ_0 are bounded rational functions of $\rho(\xi) = \sqrt{\frac{\xi - k_0}{\xi + k_0}}$ and of $\xi\beta(\xi)^{-1}$, and we always assume that $\det\Phi_0(\xi) \neq 0$, $\xi \in \ddot{\mathbb{R}}$, in order to get normal type WHOs. Given this assumption, we study then the conditions under which the operator is not normally solvable and solve the image normalization problem for the WHO.[3]

The following Fredholm criterium is well known [6] for the lifted WHO in (12). The operator W_0 is normally solvable iff

$$\det\big(\mu\Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty)\big) \neq 0, \quad \mu \in\,]0,1[. \tag{13}$$

As a consequence of $\rho(\xi) \to \pm 1$ and $\xi\beta(\xi)^{-1} \to \pm 1$ as $\xi \to \pm\infty$, respectively, this condition does not hold for a large class of WHOs in (9), and from the equivalence relation (10), the same is true for the associated operator $\mathcal{P}$. Therefore, it is necessary to obtain the image normalization of both operators.

Finally, we shall also use the zero extension operator $\ell^{(0)}$ and the following Bessel potential operators [8] for $w \in \mathbb{C}$, $k_0 \in \mathbb{C}$, $\operatorname{Im} k_0 > 0$,

$$\Lambda_\pm^w = \mathcal{F}^{-1}\lambda_\pm^w \cdot \mathcal{F} : H^s \to H^{s-\operatorname{Re} w},$$

where we introduced $\lambda_\pm(\xi) = \xi \pm k_0$, a notation often used in this context.

3. Image normalization of WHOs in scalar and matrix cases

We briefly describe the main results of our approach (for proofs see [8]) towards the normalization of the WHOs defined by (11) with lifted Fourier symbol $\Phi_0 \in \mathcal{G}C^\nu(\ddot{\mathbb{R}})^{2\times 2}$ for which the Fredholm criterium (13) doesn't hold. The method is based on two central ideas: firstly we want the domain of the operator to remain a space of locally finite energy, and secondly we change the image space in a *minimal* way. The following scalar result [8] helps to understand the method for the matrix case.

Theorem 3.1. *Let us consider the scalar WHO of normal type, which acts symmetrically, i.e., $r = s$*

$$W_s = W_s(\Phi) = r_+ A|_{H_+^s} : H_+^s \to H^s(\mathbb{R}_+).$$

Then for the critical orders [2] $s + \eta + 1/2 \in \mathbb{Z}$, where $\eta = \frac{1}{2\pi i}\int_{\mathbb{R}} d\arg\Phi$, the operator W_s is not normally solvable.

Introducing $w = \eta + i\tau$, with $\tau = \frac{1}{2\pi}\ln|\Phi(-\infty)/\Phi(+\infty)|$, we define the image normalized operator $\breve{W}_s$ by

$$\breve{W}_s = \operatorname{Rst} W_s : H_+^s \to \breve{H}^{s-i\tau}(\mathbb{R}_+),$$

where $\breve{H}^{s-i\tau}(\mathbb{R}_+) = r_+\Lambda_-^{-s+i\tau-1/2}H_+^{-1/2} \subset H^{\operatorname{Re} w}(\mathbb{R}_+)$. The image space of $\breve{W}_s$ solves the normalization problem for $\{W_s = W_s(\Phi)\colon \Phi \in \mathcal{G}C^\nu(\ddot{\mathbb{R}}),\ \nu \in\,]0,1[,\ \operatorname{im} W_s \neq \overline{\operatorname{im} W_s}\}$.

[3] In this paper we are interested in the not normally solvable cases, thus we assume first that the coefficients in (2)–(3) are such that $\det\Phi_0 \neq 0$, i.e., the matrix does not degenerate on $\ddot{\mathbb{R}}$, and for these coefficients analyze the case of not normally solvable WHOs.

The normalization in the matrix case is based on the same idea of using the jump at infinity of the lifted Fourier symbol to change the image space in a *minimal* way. The following result, which we state for the 2×2 matrix case, since this will be enough for our purpose, can be found in [8] for the $n \times n$ matrix case.

Theorem 3.2. *Consider the WHO W in (11), with $r = (r_1, r_2)$, $s = (s_1, s_2)$ and such that the corresponding lifted Fourier symbol $\Phi_0 \in \mathcal{G}C^{\nu}(\ddot{\mathbb{R}})^{2\times 2}$. Moreover, let this Fourier symbol Φ_0 have a jump at infinity. Let λ_1, λ_2 be the eigenvalues of Φ_0 and write*

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = T^{-1}\operatorname{diag}(\lambda_1, \lambda_2)T, \tag{14}$$

where $T \in \mathcal{G}\mathbb{C}^{2\times 2}$. Assuming that $\lambda_1 = e^{2i\pi w_1}$ with $\operatorname{Re} w_1 = -1/2$ ($\lambda_2 = e^{2i\pi w_2}$ with $\operatorname{Re} w_2 \neq -1/2$), the WHO W is not normally solvable for the given s_1 and $\eta_1 = -1/2$. Then we define the image normalized operator $\breve{W}$ by

$$\breve{W} : H_+^r \to Y_1 = r_+ \Lambda_-^{-s} T \ell^{(0)} \{ \breve{H}^{-i\tau}(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \}$$

with $\tau = \frac{1}{2\pi} \ln|\Phi_0(-\infty)/\Phi_0(+\infty)|$, which corresponds to the eigenvalue λ_1. The image space Y_1 of the restricted WHO $\breve{W}$ solves the normalization problem for W.

Remark that we say that the image space Y_1 solves the normalization problem for the WHO and denote by $\breve{W}$ the corresponding *image normalized* operator.

In Theorem 3.2 we assumed that the eigenvalues of Φ_0 are different and that only one of the eigenvalues, λ_1 in (14), is responsible for the jump at infinity. As we will see in next sections, it is possible that we get an eigenvalue with multiplicity two, i.e., $\lambda_2 = \lambda_1$ in (14), and in this case we should modify the image space in both components. Furthermore, very often in applications we have the eigenvalue $\lambda_1 = -1$, due to $w_1 = -1/2$, which leads to an image space of the type

$$Y_1 = r_+ \Lambda_-^{-s} \ell^{(0)} \{ \breve{H}^0(\mathbb{R}_+) \times L^2(\mathbb{R}_+) \}$$

with $\breve{H}^0(\mathbb{R}_+) = r_+ \Lambda_-^{-1/2} H_+^{-1/2}$ being a proper dense subspace of $L^2(\mathbb{R}_+)$ [8].

Finally, note that by means of the equivalence relation (10), the image normalization of a particular WHO W yields the image normalization of the operator $\mathcal{P}$ in (4).

4. Boundary-transmission problems of higher order

From this section on we analyze several examples of boundary-transmission conditions, less general than (2)–(3), but still very significant from the applications point of view. In the first place we retain only the higher order terms in the boundary-transmission conditions (2)–(3) and such that they do not contain derivatives of mixed type. This assumption is also mathematically consistent with the fact that these terms fully describe the behavior at infinity of the Fourier symbol. Consider now Problem $\mathcal{P}$ with the following higher order boundary-transmission conditions: order $m = (m_1, m_2)$ on the left half-line and order $m' = (m_3, m_4)$ on the right half-line[4]

$$\begin{cases} a_1^+ \varphi_{m_1}^+ + a_1^- \varphi_{m_1}^- + \breve{a}_1^+ \breve{\varphi}_{m_1}^+ + \breve{a}_1^- \breve{\varphi}_{m_1}^- = h_1, \\ a_2^+ \varphi_{m_2}^+ + a_2^- \varphi_{m_2}^- + \breve{a}_2^+ \breve{\varphi}_{m_2}^+ + \breve{a}_2^- \breve{\varphi}_{m_2}^- = h_2, \end{cases} \quad \text{on } \mathbb{R}_-, \tag{15}$$

$$\begin{cases} b_3^+ \varphi_{m_3}^+ + b_3^- \varphi_{m_3}^- + \breve{b}_3^+ \breve{\varphi}_{m_3}^+ + \breve{b}_3^- \breve{\varphi}_{m_3}^- = g_1, \\ b_4^+ \varphi_{m_4}^+ + b_4^- \varphi_{m_4}^- + \breve{b}_4^+ \breve{\varphi}_{m_4}^+ + \breve{b}_4^- \breve{\varphi}_{m_4}^- = g_2, \end{cases} \quad \text{on } \mathbb{R}_+, \tag{16}$$

where in general all four orders m_j, $j = 1, 2, 3, 4$, are supposed to be different. The notation $\varphi_{m_j}^{\pm}$ and $\breve{\varphi}_{m_j}^{\pm}$ stands for the traces of the normal and tangential derivatives of order m_j, respectively, with $a_j^{\pm}$, $b_j^{\pm}$, $\breve{a}_j^{\pm}$, $\breve{b}_j^{\pm}$ denoting the corresponding coefficients. From the trace theorem and the representation formula, we conclude that $h = (h_1, h_2) \in H^{1/2-m_1}(\mathbb{R}_-) \times H^{1/2-m_2}(\mathbb{R}_-)$ and $g = (g_1, g_2) \in H^{1/2-m_3}(\mathbb{R}_+) \times H^{1/2-m_4}(\mathbb{R}_+)$.

The following theorem holds for the operator $\mathcal{P}$ associated with this boundary-transmission problem and the equivalent WHO W, and is a direct consequence of Theorems 2.1 and 2.2.

Theorem 4.1. *Let $B_- = \mathcal{F}^{-1}\Phi_- \cdot \mathcal{F}$ and $B_+ = \mathcal{F}^{-1}\Phi_+ \cdot \mathcal{F}$ be the following linear bounded operators*

$$B_- : [H^{1/2}]^2 \to H^{1/2-m_1} \times H^{1/2-m_2},$$
$$B_+ : [H^{1/2}]^2 \to H^{1/2-m_3} \times H^{1/2-m_4}$$

[4] In this section we intentionally use the notation of orders m_j, $j = 1, 2, 3, 4$, corresponding to the indices of the coefficients in order not to overload the formulas. We also simplify the index notations of the coefficients, e.g. a_1^+ instead of $a_{m_1,1}^+$.

with the non-degenerated Fourier symbols

$$\Phi_- = \begin{bmatrix} (-1)^{m_1} a_1^+ \beta^{m_1} + \check{a}_1^+ (-i\xi)^{m_1} & a_1^- \beta^{m_1} + \check{a}_1^- (-i\xi)^{m_1} \\ (-1)^{m_2} a_2^+ \beta^{m_2} + \check{a}_2^+ (-i\xi)^{m_2} & a_2^- \beta^{m_2} + \check{a}_2^- (-i\xi)^{m_2} \end{bmatrix} \tag{17}$$

and

$$\Phi_+ = \begin{bmatrix} (-1)^{m_3} b_3^+ \beta^{m_3} + \check{b}_3^+ (-i\xi)^{m_3} & b_3^- \beta^{m_3} + \check{b}_3^- (-i\xi)^{m_3} \\ (-1)^{m_4} b_4^+ \beta^{m_4} + \check{b}_4^+ (-i\xi)^{m_4} & b_4^- \beta^{m_4} + \check{b}_4^- (-i\xi)^{m_4} \end{bmatrix}, \tag{18}$$

respectively, i.e., $\det \Phi_+ \Phi_-^{-1} \neq 0$*. Moreover, consider the restricted operator* $\tilde{B}_- = \operatorname{Rst} B_-$ *with Fourier symbol also given by* (17)*. Then, the operator* $\mathcal{P}$ *given by*

$$\begin{aligned} &\mathcal{P} : D(\mathcal{P}) \to H^{1/2-m_3}(\mathbb{R}_+) \times H^{1/2-m_4}(\mathbb{R}_+), \\ &\qquad \varphi \to \mathcal{P}\varphi = g, \end{aligned}$$

where $\varphi \in D(\mathcal{P})$ *has traces* $\varphi_0 = (\varphi_0^+, \varphi_0^-) = \tilde{B}_-^{-1}(v^+ + \ell^{(c)} h)$, $v^+ \in H_+^{1/2-m_1} \times H_+^{1/2-m_2}$*, is equivalent to the WHO*

$$\begin{aligned} W : H_+^{1/2-m_1} \times H_+^{1/2-m_2} &\to H^{1/2-m_3}(\mathbb{R}_+) \times H^{1/2-m_4}(\mathbb{R}_+), \\ v^+ &\to W v^+ = g - r_+ B_+ \tilde{B}_-^{-1} \ell^{(c)} h, \end{aligned}$$

i.e., $W = r_+ \mathcal{F}^{-1} \Phi \cdot \mathcal{F}$ *with Fourier symbol* $\Phi = \Phi_+ \Phi_-^{-1}$*. The equivalence relation is given by* $\mathcal{P} = W \tilde{B}_- T_0$*.*

A straightforward computation leads to the Fourier symbol of the equivalent WHO W of the form

$$\Phi = \frac{1}{\det \Phi_-} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{19}$$

with

$$\begin{aligned} \det \Phi_- = {} & \beta^{m_1+m_2} \big((-1)^{m_1} a_1^+ a_2^- - (-1)^{m_2} a_2^+ a_1^- \big) + (-i\xi)^{m_1+m_2} \big(\check{a}_1^+ \check{a}_2^- - \check{a}_1^- \check{a}_2^+ \big) \\ & + (-i\xi)^{m_1} \beta^{m_2} \big((-1)^{m_1} \check{a}_1^+ a_2^- - (-1)^{m_2} \check{a}_1^- a_2^+ \big) + (-i\xi)^{m_2} \beta^{m_1} \big((-1)^{m_1} a_1^+ \check{a}_2^- - (-1)^{m_2} a_1^- \check{a}_2^+ \big), \end{aligned}$$

and entries

$$\begin{aligned} A_{11} = {} & \beta^{m_2+m_3} \big((-1)^{m_3} a_2^- b_3^+ - (-1)^{m_2} a_2^+ b_3^- + \big(\check{a}_2^- \check{b}_3^+ + \check{a}_2^+ \check{b}_3^- \big) \big(-i\xi\beta^{-1} \big)^{m_2+m_3} \\ & + \big((-1)^{m_3} \check{a}_2^- b_3^+ + \check{a}_2^+ b_3^- \big) \big(-i\xi\beta^{-1} \big)^{m_2} + \big(a_2^- \check{b}_3^+ - (-1)^{m_2} a_2^+ \check{b}_3^- \big) \big(-i\xi\beta^{-1} \big)^{m_3} \big), \\ A_{12} = {} & \beta^{m_1+m_3} \big((-1)^{m_1} a_1^+ b_3^- - (-1)^{m_3} a_1^- b_3^+ + \big(\check{a}_1^+ \check{b}_3^- - a_1^- \check{b}_3^+ \big) \big(-i\xi\beta^{-1} \big)^{m_1+m_3} \\ & + \big((-1)^{m_3} \check{a}_1^- b_3^+ + \check{a}_1^+ b_3^- \big) \big(-i\xi\beta^{-1} \big)^{m_1} + \big((-1)^{m_1} a_1^+ \check{b}_3^- - \check{a}_1^- \check{b}_3^+ \big) \big(-i\xi\beta^{-1} \big)^{m_3} \big), \\ A_{21} = {} & \beta^{m_2+m_4} \big((-1)^{m_4} a_2^- b_4^+ - (-1)^{m_2} a_2^+ b_4^- + \big(\check{a}_2^- \check{b}_4^+ + \check{a}_2^+ \check{b}_4^- \big) \big(-i\xi\beta^{-1} \big)^{m_2+m_4} \\ & + \big((-1)^{m_4} \check{a}_2^- b_4^+ + \check{a}_2^+ b_4^- \big) \big(-i\xi\beta^{-1} \big)^{m_2} + \big(a_2^- \check{b}_4^+ - (-1)^{m_2} a_2^+ \check{b}_4^- \big) \big(-i\xi\beta^{-1} \big)^{m_4} \big), \\ A_{22} = {} & \beta^{m_1+m_4} \big((-1)^{m_1} a_1^+ b_4^- - (-1)^{m_4} a_1^- b_4^+ + \big(\check{a}_1^+ \check{b}_4^- - \check{a}_1^- \check{b}_4^+ \big) \big(-i\xi\beta^{-1} \big)^{m_1+m_4} \\ & + \big(\check{a}_1^+ b_4^- - (-1)^{m_4} \check{a}_1^- b_4^+ \big) \big(-i\xi\beta^{-1} \big)^{m_1} + \big((-1)^{m_1} a_1^+ \check{b}_4^- - a_1^- \check{b}_4^+ \big) \big(-i\xi\beta^{-1} \big)^{m_4} \big), \end{aligned}$$

where $\beta(\xi) = \sqrt{\xi^2 - k_0^2}$. The corresponding lifted Fourier symbol can be obtained based on the standard lifting procedure, i.e., taking $\Phi_0 = \operatorname{diag}(\lambda_-^{1/2-m_3}, \lambda_-^{1/2-m_4}) \Phi \operatorname{diag}(\lambda_+^{m_1-1/2}, \lambda_+^{m_2-1/2})$. We obtain explicitely

$$\Phi_0 = \frac{\rho}{\det \Phi_-} \begin{bmatrix} \frac{(\xi+k_0)^{m_1}}{(\xi-k_0)^{m_3}} A_{11} & \frac{(\xi+k_0)^{m_2}}{(\xi-k_0)^{m_3}} A_{12} \\ \frac{(\xi+k_0)^{m_1}}{(\xi-k_0)^{m_4}} A_{21} & \frac{(\xi+k_0)^{m_2}}{(\xi-k_0)^{m_4}} A_{22} \end{bmatrix}, \tag{20}$$

where $\rho(\xi) = \sqrt{\frac{\xi-k_0}{\xi+k_0}}$.

In general, the given operator $\mathcal{P}$ is not normally solvable for arbitrary orders m_j and coefficients $a_j^\pm$, $b_j^\pm$, $\check{a}_j^\pm$, $\check{b}_j^\pm$, $j = 1, 2, 3, 4$. Furthermore, if all orders are even, then the fact that $\mathcal{P}$ is not normally solvable does not depend on the coefficients. Although it will be very cumbersome to enumerate all the cases, the following example shows the power of the image normalization technique.

Theorem 4.2. *Let $m_1 = m_2 = m_3 = m_4 = m \in 2\mathbb{N}_0$, i.e., all orders are equal to m, and m is zero or an even number. Furthermore, let $\det \Phi_0 \neq 0$ in (20). Then the associated operator $\mathcal{P}$, and consequently the equivalent WHO W, are not normally solvable. In this case we consider the corresponding image normalized operator $\breve{W}$ defined by*

$$\breve{W} = \operatorname{Rst} W : \left[H_+^{1/2-m}\right]^2 \to Y_1 = r_+ \Lambda_-^{-s} \ell^{(0)} \left\{\breve{H}^0(\mathbb{R}_+) \times \breve{H}^0(\mathbb{R}_+)\right\},$$

and $s = (1/2 - m, 1/2 - m)$. The image space Y_1 of the restricted operator $\breve{W}$ solves the normalization problem for the WHO, consequently we look for solutions of $\mathcal{P}\varphi = g$, $\mathcal{P} : D(\mathcal{P}) \to Y_1$, for which $g - r_+ B_+ \tilde{B}_-^{-1} \ell^{(c)} h \in Y_1$.

Proof. The lifted Fourier symbol in (20) for $m = m_1 = m_2 = m_3 = m_4 \in 2\mathbb{N}_0$ simplifies to

$$\Phi_0 = \frac{\rho}{A} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$\begin{aligned}
A &= a_1^+ a_2^- - a_2^+ a_1^- + \left(\xi\beta^{-1}\right)^{2m} \left(\breve{a}_1^+ \breve{a}_2^- - \breve{a}_1^- \breve{a}_2^+\right) + (-i)^m \left(\xi\beta^{-1}\right)^m \left(\breve{a}_1^+ a_2^- - \breve{a}_1^- a_2^+ + a_1^+ \breve{a}_2^- - a_1^- \breve{a}_2^+\right), \\
B_{11} &= a_2^- b_3^+ - a_2^+ b_3^- + \left(\xi\beta^{-1}\right)^{2m} \left(\breve{a}_2^- \breve{b}_3^+ + \breve{a}_2^+ \breve{b}_3^-\right) + (-i)^m \left(\xi\beta^{-1}\right)^m \left(\breve{a}_2^- b_3^+ + \breve{a}_2^+ b_3^- + a_2^- \breve{b}_3^+ - a_2^+ \breve{b}_3^-\right), \\
B_{12} &= a_1^+ b_3^- - a_1^- b_3^+ + \left(\xi\beta^{-1}\right)^{2m} \left(\breve{a}_1^+ \breve{b}_3^- - a_1^- \breve{b}_3^+\right) + (-i)^m \left(\xi\beta^{-1}\right)^m \left(\breve{a}_1^- b_3^+ + \breve{a}_1^+ b_3^- + a_1^+ \breve{b}_3^- - \breve{a}_1^- \breve{b}_3^+\right), \\
B_{21} &= a_2^- b_4^+ - a_2^+ b_4^- + \left(\xi\beta^{-1}\right)^{2m} \left(\breve{a}_2^- \breve{b}_4^+ + \breve{a}_2^+ \breve{b}_4^-\right) + (-i)^m \left(\xi\beta^{-1}\right)^m \left(\breve{a}_2^- b_4^+ + \breve{a}_2^+ b_4^- + a_2^- \breve{b}_4^+ - a_2^+ \breve{b}_4^-\right), \\
B_{22} &= a_1^+ b_4^- - a_1^- b_4^+ + \left(\xi\beta^{-1}\right)^{2m} \left(\breve{a}_1^+ \breve{b}_4^- - \breve{a}_1^- \breve{b}_4^+\right) + (-i)^m \left(\xi\beta^{-1}\right)^m \left(\breve{a}_1^+ b_4^- - \breve{a}_1^- b_4^+ + a_1^+ \breve{b}_4^- - a_1^- \breve{b}_4^+\right).
\end{aligned}$$

Recall that $\rho(\xi)$, as well as $\xi\beta(\xi)^{-1}$, tends to ± 1 as ξ tends to $\pm\infty$, respectively. But here all the $\xi\beta(\xi)^{-1}$ factors are raised to an even power: $2m$ or m. Thus $\Phi_0(-\infty) = -\Phi_0(+\infty)$ and for the Fredholm criterium (13) one has

$$\mu\Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty) = (1-2\mu)\Phi_0(+\infty),$$

which degenerates for $\mu = 1/2$, i.e., Φ_0 doesn't fulfill the Fredholm criterium for $\mu = 1/2$. After some calculations we arrive at

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, the result follows from Theorem 3.2, since the jump at infinity (14) has a diagonal form with one eigenvalue $\lambda = -1$ with multiplicity two. □

We remark once again that an analogous result can be obtained for even orders m_j not necessarily all equals, see e.g. Theorem 6.2, only in that case the calculations are more complicated.

5. Boundary-transmission problems of pairwise normal type

We formulate now a particular case of boundary-transmission conditions of the form (15)–(16), namely consider on both upper banks of $\mathbb{R}_-$ and $\mathbb{R}_+$ boundary-transmission conditions with normal derivatives of a given order, say m_1, and on both lower banks of $\mathbb{R}_-$ and $\mathbb{R}_+$ boundary-transmission conditions with normal derivatives of another order, say m_2. We can similarly to the previous Section 4, define the associated operator $\mathcal{P}$ to the problem and study its normal solvability together with the normal solvability of the equivalent WHO W. Consider, together with the Helmholtz equation (1), the following boundary-transmission conditions of orders $m = (m_1, m_2)$ and $m' = (m_1, m_2)$

$$\begin{cases} a_1^+ \varphi_{m_1}^+ + a_1^- \varphi_{m_1}^- = h_1, \\ a_2^+ \varphi_{m_2}^+ + a_2^- \varphi_{m_2}^- = h_2, \end{cases} \quad \text{on } \mathbb{R}_-, \tag{21}$$

$$\begin{cases} b_1^+ \varphi_{m_1}^+ + b_1^- \varphi_{m_1}^- = g_1, \\ b_2^+ \varphi_{m_2}^+ + b_2^- \varphi_{m_2}^- = g_2, \end{cases} \quad \text{on } \mathbb{R}_+, \tag{22}$$

where $m_1 \neq m_2$, $h = (h_1, h_2) \in H^{1/2-m_1}(\mathbb{R}_-) \times H^{1/2-m_2}(\mathbb{R}_-)$ and $g = (g_1, g_2) \in H^{1/2-m_1}(\mathbb{R}_+) \times H^{1/2-m_2}(\mathbb{R}_+)$.

Here we should consider two cases: when $m_1 + m_2$ is even or zero, and when $m_1 + m_2$ is odd, due to the following necessary and sufficient conditions for the operator $\mathcal{P}$, and the equivalent WHO W, be of normal type.

Theorem 5.1. *Consider the associated operator*

$$\mathcal{P}: D(\mathcal{P}) \to H^{1/2-m_1}(\mathbb{R}_+) \times H^{1/2-m_2}(\mathbb{R}_+),$$
$$\varphi \to \mathcal{P}\varphi = g,$$

and the equivalent WHO

$$W = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : H_+^{1/2-m_1} \times H_+^{1/2-m_2} \to H^{1/2-m_1}(\mathbb{R}_+) \times H^{1/2-m_2}(\mathbb{R}_+)$$

with Fourier symbol

$$\Phi = \frac{1}{A}\begin{bmatrix} (-1)^{m_1}a_2^-b_1^+ - (-1)^{m_2}a_2^+b_1^- & (-1)^{m_1}(a_1^+b_1^- - a_1^-b_1^+)\beta^{m_1-m_2} \\ (-1)^{m_2}(a_2^-b_2^+ - a_2^+b_2^-)\beta^{m_2-m_1} & (-1)^{m_1}a_1^+b_2^- - (-1)^{m_2}a_1^-b_2^+ \end{bmatrix}, \tag{23}$$

where $A = (-1)^{m_1}a_1^+a_2^- - (-1)^{m_2}a_1^-a_2^+$. Then the operator W, and consequently the operator $\mathcal{P}$, are of normal type iff

$$\frac{a_1^+a_2^-b_1^+b_2^- + a_1^-a_2^+b_1^-b_2^+}{a_1^-a_2^+b_1^+b_2^- + a_1^+a_2^-b_1^-b_2^+} \neq (-1)^{m_1+m_2}. \tag{24}$$

Proof. The operator W is obtained as in Section 4 from definition (9), i.e., $W = r_+B_+\tilde{B}_-^{-1}$, where now the operators $\tilde{B}_-$ and B_+ have the folllowing Fourier symbols

$$\Phi_- = \begin{bmatrix} a_1^+(-\beta)^{m_1} & a_1^-\beta^{m_1} \\ a_2^+(-\beta)^{m_2} & a_2^-\beta^{m_2} \end{bmatrix},$$
$$\Phi_+ = \begin{bmatrix} b_1^+(-\beta)^{m_1} & b_1^-\beta^{m_1} \\ b_2^+(-\beta)^{m_2} & b_2^-\beta^{m_2} \end{bmatrix},$$

respectively. Note that these are particular cases of symbols (17) and (18) for zero coefficients of the tangential derivatives and $m_3 = m_1$, $m_4 = m_2$. From (23) we can obtain the lifted Fourier symbol, computing $\Phi_0 = \operatorname{diag}(\lambda_-^{1/2-m_1}, \lambda_-^{1/2-m_2}) \times \Phi\, \operatorname{diag}(\lambda_+^{m_1-1/2}, \lambda_+^{m_2-1/2})$ explicitly, i.e.,

$$\Phi_0 = \frac{1}{A}\begin{bmatrix} ((-1)^{m_1}a_2^-b_1^+ - (-1)^{m_2}a_2^+b_1^-)\rho^{1-2m_1} & (-1)^{m_1}(a_1^+b_1^- - a_1^-b_1^+)\rho^{1-m_1-m_2} \\ (-1)^{m_2}(a_2^-b_2^+ - a_2^+b_2^-)\rho^{1-m_1-m_2} & ((-1)^{m_1}a_1^+b_2^- - (-1)^{m_2}a_1^-b_2^+)\rho^{1-2m_2} \end{bmatrix}, \tag{25}$$

where $A = (-1)^{m_1}a_1^+a_2^- - (-1)^{m_2}a_1^-a_2^+$. Finally, condition $\det\Phi_0 \neq 0$ is equivalent to

$$a_1^+a_2^-b_1^+b_2^- + a_1^-a_2^+b_1^-b_2^+ + (-1)^{m_1+m_2+1}\big(a_1^-a_2^-b_1^+b_2^+ + a_1^+a_2^+b_1^-b_2^- + \big(a_1^-b_1^+ - a_1^+b_1^-\big)\big(a_2^-b_2^+ - a_2^+b_2^-\big)\big) \neq 0,$$

which can also be simplified to (24). □

Condition (24) means that for orders $m_1 + m_2 \in 2\mathbb{N}_0$ the operators $\mathcal{P}$ and W are of normal type iff $a_1^+a_2^- \neq a_1^-a_2^+$ and $b_1^+b_2^- \neq b_1^-b_2^+$. On the other hand, if $m_1 + m_2 \in 2\mathbb{N}_0 + 1$, then the operators $\mathcal{P}$ and W are of normal type iff $a_1^+a_2^- \neq -a_1^-a_2^+$ and $b_1^+b_2^- \neq -b_1^-b_2^+$. The first case gives place to the following theorem on the image normalization of W and $\mathcal{P}$.

Theorem 5.2. *Let $m_1 + m_2 \in 2\mathbb{N}_0$ in the boundary-transmission conditions* (21)–(22) *and assume that* (24) *holds. Then the operator $\mathcal{P}$ and the equivalent operator W are not-normally solvable. In this case, the image space of the image normalized operator $\breve{W}$ given by*

$$\breve{W} = \operatorname{Rst} W : \big[H_+^{1/2-m_1}\big]^2 \to r_+\Lambda_-^{-s}\ell^{(0)}\big\{\breve{H}^0(\mathbb{R}_+) \times \breve{H}^0(\mathbb{R}_+)\big\},$$

with $s = (1/2 - m_1, 1/2 - m_2)$, solves the normalization problem for the WHO. The image normalization of operator $\mathcal{P}$ is obtained by substituting W by $\breve{W}$ in the equivalence relation (10)*.*

Proof. For $m_1 + m_2 \in 2\mathbb{N}_0$ we have $(-1)^{m_1} = (-1)^{m_2}$, and the lifted Fourier symbol in (25) simplifies to

$$\Phi_0 = \frac{1}{a_1^+a_2^- - a_1^-a_2^+}\begin{bmatrix} (a_2^-b_1^+ - a_2^+b_1^-)\rho^{1-2m_1} & (a_1^+b_1^- - a_1^-b_1^+)\rho^{1-m_1-m_2} \\ (a_2^-b_2^+ - a_2^+b_2^-)\rho^{1-m_1-m_2} & (a_1^+b_2^- - a_1^-b_2^+)\rho^{1-2m_2} \end{bmatrix}.$$

Remark that $\rho(\xi)^{1-m_1-m_2}$, as well as $\rho(\xi)^{1-2m_j}$, $j = 1, 2$, tends to ± 1 as ξ tends to $\pm\infty$, respectively. Thus $\Phi_0(-\infty) = -\Phi_0(+\infty)$ and the Fredholm criterium (13) gives

$$\mu\Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty) = (1-2\mu)\Phi_0(+\infty),$$

which degenerates for $\mu = 1/2$, i.e., Φ_0 doesn't fulfill the Fredholm criterium for $\mu = 1/2$. We now arrive at

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} \frac{(a_1^+a_2^- - a_1^-a_2^+)(b_1^+b_2^- - b_1^-b_2^+)}{(a_1^-a_2^+ - a_1^+a_2^-)(b_1^+b_2^- - b_1^-b_2^+)} & 0 \\ 0 & \frac{(a_1^+a_2^- - a_1^-a_2^+)(b_1^+b_2^- - b_1^-b_2^+)}{(a_1^-a_2^+ - a_1^+a_2^-)(b_1^+b_2^- - b_1^-b_2^+)} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore the result is a consequence of Theorem 3.2, applied to the jump at infinity matrix of diagonal form with one eigenvalue $\lambda = -1$ with multiplicity two. □

For orders $m_1 + m_2 \in 2\mathbb{N}_0 + 1$ the following normalization theorem is an example when the image normalization can also depend on the coefficients.

Theorem 5.3. *Let $m_1 + m_2 \in 2\mathbb{N}_0 + 1$ in the boundary-transmission conditions* (21)–(22) *and assume that* (24) *holds. Then the operator $\mathcal{P}$ and the equivalent operator W are not-normally solvable iff there exists a solution $\theta \in [-1, 1]$ for the equation*

$$\theta^2 = \frac{(a_1^+b_1^- - a_1^-b_1^+)(a_2^+b_2^- - a_2^-b_2^+)}{(a_1^+b_2^- + a_1^-b_2^+)(a_2^-b_1^+ + a_2^+b_1^-)}. \tag{26}$$

In this case, the image space of the image normalized operator $\breve{W}$ defined by

$$\breve{W} = \operatorname{Rst} W : \left[H_+^{1/2-m_1}\right]^2 \to r_+\Lambda_-^{-s}T\ell^{(0)}\left\{\breve{H}^{-i\tau}(\mathbb{R}_+) \times L_2(\mathbb{R}_+)\right\},$$

with $s = (1/2 - m_1, 1/2 - m_2)$, solves the normalization problem for the WHO. Here T is the matrix which allows the diagonalization $\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = T^{-1}\operatorname{diag}(\lambda_1, \lambda_2)T$ for which the argument of the eigenvalue λ_1 is equal to $-\pi$ and $\tau = \frac{1}{2\pi}\log|\frac{\Phi_0(-\infty)}{\Phi_0(+\infty)}|$ corresponds to λ_1. Moreover, the image normalization of operator $\mathcal{P}$ is obtained by substituting W by $\breve{W}$ in the equivalence relation (10).

Proof. For $m_1 + m_2 \in 2\mathbb{N}_0 + 1$ we have $(-1)^{m_1} = -(-1)^{m_2}$, and the lifted Fourier symbol in (25) simplifies to

$$\Phi_0 = \frac{1}{a_1^+a_2^- + a_1^-a_2^+}\begin{bmatrix} (a_2^-b_1^+ + a_2^+b_1^-)\rho^{1-2m_1} & (a_1^+b_1^- - a_1^-b_1^+)\rho^{1-m_1-m_2} \\ (a_2^+b_2^- - a_2^-b_2^+)\rho^{1-m_1-m_2} & (a_1^+b_2^- + a_1^-b_2^+)\rho^{1-2m_2} \end{bmatrix}.$$

Remark that now, while $\rho(\xi)^{1-m_1-m_2}$ tends to one, $\rho(\xi)^{1-2m_j}$, $j = 1, 2$, tends to ± 1 as ξ tends to $\pm\infty$, respectively. Therefore, the Fredholm criterium (13) gives

$$\mu\Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty) = \frac{1}{a_1^+a_2^- + a_1^-a_2^+}\begin{bmatrix} (1-2\mu)(a_2^-b_1^+ + a_2^+b_1^-) & a_1^+b_1^- - a_1^-b_1^+ \\ a_2^+b_2^- - a_2^-b_2^+ & (1-2\mu)(a_1^+b_2^- + a_1^-b_2^+) \end{bmatrix},$$

which degenerates for

$$(1-2\mu)^2\left(a_2^-b_1^+ + a_2^+b_1^-\right)\left(a_1^+b_2^- + a_1^-b_2^+\right) - \left(a_1^+b_1^- - a_1^-b_1^+\right)\left(a_2^+b_2^- - a_2^-b_2^+\right) = 0$$

or equivalently when (26) holds, where we introduced $\theta = 1 - 2\mu$.

Since one has

$$\Phi_0(-\infty) = \frac{1}{b_1^+b_2^- + b_1^-b_2^+}\begin{bmatrix} -a_1^+b_2^- - a_1^-b_2^+ & -a_1^+b_1^- + a_1^-b_1^+ \\ -a_2^+b_2^- + a_2^-b_2^+ & -a_2^+b_1^- - a_2^-b_1^+ \end{bmatrix},$$

we get

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} \frac{(\theta^2-1)(a_1^+b_2^- + a_1^-b_2^+)(a_2^-b_1^+ + a_2^+b_1^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)} & \frac{2(a_1^+b_2^- + a_1^-b_2^+)(a_1^-b_1^+ - a_1^+b_1^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)} \\ \frac{2(a_2^-b_1^+ + a_2^+b_1^-)(a_2^-b_2^+ - a_2^+b_2^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)} & \frac{(1-\theta^2)(a_1^+b_2^- + a_1^-b_2^+)(a_2^-b_1^+ + a_2^+b_1^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)} \end{bmatrix}$$

or, after introducing the notations

$$A_{11} = \frac{(a_1^+b_2^- + a_1^-b_2^+)(a_2^-b_1^+ + a_2^+b_1^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)},$$

$$A_{12} = \frac{2(a_1^+b_2^- + a_1^-b_2^+)(a_1^-b_1^+ - a_1^+b_1^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)},$$

$$A_{21} = \frac{2(a_2^-b_1^+ + a_2^+b_1^-)(a_2^-b_2^+ - a_2^+b_2^-)}{(a_1^+a_2^- + a_1^-a_2^+)(b_1^+b_2^- + b_1^-b_2^+)},$$

we arrive at

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} (\theta^2-1)A_{11} & A_{12} \\ A_{21} & (1-\theta^2)A_{11} \end{bmatrix}.$$

The eigenvalues of the jump at infinity matrix are

$$\lambda_j = \pm\sqrt{\left(1-\theta^2\right)^2 A_{11}^2 + A_{12}A_{21}}, \quad j=1,2, \tag{27}$$

and the result is a consequence of Theorem 3.2, with the choice of T to be the matrix, possible with a permutation of columns, that yields the argument of the eigenvalue λ_1 in (27) to be equal to $-\pi$. □

6. Boundary-transmission problems with oblique derivatives

Finally, we analyze a particular case of boundary-transmission conditions of the form (15)–(16), when we have a boundary condition with normal derivatives of order m_1 on both banks of $\mathbb{R}_-$, and boundary-transmission conditions with normal and tangential derivatives of order m_2 on both banks of $\mathbb{R}_+$. That is, together with the Helmholtz equation (1), consider the following boundary-transmission conditions of orders $m = (m_1, m_1)$ on the left half-line and order $m' = (m_2, m_2)$ on the right half-line

$$\begin{cases} \varphi_{m_1}^- = h_1, \\ \varphi_{m_1}^- = h_2, \end{cases} \quad \text{on } \mathbb{R}_-, \tag{28}$$

$$\begin{cases} b_2^+\varphi_{m_2}^+ + b_2^-\varphi_{m_2}^- + \check{b}_2^+\check{\varphi}_{m_2}^+ + \check{b}_2^-\check{\varphi}_{m_2}^- = g_1, \\ c_2^+\varphi_{m_2}^+ + c_2^-\varphi_{m_2}^- + \check{c}_2^+\check{\varphi}_{m_2}^+ + \check{c}_2^-\check{\varphi}_{m_2}^- = g_2, \end{cases} \quad \text{on } \mathbb{R}_+, \tag{29}$$

where in general $m_1 \neq m_2$, $h = (h_1, h_2) \in [H^{1/2-m_1}(\mathbb{R}_-)]^2$ and $g = (g_1, g_2) \in [H^{1/2-m_2}(\mathbb{R}_+)]^2$.

The following necessary and sufficient conditions hold for the operator $\mathcal{P}$ and the equivalent WHO W be of normal type.

Theorem 6.1. *Consider the associated operator*

$$\mathcal{P} : D(\mathcal{P}) \to \left[H^{1/2-m_2}(\mathbb{R}_+)\right]^2,$$
$$\varphi \to \mathcal{P}\varphi = g,$$

and the equivalent WHO

$$W = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : \left[H_+^{1/2-m_1}\right]^2 \to \left[H^{1/2-m_2}(\mathbb{R}_+)\right]^2$$

with Fourier symbol

$$\Phi = (-1)^{m_1}\beta^{m_1-m_2}\begin{bmatrix} (-1)^{m_2}b_2^+ + \check{b}_2^+(-i\xi\beta^{-1})^{m_2} & (-1)^{m_1}(b_2^- + \check{b}_2^-(-i\xi\beta^{-1})^{m_2}) \\ (-1)^{m_2}c_2^+ + \check{c}_2^+(-i\xi\beta^{-1})^{m_2} & (-1)^{m_1}(c_2^- + \check{c}_2^-(-i\xi\beta^{-1})^{m_2}) \end{bmatrix}. \tag{30}$$

Then both operators W and $\mathcal{P}$ are of normal type iff

$$\begin{aligned} &(-1)^{m_2}\left(b_2^+c_2^- - b_2^-c_2^+\right) + \left((-1)^{m_2}\left(b_2^+\check{c}_2^- - \check{b}_2^-c_2^+\right) + \check{b}_2^+c_2^- - b_2^-\check{c}_2^+\right)\left(-i\xi\beta^{-1}\right)^{m_2} \\ &\quad + \left(\check{b}_2^+\check{c}_2^- - \check{b}_2^-\check{c}_2^+\right)\left(-i\xi\beta^{-1}\right)^{2m_2} \neq 0. \end{aligned} \tag{31}$$

Proof. As before, we obtain first the Fourier symbol in (30) from the Fourier symbols of the operators $\tilde{B}_-$ and B_+, and then come to the lifted symbol

$$\Phi_0 = (-1)^{m_1}\rho^{1-m_1-m_2}\begin{bmatrix} (-1)^{m_2}b_2^+ + \check{b}_2^+(-i\xi\beta^{-1})^{m_2} & (-1)^{m_1}(b_2^- + \check{b}_2^-(-i\xi\beta^{-1})^{m_2}) \\ (-1)^{m_2}c_2^+ + \check{c}_2^+(-i\xi\beta^{-1})^{m_2} & (-1)^{m_1}(c_2^- + \check{c}_2^-(-i\xi\beta^{-1})^{m_2}) \end{bmatrix}. \tag{32}$$

Thus, condition (31) follows from the assumption that $\det\Phi_0 \neq 0$. □

We must consider now four situations: both orders m_j are zero or even, both orders are odd, m_1 is zero or even and m_2 is odd, and the way around, m_1 is odd and m_2 is zero or even. These four cases give rise to the following four results.

Theorem 6.2. *Let $m_1, m_2 \in 2\mathbb{N}_0$ in the boundary-transmission conditions* (28)–(29) *and assume that* (31) *holds. Then the operator $\mathcal{P}$ and the equivalent operator W are not-normally solvable. In this case, the image space of the image normalized operator $\breve{W}$ defined by*

$$\breve{W} = \mathrm{Rst}\, W : \left[H_+^{1/2-m_1}\right]^2 \to r_+ \Lambda_-^{-s} \ell^{(0)} \left\{\breve{H}^0(\mathbb{R}_+) \times \breve{H}^0(\mathbb{R}_+)\right\},$$

with $s = (1/2 - m_2, 1/2 - m_2)$, solves the normalization problem for the WHO. Furthermore, the image normalization of operator $\mathcal{P}$ is obtained by substituting W by $\breve{W}$ in the equivalence relation (10).

Proof. For $m_1 = m_2$ this is a direct consequence of Theorem 4.2. For different orders $m_1 \neq m_2$ we also arrive at $\Phi_0(-\infty) = -\Phi_0(+\infty)$, since the lifted Fourier symbol simplifies to

$$\Phi_0 = \rho^{1-m_1-m_2} \begin{bmatrix} b_2^+ + i^{m_2} \breve{b}_2^+ (\xi\beta^{-1})^{m_2} & b_2^- + i^{m_2} \breve{b}_2^- (\xi\beta^{-1})^{m_2} \\ c_2^+ + i^{m_2} \breve{c}_2^+ (\xi\beta^{-1})^{m_2} & c_2^- + i^{m_2} \breve{c}_2^- (\xi\beta^{-1})^{m_2} \end{bmatrix}.$$

In this case we obtain, once more, a diagonal matrix with -1 in the diagonal entries for the jump at infinity matrix. □

Theorem 6.3. *Let $m_1, m_2 \in 2\mathbb{N}_0 + 1$ in the boundary-transmission conditions* (28)–(29) *and assume that* (31) *holds. Then the operators $\mathcal{P}$ and W are not-normally solvable if there exists a solution $\theta \in [-1, 1]$ for the equation*

$$\left(b_2^+ c_2^- - b_2^- c_2^+\right)\theta^2 + i^{m_2}\left(\breve{b}_2^+ c_2^- - b_2^+ \breve{c}_2^- + \breve{b}_2^- c_2^+ - b_2^- \breve{c}_2^+\right)\theta + \breve{b}_2^+ \breve{c}_2^- - \breve{b}_2^- \breve{c}_2^+ = 0. \tag{33}$$

In this case, the image space of the image normalized operator $\breve{W}$ given by

$$\breve{W} = \mathrm{Rst}\, W : \left[H_+^{1/2-m_1}\right]^2 \to r_+ \Lambda_-^{-s} T \ell^{(0)} \left\{\breve{H}^{-i\tau}(\mathbb{R}_+) \times L_2(\mathbb{R}_+)\right\},$$

with $s = (1/2 - m_2, 1/2 - m_2)$, solves the normalization problem for the WHO. The matrix T is chosen to be the matrix in the diagonalization $\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = T^{-1} \operatorname{diag}(\lambda_1, \lambda_2) T$ for which the argument of the eigenvalue λ_1 is equal to $-\pi$ and $\tau = \frac{1}{2\pi} \log |\frac{\Phi_0(-\infty)}{\Phi_0(+\infty)}|$ corresponds to the eigenvalue λ_1. The image normalization of the operator $\mathcal{P}$ is obtained by substituting W by $\breve{W}$ in the equivalence relation (10).

Proof. For $m_1 + m_2 \in 2\mathbb{N}_0 + 1$ we have $(-1)^{m_1} = (-1)^{m_2} = -1$, and the lifted Fourier symbol in (32) simplifies to

$$\Phi_0 = \rho^{1-m_1-m_2} \begin{bmatrix} b_2^+ + i^{m_2} \breve{b}_2^+ (\xi\beta^{-1})^{m_2} & b_2^- - i^{m_2} \breve{b}_2^- (\xi\beta^{-1})^{m_2} \\ c_2^+ + i^{m_2} \breve{c}_2^+ (\xi\beta^{-1})^{m_2} & c_2^- - i^{m_2} \breve{c}_2^- (\xi\beta^{-1})^{m_2} \end{bmatrix}.$$

Here, while $\rho(\xi)^{1-m_1-m_2}$ tends to one, $(\xi\beta(\xi)^{-1})^{m_2}$ tends to ± 1 as ξ tends to $\pm\infty$, respectively. Thus the Fredholm criterium (13) applied to the lifted Fourier symbol gives

$$\mu\Phi_0(-\infty) + (1-\mu)\Phi_0(+\infty) = \begin{bmatrix} b_2^+\theta + i^{m_2}\breve{b}_2^+ & b_2^-\theta - i^{m_2}\breve{b}_2^- \\ c_2^+\theta + i^{m_2}\breve{c}_2^+ & c_2^-\theta - i^{m_2}\breve{c}_2^- \end{bmatrix},$$

where we introduced the former notation $\theta = 1 - 2\mu$. Remark that once $i^{2m_2} = -1$, the determinant of the last matrix equals zero when (33) holds.

Since we have

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} -\frac{A + i^{m_2} B}{C} & \frac{2i^{m_2} D}{C} \\ \frac{2i^{m_2} E}{C} & \frac{-A + i^{m_2} B}{C} \end{bmatrix},$$

where we introduced the notations

$$A = b_2^- c_2^+ - b_2^+ c_2^- + \breve{b}_2^+ \breve{c}_2^- - \breve{b}_2^- \breve{c}_2^+, \qquad B = \breve{b}_2^- \breve{c}_2^+ + b_2^- \breve{c}_2^+ - \breve{b}_2^+ c_2^- - b_2^+ \breve{c}_2^-,$$
$$C = b_2^- c_2^+ - b_2^+ c_2^- + \breve{b}_2^- \breve{c}_2^- + \breve{b}_2^- \breve{c}_2^+ + i^{m_2} B, \qquad D = b_2^- \breve{c}_2^- - \breve{b}_2^- c_2^-, \qquad E = b_2^+ \breve{c}_2^+ - \breve{b}_2^+ c_2^+,$$

we get the eigenvalues

$$\lambda_j = \frac{-A \pm i\sqrt{B^2 + 4DE}}{C}, \quad j = 1, 2. \tag{34}$$

Therefore the result is a consequence of Theorem 3.2, applied to the jump at infinity matrix with the choice of the eigenvalue λ_1 in (34) to be the one for which the argument is equal to $-\pi$. □

Theorem 6.4. *Let $m_1 \in 2\mathbb{N}_0$ and $m_2 \in 2\mathbb{N}_0 + 1$ in the boundary-transmission conditions* (28)–(29) *and assume that* (31) *holds. Then the operator $\mathcal{P}$ and the equivalent operator W are not-normally solvable if there exists a solution $\theta \in [-1, 1]$ for the equation*

$$\big(\check{b}_2^- \check{c}_2^+ - \check{b}_2^+ \check{c}_2^-\big)\theta^2 - i^{m_2}\big(\check{b}_2^+ c_2^- - \check{c}_2^- b_2^+ + \check{b}_2^- c_2^+ - b_2^- \check{c}_2^+\big)\theta + b_2^- c_2^+ - b_2^+ c_2^- = 0. \tag{35}$$

In this case, the image space of the image normalized operator $\breve{W}$ given by

$$\breve{W} = \operatorname{Rst} W : \big[H_+^{1/2-m_1}\big]^2 \to r_+ \Lambda_-^{-s} T \ell^{(0)}\big\{\breve{H}^{-i\tau}(\mathbb{R}_+) \times L_2(\mathbb{R}_+)\big\},$$

with $s = (1/2 - m_2, 1/2 - m_2)$, solves the normalization problem for the WHO. The matrix T is chosen to be the matrix in the diagonalization $\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = T^{-1}\operatorname{diag}(\lambda_1, \lambda_2)T$ for which the argument of the eigenvalue λ_1 is equal to $-\pi$ and $\tau = \frac{1}{2\pi}\log|\frac{\Phi_0(-\infty)}{\Phi_0(+\infty)}|$ corresponds to the eigenvalue λ_1. The image normalization of the operator $\mathcal{P}$ can be obtained by substituting W by $\breve{W}$ in the equivalence relation (10)*.*

Proof. The proof follows the same steps as the proof of Theorem 6.3. In this case the jump at infinity is characterized by the matrix

$$\Phi_0^{-1}(-\infty)\Phi_0(+\infty) = \begin{bmatrix} \frac{A+i^{m_2}B}{C} & \frac{2i^{m_2}D}{C} \\ \frac{2i^{m_2}E}{C} & \frac{A-i^{m_2}B}{C} \end{bmatrix},$$

where the notations are the same as the ones used in the proof of Theorem 6.3. The eigenvalues are given by

$$\lambda_j = \frac{A \pm i\sqrt{B^2 + 4DE}}{C}, \quad j = 1, 2,$$

and we must choose the diagonalization which gives the value of $-\pi$ for the argument of λ_1. $\square$

Theorem 6.5. *Let $m_1 \in 2\mathbb{N}_0 + 1$ and $m_2 \in 2\mathbb{N}_0$ in the boundary-transmission conditions* (28)–(29) *and assume that* (31) *holds. Then both operators $\mathcal{P}$ and W are normally solvable operators.*

Proof. For $m_1 \in 2\mathbb{N}_0 + 1$ and $m_2 \in 2\mathbb{N}_0$ the lifted Fourier symbol reads

$$\Phi_0 = \rho^{1-m_1-m_2}\begin{bmatrix} -b_2^+ - i^{m_2}\check{b}_2^+(\xi\beta^{-1})^{m_2} & b_2^- + i^{m_2}\check{b}_2^-(\xi\beta^{-1})^{m_2} \\ -c_2^+ - i^{m_2}\check{c}_2^+(\xi\beta^{-1})^{m_2} & c_2^- + i^{m_2}\check{c}_2^-(\xi\beta^{-1})^{m_2} \end{bmatrix},$$

which has no jumps at infinity, since both $\rho(\xi)^{1-m_1-m_2}$ and $(\xi\beta(\xi)^{-1})^{m_2}$ tend to one as ξ tends to $\pm\infty$. Therefore, the corresponding WHO has always a closed image and, by the equivalence relation (10), so does the operator $\mathcal{P}$. $\square$

In the present paper we were able to achieve the image normalization of particular WHOs which arise from relevant boundary-transmission value problems for a junction of two half-planes. For theoretical and practical reasons it is most important to be able to answer further questions about the invertibility or the Fredholm properties of these operators. We plan to do this in a future work.

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